Dirac-type characterizations of graphs without long chordless cycles

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Abstract

We call a chordless path $v_1v_2 \ldots v_i$ *simplicial* if it does not extend into any chordless path $v_0v_1v_2 \ldots v_iv_{i+1}$. Trivially, for every positive integer k, a graph contains no chordless cycle of length $k+3$ or more if each of its nonempty induced subgraphs contains a simplicial path with at most k vertices; we prove the converse. The case of $k=1$ is a classic result of Dirac.

Keywords: triangulated graphs, induced paths, forbidden induced cycles.

The graphs that we consider are undirected, simple, and finite. We use the phrase *chordless cycle* in preference to *induced cycle* and we use the phrase *chordless path* in preference to *induced path.* We specify each path by simply listing its vertices in their natural order. As usual, we let the symbol P_j stand for a path with precisely j vertices.

A vertex in a graph is called *simplicial* if all of its neighhbors are pairwise adjacent. A classic theorem of Dirac [1] asserts that

(D1) a graph contains no chordless cycle of length four or more if and only if each of its induced subgraphs either is a clique or contains two nonadjacent simplicial vertices;

its corollary states that

(D2) a graph contains no chordless cycle of length four or more if and only if each of its nonempty induced subgraphs contains a simplicial vertex.

The purpose of this note is to generalize (D1) and (D2). Towards this end, let us call a chordless path $v_1 v_2 \dots v_i$ with at least one vertex *simplicial* if it does not extend into any chordless path $v_0 v_1 v_2 \dots v_i v_{i+1}$; note that "a simplicial vertex" and "a simplicial P_1" are synonymous. Our generalizations of (D1) and (D2) go as follows.

THEOREM 1 *For every positive integer k, a graph contains no chordless cycle of length $k+3$ or more if and only if each of its nonempty induced subgraphs G has the following property:*

> *If G contains a vertex v and a chordless P_k with all k vertices distinct from v and nonadjacent to v, then it contains a simplicial P_k with all k vertices distinct from v and nonadjacent to v.*

THEOREM 2 *For every positive integer k, a graph contains no chordless cycle of length $k+3$ or more if and only if each of its nonempty induced subgraphs contains a simplicial path with at most k vertices.*

For each vertex v of a graph G, let $M(v)$ denote the set of all vertices nonadjacent to v and distinct from v. The following lemma goes a long way towards our two theorems.

LEMMA. *Let k be a positive integer and let G be a graph with no chordless cycle of length $k+3$ or more.*

(A) If G contains a vertex v and a chordless P_k with all k vertices in $M(v)$, then it contains a simplicial P_k with all k vertices in $M(v)$.

(B) If G contains a chordless P_k, then it contains a simplicial P_k.

PROOF. Consider the following statements.

A_n: Let G be a graph with at most n vertices that contains no chordless cycle of length $k+3$ or more. If G contains a vertex v and a chordless P_k with all k vertices in $M(v)$, then it contains a simplicial P_k with all k vertices in $M(v)$.

B_n: Let G be a graph with at most n vertices that contains no chordless cycle of length $k+3$ or more. If G contains a chordless P_k, then it contains a simplicial P_k.

Each of A_n and B_n holds vacuously when $n < k$; we propose to show that $A_n \Rightarrow B_n$ for all n and that $(A_n \& B_n) \Rightarrow A_{n+1}$ for all n.

To see that $A_n \Rightarrow B_n$, consider a graph G with at most n vertices that contains no chordless cycle of length $k+3$ or more; let $v_1 v_2 \ldots v_k$ be a chordless path in G. If this path is simplicial, then we are done; else it extends to some chordless path $v_0 v_1 v_2 \ldots v_k v_{k+1}$ and we are done by A_n with v_{k+1} in place of v.

To show that $(A_n \& B_n) \Rightarrow A_{n+1}$, consider a graph G with at most $n+1$ vertices that contains no chordless cycle of length $k+3$ or more; let v be a vertex of G such that G contains a chordless P_k with all k vertices in $M(v)$. Let F denote the subgraph of G induced by $M(v)$. Assumption B_n guarantees that F contains a simplicial P_k, say $v_1 v_2 \ldots v_k$. If this P_k is simplicial in G, then we are done; else it extends to some chordless path $v_0 v_1 v_2 \ldots v_k v_{k+1}$ in G. Since $v_1 v_2 \ldots v_k$ is simplicial in F, at least one of v_0 and v_{k+1} lies outside F; since G contains no chordless cycle of length $k+3$, at most one of v_0 and v_{k+1} is adjacent to v; hence symmetry allows us to assume that v_0 is adjacent to v and v_{k+1} is not. Now let Q be a minimal set of vertices of G such that vv_0 is in one component of $G-Q$ and $v_2 \dots v_k v_{k+1}$ is in another; let S be the component of $G-Q$ that contains vv_0 and let H be the graph obtained from G by shrinking S into a single vertex w. Trivially, H has at most n vertices and contains no chordless cycle of length $k+3$ or more; minimality of Q guarantees that each vertex of Q has at least one neighbor in S, and so Q is the set of all the neighbors of w in H. The existence of the desired simplicial P_k is guaranteed by A_n with H in place of G and w in place of v. □

PROOF OF THEOREM 1.

The "if" part: Every chordless cycle of length $k+3$ or more contains a vertex v and a chordless P_k with all k vertices distinct from v and nonadjacent to v, but it does not contain any simplicial P_k.

The "only if" part follows from part (A) of the Lemma. □

PROOF OF THEOREM 2.

The "if" part: No chordless cycle of length $k+3$ or more contains a simplicial path with at most k vertices.

The "only if" part: Let H be a graph that contains no chordless cycle of length $k+3$ or more, let G be a nonempty induced subgraph of H, and let $w_1 w_2 \dots w_j$ be a maximal chordless path in G. If $j \geq k$, then we appeal to part (B) of the Lemma; else we are done simply because $w_1 w_2 \dots w_j$, being maximal in G, is simplicial in G. □

A few results in a similar vein go as follows. Golumbic [3] showed that every bipartite graph containing no chordless cycle of length six or more contains a simplicial P_2 (unless it contains no edge at all); clearly, this theorem is a special case of our Theorem 2. Eschen and Sritharan [2] generalized Golumbic's theorem in a different direction, concerning graphs containing no chordless cycles of length five or more and no complements of chordless cycles up to some fixed length. Hayward [4] proved that every graph containing no chordless cycles of length five or more and no complement of P_5 has a vertex that is not the middle of any P_5.

References

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