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RECOGNIZING DART-FREE PERFECT GRAPHS*

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Abstract. A graph G is called a *Berge graph* if neither G nor its complement contains a chordless cycle whose length is odd and at least five; what we call a *dart* is the graph with vertices u, v, w, x, y and edges uv, vw, uy, vy, wy, xy; a graph is called *dart-free* if it has no induced subgraph isomorphic to the dart. We present a polynomial-time algorithm to recognize dart-free Berge graphs; this algorithm uses as a subroutine the polynomial-time algorithm for recognizing claw-free Berge graphs designed previously by Chvátal and Sbihi [*J. Combin. Theory Ser. B*, 44 (1988), pp. 154–176].

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1. Introduction. Claude Berge proposed the notion of a *perfect graph*, which is a graph G such that, for each induced subgraph F of G, the chromatic number of F equals the largest number of pairwise adjacent vertices in F; he made the following conjecture.

CONJECTURE 1.1 (The Strong Perfect Graph Conjecture). *A graph is perfect if and only if it contains, as an induced subgraph, no odd hole chordless cycle whose number of vertices is odd and at least five and no complement of such a cycle.*

Berge publicized the Strong Perfect Graph Conjecture as early as April 1960—at the Second International Meeting on Graph Theory, organized by Horst Sachs at the University of Halle-Wittenberg—but published it only three years later [1]; the first widely available reference to it is [2]. For an account of the early history of the conjecture, see [3, 4].

A *hole* is a chordless cycle with at least four vertices; an *antihole* is the complement of a hole; holes and antiholes are called *even* or *odd* according to the parity of their number of vertices. Following Chvátal and Sbihi [6], we shall call a graph a *Berge graph* if it contains no odd hole and no odd antihole: in these terms, the Strong Perfect Graph Conjecture asserts that a graph is perfect if and only if it is a Berge graph. Its "only if" part is trivial and its "if" part remains open.

Progress towards proving the Strong Perfect Graph Conjecture is often made by proving that all graphs in some restricted class of Berge graphs are perfect. A popular way of creating restricted classes of Berge graphs for this purpose is to forbid a single induced subgraph F: the resulting class consists of all Berge graphs that are F-*free* in the sense of containing no induced subgraph isomorphic to F. In particular, with a *claw* defined as the first graph in Figure 1.1, Parathasarathy and Ravindra [10]

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proved that all claw-free Berge graphs are perfect; with a *diamond* defined as the second graph in Figure 1.1, Tucker [12] proved that all diamond-free Berge graphs are perfect; Sun [11] strengthened both of these results by proving that, with a *dart* defined as the third graph in Figure 1.1, all dart-free Berge graphs are perfect.

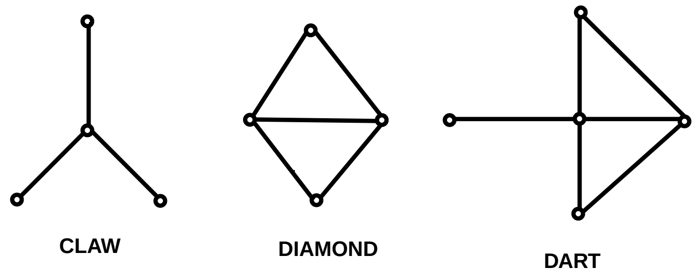

FIG. 1.1. *The claw, the diamond, and the dart.*

Another fundamental conjecture concerning perfect graphs is the following.

CONJECTURE 1.2. *There is a polynomial-time algorithm to recognize perfect graphs.*

Again, progress towards proving this conjecture is often made by designing a polynomial-time algorithm to recognize members of some restricted class of perfect graphs. In particular, Chvátal and Sbihi [6] designed a polynomial-time algorithm to recognize claw-free perfect graphs; Fonlupt and Zemirline [8, 9] designed a polynomial-time algorithm to recognize diamond-free perfect graphs; the subject of the present paper is a polynomial-time algorithm to recognize dart-free Berge graphs.

For a recent survey of results on perfect graphs and related subjects, see [5].

2. Theorems. Many theorems elucidate the structure of objects in some class $\mathcal{C}$ in terms of some proper subclass $\mathcal{C}_0$ of $\mathcal{C}$: they assert that, unless an object in $\mathcal{C}$ is primitive in the sense of belonging to $\mathcal{C}_0$, it must have a structural fault of a prescribed type. We follow this paradigm in three iterations.

First, let us call a graph *friendly* if, for each of its vertices x such that x is the center of a claw, the subgraph of G induced by the neighbors of x consists of vertex-disjoint cliques. Trivially, every friendly graph is dart-free; however, the converse is false (see either of the two graphs in Figure 2.1).

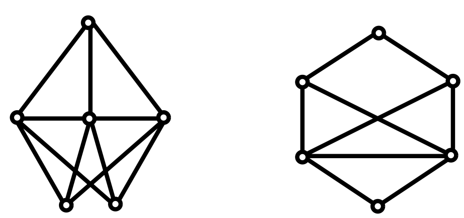

FIG. 2.1. *Two unfriendly dart-free graphs.*

Our first theorem specifies structural faults appearing in every dart-free graph that is not friendly. One of these faults is the presence of *adjacent twins*, meaning two adjacent vertices such that no vertex distinct from both is adjacent to precisely one of them.

Theorem 2.1. *Unless a dart-free graph is friendly, it has at least one of the following properties:*

(i) *it is disconnected;*
(ii) *its complement is disconnected;*
(iii) *it contains adjacent twins.*

Next, let us call a graph a *bat* if it consists of a chordless path $a_1a_2\dots a_m$ and an additional vertex z that is adjacent to a_1, a_i, a_{i+1}, a_m for some i with $3 \le i \le m-3$ and adjacent to no other a_j. Occasionally, we shall refer to z as the *head* of the bat and to a_1, a_m as its *wing-tips*; see Figure 2.2.

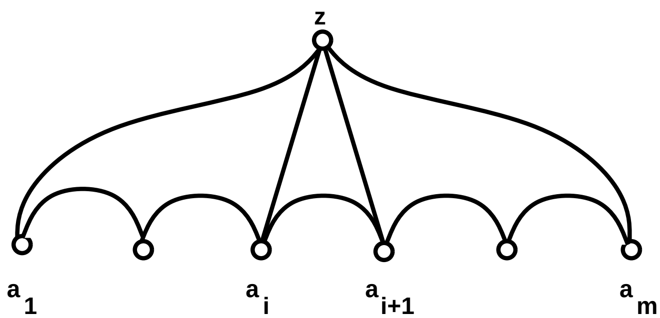

Fig. 2.2. *A bat.*

There are infinitely many bats; we call a graph *bat-free* if it contains none of them (as an induced subgraph). Our second theorem specifies structural faults appearing in every friendly Berge graph that is not bat-free. One of these faults is a clique-cutset; to describe the other fault, we need a few definitions. By a *z-edge*, we mean any edge whose endpoints are both adjacent to a vertex z; for any graph G and any vertex z of G, we let $G{*}z$ denote the graph obtained from $G - z$ by removing all the z-edges; we say that G has a *rosette* centered at z if $G{*}z$ is disconnected and the subgraph of G induced by all the neighbors of z consists of vertex-disjoint cliques.

Theorem 2.2. *Every friendly graph containing no odd hole has at least one of the following properties:*

(i) *it is bat-free;*
(ii) *it has a clique-cutset that is a maximal clique;*
(iii) *it has a rosette.*

Any of the three graphs in Figure 2.4 shows that property (i) in this theorem cannot be dropped; the two graphs in Figure 2.3 show that neither (ii) nor (iii) can be dropped.

The two notions of a clique-cutset and a rosette may appear unrelated in terms of G; in terms of a certain graph which we call the *clique graph of* G, they are nearly dual to each other. The clique graph of G is bipartite; its white vertices are the vertices of G and its red vertices are the maximal cliques in G (here, as usual, "maximal" is meant with respect to set-inclusion rather than size); a white vertex z is adjacent to a red vertex C if and only if $z \in C$. Trivially, the removal of a red vertex C and all its neighbors disconnects the clique graph of G if and only if $G - C$ is disconnected; trivially, the removal of a white vertex z and all its neighbors disconnects the clique graph of G if and only if $G{*}z$ is disconnected.

Theorem 2.2 is closely related to the following theorem of Conforti and Rao [7]: *If a bipartite graph H containing no cycle of length four and no chordless cycle of*

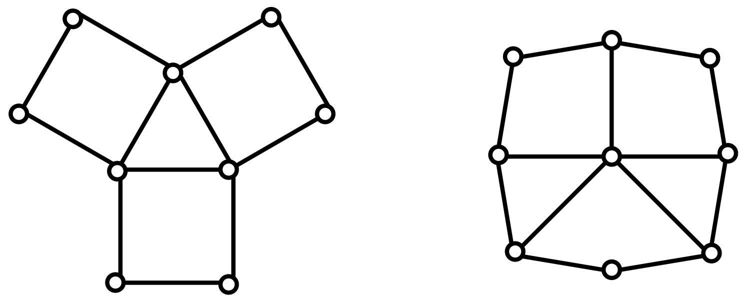

FIG. 2.3. *Neither* (ii) *nor* (iii) *in Theorem* 2.2 *can be dropped.*

length congruent to 2 *modulo* 4 *contains a cycle of length congruent to* 2 *modulo* 4*, then* H *has a vertex* v *such that the removal of* v *and all its neighbors disconnects* H*.* We are going to sketch a derivation of the Conforti–Rao theorem from a weaker form of Theorem 2.2, where "friendly" is replaced by "diamond-free."

Given a bipartite graph H that satisfies the hypothesis of the Conforti–Rao theorem, color the vertices in one part of H red, color the vertices in the other part of H white, and consider the graph G defined as follows: the vertices of G are the white vertices of H and two vertices of G are adjacent if and only if they have a common neighbor in H. Since H contains no chordless cycle of length six, H is the clique graph of G; since H contains no cycle of length four, G is diamond-free; since H contains no chordless cycle of length congruent to 2 modulo 4, G contains no odd hole. The shortest cycle of length congruent to 2 modulo 4 in H has precisely one chord (cf. Lemma 3.1 of Conforti and Rao [7]) and induces a bat in G. Now Theorem 2.2 (with "friendly" replaced by "diamond-free") guarantees that H has a vertex v such that the removal of v and all its neighbors disconnects H.

Our third theorem specifies a structural fault appearing in every friendly Berge graph that is neither bipartite nor claw-free: by a *separator* in a graph G, we mean any cutset consisting of at most two vertices except cutsets S such that S consists of two nonadjacent vertices, $G - S$ has precisely two components, and one of these components consists of a single vertex.

THEOREM 2.3. *Every bat-free friendly graph containing no odd hole has at least one of the following properties:*
(i) *it is bipartite;*
(ii) *it is claw-free;*
(iii) *it has a separator.*

Again, none of the three properties in this theorem can be dropped: see the three graphs in Figure 2.4.

Our polynomial-time algorithm to recognize friendly Berge graphs evolves from Theorems 2.1, 2.2, and 2.3; in analyzing it, we shall use a strengthening of Theorem 2.2. There, a bat in a friendly graph G is called *fragile* if, with C standing for the unique maximal clique of G that contains the triangle of the bat, the two wing-tips of the bat belong to distinct components of $G - C$.

THEOREM 2.4. *Let* G *be a friendly graph containing no odd hole. If* G *contains a bat with head* z *such that the two wing-tips of the bat belong to the same component of* $G * z$*, then* G *and* z *have at least one of the following properties:*
(i) G *contains a fragile bat with head* z*;*
(ii) G *contains a clique-cutset* C *such that* $z \in C$ *and some component of* $G - C$

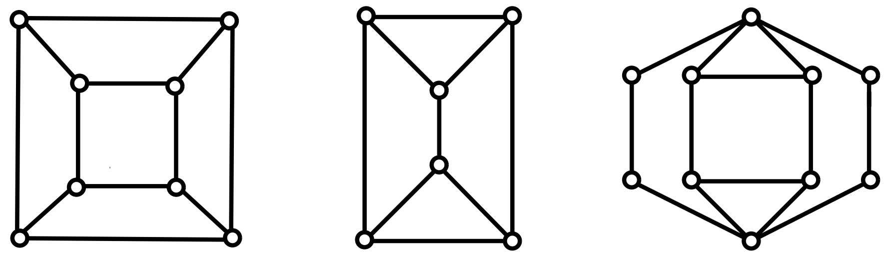

FIG. 2.4. *None of the three properties in Theorem* 2.3 *can be dropped.*

includes no neighbor of z.

Neither of the two properties in this theorem can be dropped: see the two graphs in Figure 2.5.

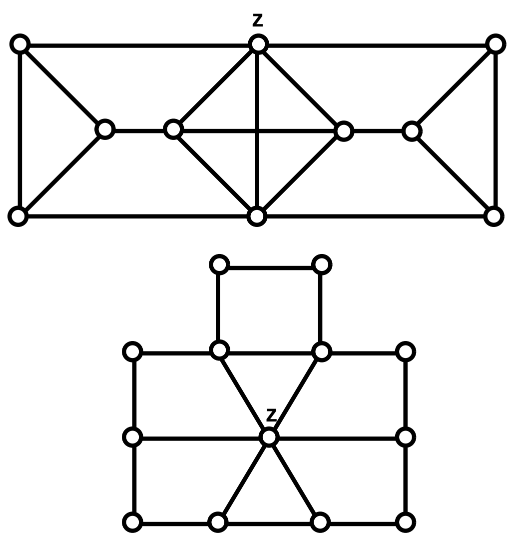

FIG. 2.5. *Neither of the two properties in Theorem* 2.3 *can be dropped.*

To derive Theorem 2.2 from Theorem 2.4, consider an arbitrary friendly graph G containing no odd hole. We may assume that G contains a bat (else it has property (i) of Theorem 2.2) and that, with z standing for the head of the bat, the two wing-tips of the bat belong to the same component of $G*z$ (else G has property (iii) of Theorem 2.2). Now G has property (i) of Theorem 2.4 or else it has property (ii) of Theorem 2.4. In the former case, the unique maximal clique of G that contains the triangle of the bat is a cutset of G; in the latter case, the unique maximal clique of G that contains C is a cutset of G.

Theorems 2.1, 2.3, and 2.4 will be proved in section 3; the algorithm and its analysis are the subject of section 4.

3. Proofs. All our subgraphs are induced. When A is a path with vertices $a_1, a_2, \ldots, a_m$ and edges $a_1a_2, a_2a_3, \ldots, a_{m-1}a_m$, we write $A = a_1a_2 \ldots a_m$ and borrow the notation used for intervals: $A[a_i, a_j] = a_i a_{i+1} \dots a_j$, $A[a_i, a_j) = a_i a_{i+1} \dots a_{j-1}$, etc. Any path of the form $a_1 a_2 \dots a_k$ will be called a *prefix* of A and any path of the form $a_k a_{k+1} \dots a_m$ will be called a *suffix* of A.

3.1. Proof of Theorem 2.1. Consider an arbitrary graph G such that G is not friendly, G is connected, its complement $\overline{G}$ is connected, and G contains no adjacent twins. We shall find a dart in G.

It is a routine matter to verify that a graph is not friendly if and only if it contains (as an induced subgraph) either a dart or else the graph with vertices u_1, u_2, u_3, v_1, v_2 and the seven edges v_1v_2, u_iv_j $(i = 1, 2, 3; j = 1, 2)$. In particular, we may assume that our G contains an induced subgraph of the second kind. Now consider the subgraph of G induced by all the common neighbors of u_1, u_2, and u_3; let Q denote the component of this subgraph that contains v_1 and v_2.

CASE 1. *There are vertices x, w_1, w_2 such that $x \notin Q, w_1 \in Q, w_2 \in Q$, and $xw_1 \in E, xw_2 \notin E$.*

Since Q is connected, we may assume that $w_1w_2 \in E$; since $x \notin Q$, we have $xu_i \notin E$ for at least one i. Now the subgraph of G induced by $u_1, u_2, u_3, w_1, w_2, x$ contains an induced dart.

CASE 2. *The set of vertices of $G - Q$ partitions into sets A and B such that $xw \in E$ whenever $x \in A, w \in Q$ and such that $xw \notin E$ whenever $x \in B, w \in Q$.*

Since $\overline{G}$ is connected, we have $B \neq \emptyset$; in turn, since G is connected, there is an edge ab with $a \in A, b \in B$. Since v_1 and v_2 are not adjacent twins, some vertex v is adjacent to precisely one of them; note that $v \in Q$. Now a, b, v, v_1, v_2 induce a dart in G. $\square$

3.2. Two simple lemmas.

LEMMA 3.1. *Let G be a graph with at least four vertices and with a unique triangle $v_1v_2v_3$. If all three graphs $G - \{v_1, v_2\}$, $G - \{v_1, v_3\}$, $G - \{v_2, v_3\}$ are connected, then G contains an odd hole.*

Proof. Let Q be any component of $G - \{v_1, v_2, v_3\}$. We may assume that Q is bipartite (else Q contains an odd hole and we are done); thus the set of vertices of Q splits into stable sets S_1 and S_2. Since each $G - \{v_i, v_j\}$ is connected, each of the three vertices v_k must have a neighbor in $S_1 \cup S_2$. Hence two of the three vertices, say v_1 and v_2, must have a neighbor in the same S_i. Since Q is connected, it follows that the subgraph of G induced by $Q \cup \{v_1, v_2\}$ is not bipartite, and so it contains an odd hole. $\square$

LEMMA 3.2. *Let an edge e of a graph G extend into no triangle. If G is friendly, then $G-e$ is friendly; if G contains no odd hole, then $G-e$ contains no odd hole.* $\square$

3.3. Proof of Theorem 2.3. By a *pseudobat*, we shall mean any graph with vertices $a_1, a_2, \dots, a_m$ and z such that

(i) $a_1 a_2 \dots a_m$ is a (not necessarily chordless) path;

(ii) z is adjacent to a_1, a_i, a_{i+1} and a_m for some i with $3 \leq i \leq m - 3$ (and possibly to other vertices a_j);

(iii) $za_1a_ia_m$ is a claw.

(In a friendly graph, condition (iii) is equivalent to saying that both $za_1a_ia_m$ and $za_1a_{i+1}a_m$ are claws.)

LEMMA 3.3. *Let G be a friendly graph containing no odd hole; let z be a vertex of G that belongs to no triangle; let $u_1u_2 \dots u_r$, $v_1v_2 \dots v_s$, and $w_1w_2 \dots w_t$ be vertex-disjoint paths in $G - z$ such that u_1, v_1, and w_1 are neighbors of z and $u_rv_sw_t$ is a triangle. Then G contains an induced pseudobat.*

Proof. The proof is by induction on the number of vertices of G. The induction hypothesis allows us to assume that the three paths are chordless and that together they cover all the vertices of $G - z$. Lemma 3.1 guarantees that G contains a triangle T other than $u_r v_s w_t$. Trivially, T meets at least two of the three chordless paths; the induction hypothesis allows us to assume that T meets at most two of the paths. Hence symmetry allows us to assume that T is $v_j w_k w_{k+1}$ for some j and k; in turn, the induction hypothesis allows us to assume that $j = s$ (consider $G - v_s$).

CASE 1. $v_s w_{t-1} \in E$.

Since G is friendly, at least two of the three vertices u_r, v_{s-1}, w_{t-1} must be adjacent (consider the neighborhood of v_s), and so we are done by the induction hypothesis applied to $G - w_t$.

CASE 2. $v_s w_{t-1} \notin E$.

Write $v_0 = z$. The induction hypothesis allows us to assume that v_{s-1} is adjacent neither to w_k nor to w_t (consider $G - w_{t-1}$). Thus the path $v_{s-1} \dots v_2 v_1 v_0 w_1 w_2 \dots w_t$ along with the additional vertex v_s induces a pseudobat in G. ∎

LEMMA 3.4. *Let G be a friendly graph containing no odd hole. If G contains an induced pseudobat, then it contains an induced bat.*

Proof. The proof is by induction on the number of edges of G. Let $a_1, a_2, \dots, a_m$ and z be as in our definition of a pseudobat. Note that $a_1 a_{i+1} \notin E$ and $a_{i+1} a_m \notin E$ since G is friendly (consider the neighborhood of z). The induction hypothesis allows us to assume that G is the pseudobat and that the two paths $a_1 a_2 \dots a_i$ and $a_{i+1} a_{i+2} \dots a_m$ are chordless.

CASE 1. *z is adjacent to some a_j other than a_1, a_i, a_{i+1}, a_m.*

Symmetry allows us to assume that $2 \le j \le i-1$. If $j = i-1$, then $a_{i-1} a_{i+1} \in E, a_1 a_{i-1} \notin E, a_{i-1} a_m \notin E$ (consider the neighborhood of z), and we are done by the induction hypothesis applied to $G - a_i$. Thus we may assume that $j \le i-2$; in particular, $a_j a_i \notin E$ (since the path $a_1 a_2 \dots a_i$ is chordless). If $a_j a_m \notin E$, then we are done by the induction hypothesis applied to the pseudobat induced in G by $a_j a_{j+1} \dots a_m$ and z; if $a_j a_m \in E$, then $a_1 a_j \notin E$, $a_j a_{i+1} \notin E$, $a_1 a_{i+1} \notin E$ (consider the neighborhood of z), and we are done by the induction hypothesis applied to the pseudobat induced in G by $a_1 a_2 \dots a_j a_m a_{m-1} \dots a_{i+1}$ and z.

CASE 2. *z is adjacent to no a_j other than a_1, a_i, a_{i+1}, a_m.*

We may assume that the path $a_1 a_2 \dots a_m$ has at least one chord (else G is a bat and we are done). If no chord of this path extends into a triangle, then G has an odd hole since conditions of Lemma 3.1 are satisfied for G or for the subgraph induced in G by $a_1 a_2 \dots a_i a_{i+1}$ and z or for the subgraph induced in G by $a_i a_{i+1} \dots a_m$ and z. Thus we may assume that a chord e extends into a triangle, T; since $z \notin T$, symmetry allows us to assume that T is $a_j a_k a_{k+1}$ for some j and k with $1 \le j \le i$ and $i+1 \le k \le m-1$. Among all such triangles, choose one with j as small as possible. If $j \le i-1$, then Lemma 3.3 guarantees that $G - a_i$ contains an induced pseudobat (in which case we are done by the induction hypothesis). Thus we may assume that $j = i$.

SUBCASE 2.1. $a_i a_{i+2} \notin E$.

Now $k \ge i+3$ and (since $z a_1 a_i a_m$ is a claw) $k \le m-2$, and so $z a_k \notin E, z a_{k+1} \notin E$; minimality of j implies that $a_{i-1} a_k \notin E$ or $a_{i-1} a_{k+1} \notin E$ (or both); hence $a_i a_{i-1} z a_k$ or $a_i a_{i-1} z a_{k+1}$ is a claw. It follows that $a_i a_{i-1} z a_k$ is a claw (consider the neighborhood of a_i), and so we are done by the induction hypothesis applied to the pseudobat induced in G by $a_{i-1} \dots a_2 a_1 z a_{i+1} a_{i+2} \dots a_k$ and a_i.

SUBCASE 2.2. $a_i a_{i+2} \in E$.

Now $a_{i-1}a_{i+2} \in E$ (consider the neighborhood of a_i), and so minimality of j implies $a_{i-1}a_{i+1} \notin E$, $a_{i-1}a_{i+3} \notin E$; since the path $a_{i+1}a_{i+2}\dots a_m$ is chordless, $a_{i+1}a_{i+3} \notin E$. Thus G is not friendly (consider the neighborhood of a_{i+2}), a contradiction. □

Proof of Theorem 2.3. Consider an arbitrary bat-free friendly graph G that contains no odd hole, is not bipartite, and is not claw-free. We shall find a separator in G.

For each vertex x of G, write $x \in A$ if x is in a triangle; write $x \in B$ if x is the center of a claw. By assumption, $A \neq \emptyset$ and $B \neq \emptyset$.

CASE 1. $A \cap B \neq \emptyset$.

Let z be a vertex in $A \cap B$. Since G is friendly, z has neighbors c_1, c_2, d_1, d_2 such that $c_1c_2 \notin E, d_1d_2$ and $c_id_j \notin E$ for all choices of i and j. Lemma 3.4 guarantees that $G - z$ contains no two vertex-disjoint paths from $\{c_1, c_2\}$ to $\{d_1, d_2\}$; now, by Menger's theorem, $G - z$ contains a vertex w such that $G - \{z, w\}$ contains no path from $\{c_1, c_2\}$ to $\{d_1, d_2\}$. Observe that $\{z, w\}$ is a separator in G.

CASE 2. $A \cap B = \emptyset$.

Let T be a triangle in G, let z be a vertex in B and let N denote the neighborhood of z. Lemmas 3.3 and 3.4 guarantee that G contains no three vertex-disjoint paths from N to T; now, by Menger's theorem, G contains a set S of at most two vertices such that $G - S$ contains no path from N to T. Observe that S is a separator in G. □

3.4. Proof of Theorem 2.4. The following simple fact will be used quite a few times.

LEMMA 3.5. *Let G be a graph with a chordless path $x_1x_2\dots x_n$ such that $n \geq 5$; let z be a vertex of G such that*

- *$zx_j \in E$ if and only if j is one of $1, 2, n-1, n$*

and such that

- *$x_2x_3\dots x_{n-1}$ is a connected component of $G - \{z, x_1, x_n\}$.*

If G is friendly, then $G - \{zx_1, zx_2, zx_{n-1}, zx_n\}$ is friendly; if G contains no odd hole, then $G - \{zx_1, zx_2, zx_{n-1}, zx_n\}$ contains no odd hole.

Proof. First of all, note that

(i) zx_1x_2 is the only triangle containing zx_1

(any other triangle zx_1y would form a dart with x_2 and x_{n-1}); similarly,

(ii) zx_nx_{n-1} is the only triangle containing zx_n.

Now write $F = G - \{zx_2, zx_{n-1}\}$. It is a routine matter to verify, using (i) and (ii), that F is friendly. The only hole in F that has a chord in G is $zx_1x_2\dots x_n$; if this hole is odd, then $zx_2x_3\dots x_{n-1}$ is an odd hole in G. The rest follows from (i) and (ii) by two applications of Lemma 3.2. □

Unless indicated otherwise, we shall use $a_1, \dots, a_i, a_{i+1}, \dots, a_m$ and z to denote the vertices of a generic bat just as we did in its definition: the bat consists of a chordless path $a_1a_2\dots a_m$ and an additional vertex z that is adjacent to a_1, a_i, a_{i+1}, a_m for some i with $3 \leq i \leq m-3$ and to no other a_j. Trivially, if a friendly graph contains a bat, then the triangle za_ia_{i+1} in the bat extends into a unique maximal clique; we shall often rely tacitly on a lemma that guarantees a stronger conclusion under the additional assumption that the friendly graph contains no odd hole.

LEMMA 3.6. *If a friendly graph G containing no odd hole contains a bat, then each edge of the triangle in the bat extends into a* unique *maximal clique of G.*

Proof. Since za_ia_{i+1} extends into a unique maximal clique, our task reduces to proving that no vertex of G other than z, a_i, a_{i+1} is adjacent to precisely two of

z, a_i, a_{i+1}. Assume the contrary. Symmetry allows us to distinguish between two cases.

CASE 1. *G includes a vertex x such that $xz \in E$, $xa_i \in E$, $xa_{i+1} \notin E$, and $x \neq a_{i+1}$.*

We must first have $xa_{i-1} \in E$ (else $a_{i-1}a_ia_{i+1}zx$ is a dart) and $xa_1 \in E$ (else $a_1a_ia_{i+1}zx$ is a dart) and $xa_m \in E$ (else $a_mzxa_ia_{i+1}$ is a dart), and then $i = 3$ (else $a_1a_{i-1}a_mzx$ is a dart). However, then $a_1a_2a_3a_mx$ is a dart, a contradiction.

CASE 2. *G includes a vertex x such that $xa_i \in E$, $xa_{i+1} \in E$, $xz \notin E$, and $x \neq z$.*

We must first have $xa_{i-1} \in E$ (else $a_{i-1}a_ia_{i+1}xz$ is a dart) and $xa_{i+2} \in E$ (else $a_{i+2}a_{i+1}a_ixz$ is a dart), and then $xa_j \notin E$ whenever $1 \leq j \leq i-2$ (else $a_jxa_{i+1}a_ia_{i+2}$ is a dart). However, then one of the holes $za_1a_2 \dots a_iz$ and $za_1a_2 \dots a_{i-1}xa_{i+1}z$ is odd, a contradiction. □

A path in G will be called *x-sparse* if it is chordless and contains no x-edge.

LEMMA 3.7. *Let a friendly graph containing no odd hole contain a bat. Then every z-sparse path joining the two wing-tips of the bat passes through the maximal clique that contains the triangle of the bat.*

Proof. Consider a counterexample G with the smallest number of edges. Some portion $b_1b_2 \dots b_n$ of the z-sparse path is vertex-disjoint from the bat, with b_1 having at least one neighbor in $A[a_1, a_{i-1}]$ and with b_n having at least one neighbor in $A[a_{i+2}, a_m]$. Write $B = b_1b_2 \dots b_n$. Since B is a portion of a z-sparse path from a_1 to a_m, neither a_1b_1 nor b_na_m is a z-edge. Minimality of G guarantees that

- every vertex of G belongs either to the bat or to B;
- no edges of G go from $A[a_1, a_{i-1}]$ to $B - b_1$;
- no edges of G go from $A[a_{i+2}, a_m]$ to $B - b_n$.

Furthermore, Lemma 3.2 and minimality of G guarantee that

- every edge of G that belongs neither to the bat nor to B
  has at least one of the four vertices a_i, a_{i+1}, b_1, b_n for an endpoint:

all other edges would have the form zb_j with $1 < j < n$, and so they could be removed.

CASE 1. *There are subscripts s, t such that $s < t$, $a_ib_t \in E$, $a_{i+1}b_s \in E$.*

Let t be as small as possible subject to the assumption of this case; once t has been fixed, let s be as large as possible subject to the assumption of this case. By Lemma 3.2 and minimality of G, each of the two edges $a_ib_t, a_{i+1}b_s$ extends into a triangle; since $a_ib_s \notin E$ and $a_{i+1}b_t \notin E$, it follows that $a_{i+1}b_{s-1}, a_ib_{t+1} \in E$.

SUBCASE 1.1. *$a_{i+1}b_r \in E$ for some r with $r < s - 1$.*

Let r be as large as possible subject to the assumption of this case. Since G is friendly, we have $r \leq s - 3$ (consider the neighborhood of a_{i+1}); maximality of r guarantees that $a_{i+1}b_j \notin E$ whenever $r < j < s - 1$; minimality of t guarantees that $a_ib_j \notin E$ whenever $r < j < s-1$. By Lemma 3.2 and minimality of G, the edge $a_{i+1}b_r$ extends into a triangle; since $a_ib_r \notin E$, it follows that $a_{i+1}b_{r-1} \in E$. However, then $G, B[b_{r-1}, b_s]$, and a_{i+1} satisfy the hypothesis of Lemma 3.5, and so the minimality of G is contradicted.

SUBCASE 1.2. *$a_{i+1}b_r \in E$ for no r with $r < s - 1$.*

Let F denote the subgraph of G induced by $A[a_1, a_i]$ and $B[b_1, b_{s-1}]$. By assumption of this case, no vertex of F other than a_i and b_{s-1} is adjacent to a_{i+1}; trivially, F contains a chordless path P from a_i to b_{s-1}. By Lemma 3.1, the subgraph of G induced by the union of P, a_{i+1}, and $B[b_s, b_t]$ contains an odd hole.

CASE 2. *There are no subscripts s, t such that $s < t$, $a_ib_t \in E$, $a_{i+1}b_s \in E$.*

By assumption of this case, there is a subscript j such that a_i is adjacent to none of $b_j, b_{j+1}, \dots, b_n$ and such that a_{i+1} is adjacent to none of $b_1, b_2, \dots, b_{j-1}$. Trivially,

$G - \{z, a_i\}$ contains a chordless path C from a_1 to a_{i+1} and $G - \{z, a_{i+1}\}$ contains a chordless path D from a_i to a_m. Since a_1 and a_{i+1} belong to distinct components of $G - \{z, a_i, b_j\}$, path C must pass through b_j; since a_i and a_m belong to distinct components of $G - \{z, a_{i+1}, b_j\}$, path D must also pass through b_j. Write

$$C_1 = C[a_1, b_j], \quad C_2 = C[b_j, a_{i+1}], \quad D_1 = D[a_i, b_j], \quad D_2 = D[b_j, a_m];$$

let c_k and d_k denote the number of edges in C_k and D_k, respectively. Since at least one of the four numbers $c_1 + c_2 + 2$, $d_1 + d_2 + 2$, $c_1 + d_2 + 2$, $d_1 + c_2 + 1$ is odd, at least one of the four cycles Cz, Dz, C_1D_2z, D_1C_2 is odd. Since none of the four subgraphs of G induced by these four cycles contains a triangle (in particular, neither za_ib_1 nor $za_{i+1}b_n$ is a triangle as the neighborhood of z consists of vertex-disjoint cliques), it follows that G contains an odd hole. $\square$

By a *z-splitter*, we mean a bat (with head z) along with a z-sparse path between the two wing-tips of the bat that passes through one of a_i, a_{i+1}.

LEMMA 3.8. *Let G be a friendly graph containing no odd hole and let z be a vertex of G such that G contains a z-splitter. Then G contains a fragile bat with head z.*

Proof. Among all counterexamples (G, z), choose one with G having as few edges as possible. In this G and for this z, consider a z-splitter with as few vertices as possible.

Symmetry allows us to assume that the z-sparse path between the two wing-tips of the bat in the z-splitter passes through a_{i+1}; now minimality of the z-splitter implies that this path has the form $XBA[a_{i+1}, a_m]$ with X a prefix of $A[a_1, a_i)$. Write $B = b_1b_2 \dots b_n$. Minimality of the z-splitter guarantees that

- B is chordless and vertex-disjoint from A;
- no vertex in $A[a_1, a_i)$ has a neighbor in $B(b_1, b_n]$;
- a_{i+1} is adjacent to no vertex in $B[b_1, b_n)$.

We claim that

(α) some vertex in $A(a_1, a_i)$ is adjacent to b_1.

To justify this claim, assume the contrary: no vertex in $A(a_1, a_i)$ has a neighbor in B. With F standing for the graph induced in G by $\{z\} \cup A[a_1, a_{i+1}] \cup B$, Lemma 3.1 guarantees that za_ia_{i+1} is not the only triangle in F; as $XBA[a_{i+1}, a_m]$ is a z-sparse path, all other triangles in F must have the form $a_ib_kb_{k+1}$ with $1 \le k < n$. Let k be the smallest subscript such that $a_ib_k \in E$, write $b_0 = a_1$, and let j be the largest subscript such that $zb_j \in E$ and $0 \le j < k$. Now the path $b_j \dots b_ka_i \dots a_m$ and z induce in G a bat with head z, whose wing-tips b_j and a_m are joined by the z-sparse path $b_j \dots b_na_{i+1} \dots a_m$. Hence minimality of our z-splitter is contradicted: a proper subset of its vertex-set (a_2 is missing) induces another z-splitter. This contradiction completes the justification of (α).

Furthermore, we claim that

(β) z has no neighbor in B.

To justify this claim, assume the contrary and let k be the smallest subscript such that $zb_k \in E$; note that (since $XBA[a_{i+1}, a_m]$ is z-sparse) $k < n$. If $a_ib_s \in E$ for some s such that $1 \le s \le k$, then let s be the largest subscript with this property, note that (since $a_{i+1}b_k \notin E$) $s < k$, and set $r = i$; else let r be the largest subscript such that $a_rb_1 \in E$ and set $s = 1$. In either case, the path $b_k \dots b_sa_r \dots a_m$ and z induce in G a bat with head z, whose wing-tips b_k and a_m are joined by the z-sparse path $b_k \dots b_na_{i+1} \dots a_m$. Hence minimality of our z-splitter is contradicted: a proper subset of its vertex-set (a_1 is missing) induces another z-splitter. This contradiction completes the justification of (β).

Lemma 3.7 guarantees that

• no edge of G goes from $A[a_1, a_i) \cup B$ to $A(a_{i+1}, a_m]$.

Let K denote the maximal clique in G that contains the triangle za_ia_{i+1}. Since (G, z) is a counterexample, a_1 and a_m belong to the same component of $G - K$. Hence there is a path $c_1c_2 \ldots c_p$ in $G - K$ such that c_1 has a neighbor in $A[a_1, a_i) \cup B$ and c_p has a neighbor in $A(a_{i+1}, a_m]$; among all such paths, choose one with p as small as possible and write $C = c_1c_2 \ldots c_p$. Minimality of p guarantees that

• C is chordless and vertex-disjoint from $A[a_1, a_i) \cup B \cup A(a_{i+1}, a_m]$;

• no vertex of $C(c_1, c_p]$ has a neighbor in $A[a_1, a_i) \cup B$;

• no vertex of $C[c_1, c_p)$ has a neighbor in $A(a_{i+1}, a_m]$.

With G_0 standing for the subgraph of G induced by the union of the splitter and C, observe that G_0 has no clique-cutset K' such that $z \in K'$. Hence (G_0, z) is also a counterexample; now mimimality of G guarantees that

• G has no vertices outside the splitter and C.

Lemma 3.2 and minimality of G guarantee that

• every edge a_ic_j extends into a triangle;

• every edge $a_{i+1}c_j$ extends into a triangle;

note that

• if a_ic_jx is a triangle and $j > 1$, then $x = c_{j-1}$ or $x = c_{j+1}$;

• if $a_{i+1}c_jx$ is a triangle and $1 < j < p$, then $x = c_{j-1}$ or $x = c_{j+1}$.

There is a chordless path from a_1 to a_m of the form YCZ such that Y is a prefix of XB and Z is a suffix of $A(a_{i+1}, a_m]$; Lemma 3.7 guarantees that this path is not z-sparse; since neither Y nor Z contains a z-edge, it follows that C includes at least one neighbor of z. Let v denote the first neighbor of z on C.

We claim that

(γ) no edge of G goes from $A[a_1, a_{i+1}) \cup B \cup C[c_1, v)$ to $A(a_{i+1}, a_m] \cup C(v, c_p]$.

To justify this claim, assume the contrary. This assumption implies that some edge of G joins a_i to a vertex in $C(v, c_p]$; it follows that $C[v, c_p]$ contains an a_i-edge. There is a suffix Y of $A(a_{i+1}, a_m]$ such that $C[v, c_p]Y$ is a chordless path; write $D = C[v, c_p]Y$ and note that (as D is vertex-disjoint from K) each vertex of D is adjacent to at most one of z, a_i, and a_{i+1}. Consider a minimal portion $D[x, y]$ of D such that x, y are neighbors of z and such that $D[x, y]$ contains an a_i-edge; note that no vertex of $D(x, y)$ is adjacent to z. If $D[x, y]$ contains precisely one a_i-edge, then the subgraph of G induced by $D[x, y] \cup \{z, a_i\}$ satisfies the hypothesis of Lemma 3.1, and so G contains an odd hole. Hence $D[x, y]$ contains at least two a_i-edges. These two a_i-edges lie on $C[v, c_p]$; consider a minimal portion $C[c_r, c_s]$ of $C[v, c_p] \cap D(x, y)$ that contains two a_k-edges with $k = i$ or $k = i + 1$. Lemma 3.5 and minimality of G guarantee that $C(c_r, c_s)$ is not a connected component of $G - \{a_k, c_r, c_s\}$; it follows that some vertex of $C(c_r, c_s)$ is adjacent to a_t with $\{t, k\} = \{i, i + 1\}$; now $C(c_r, c_s)$ contains precisely one a_t-edge and no a_k-edge. However, then Lemma 3.1 guarantees that the subgraph of G induced by $C(c_r, c_s) \cup \{a_i, a_{i+1}\}$ contains an odd hole; this contradiction completes the justification of (γ).

The subgraph of G induced by $A[a_{i+1}, a_m] \cup C[v, c_p]$ contains a chordless path from a_{i+1} to v; let P denote this path. We propose to find, in the subgraph of G induced by $A[a_1, a_{i+1}] \cup B \cup C[c_1, v]$, chordless paths P_0, P_1 from v to a_{i+1} such that P_0 has an even number of edges and P_1 has an odd number of edges. This will complete the proof of Lemma 3.8: one of the cycles P_0P and P_1P is odd and, by virtue of (γ), chordless.

In finding P_0 and P_1, we shall rely tacitly on the fact that

- the subgraph of G induced by $A(a_1, a_i) \cup B$
  is connected and includes no neighbor of z,

which is guaranteed by (α) and (β).

CASE 1. *At least one of a_i and a_{i+1} has a neighbor in $C[c_1, v)$.*

Since the subgraph of G induced by $A[a_1, a_i) \cup B \cup C[c_1, v] \cup \{a_{i+1}\}$ is connected, it contains a chordless path P_0 from v to a_{i+1}; by assumption of this case, $c_1 \neq v$, and so $a_1v \notin E$; it follows that P_0 is z-sparse. Since the subgraph of G induced by P_0 and z contains no triangle, it must be bipartite; in particular, the cycle P_0z must be even; hence P_0 has an even number of edges.

Let u be the last vertex on $C[c_1, v)$ adjacent to one of a_i, a_{i+1}. If $ua_{i+1} \in E$ and u has a neighbor in $A[a_1, a_i)$, then $u = c_1$; in this case, let w denote the last neighbor of u on $A[a_1, a_i]$; Lemma 3.1 guarantees that the subgraph of G induced by z, $A[w, a_{i+1}]$, and $C[u, v]$ contains an odd hole. If $ua_{i+1} \in E$ and u has no neighbor in $A[a_1, a_i)$, then $A[a_1, a_{i+1}]C[u, v]$ and z induce a bat whose wing-tips are joined by a z-sparse path in the subgraph of G induced by $A[a_1, a_i) \cup B \cup C[c_1, v]$, contradicting Lemma 3.7. Hence $ua_i \in E$. With Q standing for $C[u, v]$ reversed, write $P_1 = Qa_ia_{i+1}$; since $P_1[v, a_i]z$ is a hole, P_1 has an odd number of edges.

CASE 2. *Neither a_i nor a_{i+1} has a neighbor in $C[c_1, v)$.*

SUBCASE 2.1. *c_1 has a neighbor in $A(a_1, a_i) \cup B[b_1, b_n)$.*

By assumption of this subcase, the subgraph of G induced by $A(a_1, a_i) \cup B \cup \{a_{i+1}\} \cup C[c_1, v]$ contains a chordless path P_0 from v to a_{i+1} and the subgraph of G induced by $A(a_1, a_i] \cup B[b_1, b_n) \cup C[c_1, v]$ contains a chordless path Q from v to a_i. Observe that each of P_0z, Qz is a hole, and so each of P_0, Q has an even number of edges. Set $P_1 = Qa_{i+1}$ and observe that P_1 is a chordless path.

SUBCASE 2.2. *c_1 has no neighbor in $A(a_1, a_i) \cup B[b_1, b_n)$.*

The subgraph of G induced by $A(a_1, a_i] \cup B$ contains a chordless path Q from b_n to a_i. Since G contains no odd hole, its subgraph induced by $Q \cup C[c_1, v] \cup \{z, a_{i+1}\}$ must not satisfy the hypothesis of Lemma 3.1; under the assumption of this subcase, this means $c_1b_n \notin E$. Hence a_1 is the unique neighbor of c_1 in $A[a_1, a_i) \cup B$.

The subgraph of G induced by $A[a_1, a_i) \cup B \cup \{a_{i+1}\}$ contains a chordless path S from a_1 to a_{i+1}. Observe that each of $A[a_1, a_i]z$, Sz is a hole, and so each of $A[a_1, a_i]$, S has an even number of edges. With R standing for $C[c_1, v]$ reversed, each of $RA[a_1, a_{i+1}]$ and RS is a chordless path; one of these paths has an even number of edges and the other has an odd number of edges. ∎

A path $w_1w_2 \dots w_m$ in G will be called *z-special* if

- $w_1w_2 \dots w_m$ is a chordless path in $G * z$,
- w_1 and w_m are neighbors of z,
- z has at least two neighbors in $w_2w_3 \dots w_{m-1}$,
  the set N of these neighbors forms a clique,
  and $w_1w \notin E, w_mw \notin E$ whenever $w \in N$.

(Note that w_1 and w_m may or may not be adjacent.)

LEMMA 3.9. *Let G be a friendly graph containing no odd hole and let z be a vertex of G such that z is the center of a claw. If G contains a z-special path, then G and z have at least one of the following properties:*

(i) *G contains a fragile bat with head z;*

(ii) *G contains a clique-cutset K such that $z \in K$ and some component of $G - K$ includes no neighbor of z.*

Proof. Note that the neighborhood of z consists of vertex-disjoint cliques. We shall proceed by induction on the number of neighbors of z on the z-special path. Let A denote the z-special path; write $A = a_1 a_2 \dots a_m$; let a_i be the first neighbor of z on $A(a_1, a_m)$ and let a_j be the last neighbor of z on $A(a_1, a_m)$; let K denote the maximal clique in G that contains z and all its neighbors on $A(a_1, a_m)$. Write

$$A_O = A[a_1, a_i) \cup A(a_j, a_m] \text{ and } A_I = A(a_i, a_j) - K.$$

(The subscripts O and I are mnemonic for "outside" and "inside".) If A_O and A_I belong to the same connected component of $G - K$, then $G - K$ contains a path $b_1 b_2 \dots b_n$ such that b_1 has a neighbor in A_O and b_n has a neighbor in A_I. If A_O and A_I belong to distinct connected components of $G - K$, then consider a component Q of $G - K$ that includes a vertex of A_I: unless (ii) holds (in which case we are done), Q includes a neighbor of z, and so Q contains a path $b_1 b_2 \dots b_n$ such that b_1 is a neighbor of z and b_n has a neighbor in A_I. Hence we may assume in any case that $G - K$ contains a path $b_1 b_2 \dots b_n$ such that

- b_1 has a neighbor in $A_O \cup \{z\}$;
- b_n has a neighbor in A_I;

taking n as small as possible, we may assume further that

- $b_1 b_2 \dots b_n$ is chordless and vertex-disjoint from A;
- no b_k with $k > 1$ has a neighbor in $A_O \cup \{z\}$;
- no b_k with $k < n$ has a neighbor in A_I.

Write $B = b_1 b_2 \dots b_n$ and note that

- every neighbor of z in $A \cup B$ belongs to $K \cup \{a_1, a_m, b_1\}$.

Let F denote the subgraph of $G * z$ induced by $A \cup B$.

CASE 1. *b_1 has no neighbor in A_O.*

F contains a chordless path C from b_1 to a_1 that avoids $A[a_j, a_m]$; by assumption of this case, C must pass through a_i; if some other interior vertex of C also belongs to K, then C is z-special and we are done by the induction hypothesis; hence we may assume that b_1, a_i, a_1 are the only neighbors of z on C; in particular, C is z-sparse. Similarly, F contains a z-sparse path D from b_1 to a_m such that D avoids $A[a_1, a_i]$ and such that b_1, a_j, a_m are the only neighbors of z on D.

If a_j has no neighbor on $C[b_1, a_i)$, then $C[b_1, a_i]$, $A[a_j, a_m]$, and z induce in G a bat with head z, whose wing-tips are joined by the z-sparse path D, and so (i) follows by Lemma 3.8. If a_j has a neighbor on $C[b_1, a_i)$, then let x be the first neighbor of a_j on $C[b_1, a_i)$; note that x is not the last vertex on $C[b_1, a_i)$ (else $x a_i z a_j a_{j+1}$ would be a dart); now $C[b_1, x] a_j$, $A[a_1, a_i]$, and z induce in G a bat with head z, whose wing-tips are joined by the z-sparse path C, and so (i) follows again by Lemma 3.8.

CASE 2. *At least one of $z b_1 a_1$ and $z b_1 a_m$ is a triangle.*

Symmetry allows us to assume that $z b_1 a_1$ is a triangle.

SUBCASE 2.1. *b_1 has no neighbor in $A(a_1, a_i)$.*

F contains a chordless path C from b_1 to a_1 that avoids $A[a_j, a_m]$; by assumption of this subcase, C must pass through a_i. Lemma 3.1 applied to the subgraph of G induced by $C \cup \{z\}$ guarantees that a_i is not the only interior vertex of C that belongs to K. Hence C is z-special and we are done by the induction hypothesis.

SUBCASE 2.2. *b_1 has a neighbor in $A(a_1, a_i)$ and it has a neighbor in $A(a_j, a_m]$.*

Let u be the last neighbor of b_1 on $A(a_1, a_i)$ and let v be the first neighbor of b_1 on $A(a_j, a_m]$; Lemma 3.1 guarantees that $z a_i a_j$ is not the only triangle in the subgraph of G induced by $A[u, a_i] \cup A[a_j, v] \cup \{b_1, z\}$. Hence $v = a_m$, which brings us back to a mirror image of Subcase 2.1.

SUBCASE 2.3. *b_1 has a neighbor in $A(a_1, a_i)$ and it has no neighbor in $A(a_j, a_m]$.*

F contains a chordless path C from b_1 to a_m that avoids $A[a_1, a_i]$; by assumption of this subcase, C must pass through a_j. If C is z-special, then we are done by the induction hypothesis; hence we may assume that a_j is the only interior vertex of C that belongs to K. With x standing for the last neighbor of b_1 on $A(a_1, a_i)$, observe that $b_1A[x, a_i]A[a_j, a_m]$ and z induce a bat in G, whose wing-tips are joined by the z-sparse path C; hence (i) follows by Lemma 3.8.

CASE 3. *b_1 has a neighbor in A_O; neither zb_1a_1 nor zb_1a_m is a triangle.*

By assumption of this case, F contains a chordless path C from a_1 to a_m that avoids at least one of a_i and a_j and has the following property: *If b_1 appears on C at all, then it appears either before all the vertices of $C \cap K$ or after all the vertices of $C \cap K$.* If C includes at least two vertices of K, then C is z-special or else one of $C[a_1, b_1]$, $C[b_1, a_m]$ is z-special and we are done by the induction hypothesis; hence we may assume that

- C includes at most one vertex of K.

Note that

- if C includes a vertex of K, then this vertex is a_i or a_j.

The assumption of this case guarantees that

- C has no chords in G except possibly a_1a_m.

If $a_1a_m \notin E$, then $A[a_1, a_i]$, $A[a_j, a_m]$, and z induce a bat with head z, whose wing-tips are joined by the z-sparse path C; Lemma 3.7 guarantees that C includes a vertex of K; hence the bat and C constitute a splitter and (i) follows by Lemma 3.8. We propose to complete the proof by deriving a contradiction from the assumption that

- $a_1a_m \in E$.

Since C includes no z-edge, Lemma 3.1 applied to the subgraph of G induced by $C \cup \{z\}$ guarantees that

- no interior vertex of C is adjacent to z.

Now all the vertices of C come from $(A \cup B) - K$, which is $A_O \cup A_I \cup B$. With F_0 standing for the subgraph of $G * z$ induced by $A_O \cup A_I \cup B$, observe that $A[a_1, a_i)$ is a connected component of $F_0 - b_1$ and that $A(a_j, a_m]$ is another; it follows that

- $C = A[a_1, x]b_1A[y, a_m]$

for some x on $A[a_1, a_i)$ and some y on $A(a_j, a_m]$.

SUBCASE 3.1. *$x = a_{i-1}$ and $y = a_{j+1}$.*

Let H_1 denote the subgraph of G induced by $A[a_1, a_i] \cup A[a_j, a_m]$ and let H_2 denote the subgraph of G induced by C. By definition, H_1 is a hole of length at least six; by assumption of this subcase, H_2 is a hole whose length is one less than the length of H_1. Hence one of H_1, H_2 is odd, a contradiction.

SUBCASE 3.2. *$x \neq a_{i-1}$ or $y \neq a_{j+1}$.*

Symmetry allows us to assume that $x \neq a_{i-1}$. Observe that B extends into a chordless path D in F such that D leads from b_1 to some vertex in $A \cap K$ other than a_j and such that no interior vertex of D has a neighbor in $A_O \cup \{z\}$. Lemma 3.1 is contradicted by the subgraph of G induced by C, z, and D. □

LEMMA 3.10. *Let G be a friendly graph containing no odd hole and let xyz be a triangle in G such that z is the center of a claw; let K be the maximal clique in G that contains the triangle xyz. If $G * z$ contains a chordless path P from x to y such that some neighbor of z on P does not belong to K, then G and z have at least one of the following properties:*

(i) *G contains a fragile bat with head z;*

(ii) *G contains a clique-cutset K' such that $z \in K'$ and some component of $G - K'$ includes no neighbor of z.*

Proof. The neighborhood of z consists of vertex-disjoint cliques $K_1, K_2, \ldots, K_m$ such that $K = K_1$. By assumption, $G * z$ contains a chordless path P from x to y such that some neighbor of z on P does not belong to K_1. Let $P[a, f]$ be a minimal segment of P such that a, f belong to the same K_r and some vertex of $P(a, f)$ belongs to a K_s with $s \neq r$; write $Q = P[a, f]$. By minimality of Q, no interior vertex of Q belongs to K_r; Lemma 3.1 guarantees that the subgraph of G induced by Q and z contains a triangle other than zaf; hence some two interior vertices of Q belong to the same K_t. Let c and d be the first and the last vertex in Q that belong to K_t; note that (by minimality of Q) all the neighbors of z in $Q[c, d]$ belong to K_t. Let b be the last neighbor of z in $Q[a, c)$ and let e be the first neighbor of z in $Q(d, f]$. Then $Q[b, e]$ is a z-special path and the desired conclusion follows by Lemma 3.9. ◻

Proof of Theorem 2.4. Note that the assumption

"a_1 and a_m belong to the same component of $G * z$"

is equivalent to the assumption

"a_i and a_{i+1} belong to the same component of $G * z$."

Write $A = a_1 a_2 \ldots a_m$ and let K denote the maximal clique in G that contains the triangle $z a_i a_{i+1}$. We may assume that the bat is not fragile: $G - K$ contains a path from a_1 to a_m. Hence $G - a_i a_{i+1}$ contains a path P from a_i to a_{i+1} with all interior vertices outside K; write $P = p_1 p_2 \ldots p_t$.

CASE 1. *P contains no z-edge.*

Now P is a path in $G * z$; we may assume that P is a chordless path in $G * z$ (but not necessarily a chordless path in $G - a_i a_{i+1}$); then Lemma 3.10 allows us to assume that no interior vertex of P is adjacent to z. Since $G * z$ contains the path $A[a_1, a_i] P A[a_{i+1}, a_m]$, it contains a chordless path Q from a_1 to a_m such that all neighbors of z on Q come from $\{a_1, a_i, a_{i+1}, a_m\}$; Lemma 3.7 guarantees that Q contains at least one of a_i, a_{i+1}. If Q contains precisely one of a_i, a_{i+1}, then the desired conclusion follows from Lemma 3.8; else the desired conclusion follows from Lemma 3.9.

CASE 2. *P contains precisely one z-edge.*

Let $p_j p_{j+1}$ be the unique z-edge on P and let K' denote the maximal clique in G that contains the triangle $z p_j p_{j+1}$. Since $G * z$ contains $P[p_1, p_j]$, $P[p_{j+1}, p_t]$, and a path from p_1 to p_t, it contains a chordless path Q from p_j to p_{j+1}. Lemma 3.10 allows us to assume that all the neighbors of z on Q come from K'. However, then the walk $P[p_1, p_{j-1}] Q P[p_{j+2}, p_t]$ in $G * z$ contains no vertices of K other than a_i and a_{i+1}, which brings us back to Case 1.

CASE 3. *P contains at least two z-edges.*

We may assume that P is a chordless path in $G - a_i a_{i+1}$. By assumption of this case, z has at least four neighbors on $P(p_1, p_t)$; with p_r standing for the first neighbor of z on $P(p_1, p_t)$ and with p_s standing for the last neighbor of z on $P(p_1, p_t)$, let us write

$$B = b_1 b_2 \ldots b_n = p_r p_{r-1} \ldots p_1 p_t p_{t-1} \ldots p_s \quad \text{and} \quad b_j = p_1, b_{j+1} = p_t.$$

Now B and z induce in G a bat; note that $b_j = a_i$, $b_{j+1} = a_{i+1}$, and so K is the maximal clique in G that contains the triangle $z b_j b_{j+1}$ in this new bat.

By assumption, $G * z$ contains a chordless path Q from b_j to b_{j+1}; Lemma 3.10 allows us to assume that all the neighbors of z on Q come from K. Since $G * z$ contains the path $B[b_1, b_j]QB[b_{j+1}, b_n]$, it contains a chordless path C from b_1 to b_n such that all vertices of C come from $B \cup Q$; write $C = c_1c_2 \dots c_p$. Lemma 3.7 allows us to assume that C contains at least one vertex of K; Lemma 3.9 allows us to assume that C contains at most one vertex of K; let x be the unique vertex in $C \cap K$.

Lemma 3.7 allows us to assume that

∘ no vertex in $C[c_1, x)$ has a neighbor in $B(b_{j+1}, b_n]$;

∘ no vertex in $C(x, c_p]$ has a neighbor in $B[b_1, b_j)$;

in turn, Lemma 3.8 allows us to assume that

• x is distinct from b_j and b_{j+1};

∘ no vertex in $C[c_1, x)$ is adjacent to b_{j+1};

∘ no vertex in $C(x, c_p]$ is adjacent to b_j.

Now if x has no neighbor in $B(b_1, b_j)$, then $B[b_1, b_j]C[x, c_p]$ and z induce a bat; since this bat and C constitute a z-splitter, Lemma 3.8 allows us to assume that

∘ x has a neighbor in $B(b_1, b_j)$;

∘ x has a neighbor in $B(b_{j+1}, b_n)$.

Letting u denote the first neighbor of x in $B(b_1, b_j)$ and letting v denote the last neighbor of x in $B(b_{j+1}, b_n)$, we may just as well assume that

• $C = B[b_1, u]xB[v, b_n]$.

Since $C \cup \{z\}$ induces a cycle in G and since zx is the unique chord of this cycle, C has an even number of edges; $B \cup \{z\}$ induces a bat in G, and so B has an odd number of edges; let us note for future reference that

• the lengths of B and C differ in parity.

Recall that the vertex-set of P induces in G a hole H such that

• B is a segment of H.

We propose to complete the proof by contradicting the assumption that the subgraph of G induced by $H \cup \{z, x\}$ is friendly and contains no odd hole.

For this purpose, first note that (since the neighborhood of z consists of vertex-disjoint cliques and since b_j, b_{j+1} are the only vertices of K on H),

• z and x have no common neighbor on H other than b_j and b_{j+1}.

H is a concatenation of two paths: path B from b_1 to b_n and its counterpart $\overline{B}$ ($= H - B(b_1, b_n)$) from b_n to b_1. Starting with the subgraph F of G induced by $H \cup \{z, x\}$, keep reducing F as long as possible by the following two operations:

(α) If $\overline{B}$ contains a vertex w such that
w is adjacent to x and the edge xw extends into no triangle,
then remove the edge xw from F.

(β) If $\overline{B}$ contains a segment $w_1w_2 \dots w_d$ such that
w_1, w_2, w_{d-1}, w_d are adjacent to x,
none of $w_3, \dots, w_{d-2}$ are adjacent to x,
and none of $w_1, w_2, \dots, w_d$ are adjacent to z,
then remove the four edges $xw_1, xw_2, xw_{d-1}, xw_d$ from F.

Lemma 3.2 guarantees that the invariant

• F is a friendly graph containing no odd hole

is maintained after each application of (α); Lemma 3.5 guarantees that this invariant is maintained after each application of (β). Now let us distinguish between two subcases.

SUBCASE 3.1. *x has a neighbor on $\overline{B}$.*

Since transformation (α) is not (any more) applicable to F, the assumption of this subcase guarantees that $\overline{B}$ contains an x-edge, e. Let $\overline{B}_0$ be a minimal segment of $\overline{B}$ that contains e, begins at a neighbor of z, and ends at a neighbor of z. Since transformation (β) is not (any more) applicable to F, edge e is the only x-edge in $\overline{B}_0$. However, then the subgraph of F induced by $\overline{B}_0 \cup \{z, x\}$ contradicts Lemma 3.1.

SUBCASE 3.2. *x has no neighbor on $\overline{B}$.*

The concatenation of B and $\overline{B}$ is H; by assumption of this subcase, the concatenation of C and $\overline{B}$ is another hole; since the lengths of B and C differ in parity, one of these two holes is odd, a contradiction. □

4. The algorithm. Theorem 2.3 suggests a recursive algorithm that, given any friendly bat-free graph G, finds out whether or not G is a Berge graph. This algorithm, TEST(G), uses a procedure DECOMP1(G; S) that, given any friendly graph G and a separator S in G, either finds an odd hole in G at once or else constructs friendly graphs G_1 and G_2 such that

- if G is a bat-free Berge graph,
  then both of G_1 and G_2 are bat-free Berge graphs;
- if G is not a Berge graph,
  then at least one of G_1 and G_2 is not a Berge graph.

DECOMP1(G; S) goes as follows.

By definition, the set of vertices of $G - S$ splits into disjoint nonempty sets V_1 and V_2 such that no edge of G joins a vertex in V_1 to a vertex in V_2; if S consists of two nonadjacent vertices, then each V_i includes at least two vertices. Let F_i denote the subgraph of G induced by $V_i \cup S$. If S is a clique, then DECOMP1($G; S$) simply returns F_1 and F_2. If S consists of nonadjacent vertices u and v, then we may assume that both $G - u$ and $G - v$ are connected (else a one-point separator could be used in place of S) and find a chordless path P_i from u to v in each F_i; now if both P_i have an odd number of edges, then DECOMP1($G; S$) returns graphs G_1 and G_2 such that each G_i is F_i with one additional edge, uv; if both P_i have an even number of edges, then DECOMP1($G; S$) returns graphs G_1 and G_2 such that each G_i is F_i with one additional vertex, w, and two additional edges, uw and vw; if one P_i has an odd number of edges and the other P_i has an even number of edges, then DECOMP1($G; S$) returns the odd hole $P_1 \cup P_2$.

TEST(G):
(Step 1) **if** G is bipartite **then** return *true* **end**
(Step 2) **if** G is claw-free
then if G is a Berge graph **then** return *true* **else** return *false* **end**
end
(Step 3) **if** G has no separator **then** return *false* **end**
(Step 4) S = separator in G;
if DECOMP1($G; S$) returns graphs G_1 and G_2
then return TEST(G_1)$\wedge$TEST(G_2);
else return *false*;
end

LEMMA 4.1. *Let G be a friendly graph. If* TEST(G) *returns* true*, then G is a Berge graph; if* TEST(G) *returns* false*, then G is not a Berge graph or else G contains a bat.*

Proof. We use induction on the number of vertices of G. If TEST(G) returns *false* in Step 3, then Theorem 2.3 guarantees that G contains an odd hole or a bat. If DECOMP1(G; S) returns graphs G_1 and G_2 in Step 4, then G_1 and G_2 are both friendly; if both of them are Berge graphs, then G is a Berge graph; if at least one of them is not a Berge graph, then G is not a Berge graph; if at least one of them contains a bat, then G contains a bat. (When S consists of vertices u and v that are nonadjacent in G and adjacent in G_i, it may help to keep in mind that u and v have no common neighbor.) ◻

LEMMA 4.2. TEST(G) *runs in polynomial time.*

Proof. Each step of the algorithm, except for the recursive calls to evaluate TEST(G_1) and TEST(G_2) in Step 4, can be executed in polynomial time; in particular, Step 2 can be executed in polynomial time by the algorithm of Chvátal and Sbihi [6]. Thus our task reduces to showing that the number $t(G)$ of nodes in the recursion tree of TEST(G) does not exceed some polynomial in n, the number of vertices of the input graph G.

For this purpose, first note that

$$t(G) = 1 \text{ whenever } n \leq 4$$

(since every graph with at most four vertices is bipartite or claw-free). Now induction on n shows that

$$t(G) \leq 4n - 17 \text{ whenever } n \geq 5:$$

if G_1, G_2 are the graphs returned by DECOMP1(G;S) in Step 4, then

$$t(G) = 1 + t(G_1) + t(G_2)$$

and, with n_i standing for the number of vertices of G_i,

$$n_1 \leq n - 1, \; n_2 \leq n - 1, \text{ and } n_1 + n_2 \leq n + 4. \qquad \square$$

Our main algorithm, given any friendly graph G, returns *true* if and only if G is a Berge graph. This algorithm, BERGE(G), is also recursive; it evolves from Theorem 2.2 much as Algorithm TEST(G) evolves from Theorem 2.3.

If G, the input of BERGE(G), contains a clique-cutset C, then we call a procedure DECOMP2($G; C$) which goes as follows: Let $F_1, F_2, \ldots, F_k$ be the components of $G - C$; let G_i denote the subgraph of G induced by $F_i \cup C$; return $G_1, G_2, \ldots, G_k$.

If G contains a rosette centered at z (but G contains no clique-cutset), then we call a more complicated procedure DECOMP3($G; z$) which goes as follows: Let $F_1, F_2, \ldots, F_k$ be the components of $G * z$; let G_i denote the subgraph of G induced by $F_i \cup \{z\}$. Let N denote the subgraph of G induced by all the neighbors of z; let H be the graph obtained from N by adding pairwise nonadjacent vertices $w_1, w_2, \ldots, w_k$ along with all the edges xw_i such that $x \in N \cap F_i$. Return $G_1, G_2, \ldots, G_k$ and H.

DECOMP3 is illustrated in Figure 4.1. There, G is not a Berge graph but G_1, G_2, G_3 are; the unique odd hole in G is contained in none of G_1, G_2, G_3, but it reappears in H.

In general, H may be seen as a device for detecting those odd holes in G that are contained in none of $G_1, G_2, \ldots, G_k$: at first, one might be tempted to conjecture that G is a Berge graph if and only if $G_1, G_2, \ldots, G_k$ and H are all Berge graphs. Unfortunately, this is not the case: two counterexamples are shown in Figure 4.2. Fortunately, these two counterexamples are harmless: each of them has a clique-cutset, and so it can be subjected to DECOMP2 instead of DECOMP3.

We make it our policy to subject G to DECOMP3 only if it has no clique-cutset. Under this assumption (as we shall prove later), G is a Berge graph if and only if $G_1, G_2, \ldots, G_k$ and H are all Berge graphs *and* none of $G_1, G_2, \ldots, G_k$ contains a fragile bat with head z. The proviso involving fragile bats cannot be dropped: see Figure 4.3.

A straightforward algorithm NoFB$(F; z)$, given any graph F along with a vertex z of F such that the neighborhood of z in F consists of vertex-disjoint cliques, returns *true* if and only if F contains no fragile bat with head z. There, as usual, $N(v)$ denotes the neighborhood of v.

```
NoFB(F; z):
for  all z-edges xy
do   C = the maximal clique of F that contains xyz;
     for  all choices of distinct components A, B of F − C
     do   if     A − N(y) contains a path from N(z) to N(x) and
                 B − N(x) contains a path from N(z) to N(y)
          then return false;
          end
     end
end
return true;
```

```
Berge(G):
(Step 1)  if    G has a clique-cutset, C
          then G1, G2, ..., Gk = the output of Decomp2(G; C);
                return Berge(G1) ∧ Berge(G2) ∧ ··· ∧ Berge(Gk);
          end
(Step 2)  if    G has a rosette, centered at some vertex z
          then G1, G2, ..., Gk, H = the output of Decomp3(G; z);
                return Berge(G1) ∧ Berge(G2) ∧ ··· ∧ Berge(Gk)
                        ∧Test(H)
                        ∧NoFB(G1; z) ∧ NoFB(G2; z) ∧ ... NoFB(Gk; z);
          end
(Step 3)  return Test(G);
```

THEOREM 4.3. *Given any friendly graph G, algorithm* BERGE(G) *returns* true *if and only if G is a Berge graph.*

Proof. We use induction on the number of vertices of G.

CASE 1. BERGE(G) *returns in Step* 1.

This case is trivial.

CASE 2. BERGE(G) *returns in Step* 2.

It is easy to see that H is friendly and bat-free; hence Lemma 4.1 guarantees that H is a Berge graph if and only if TEST$(H) =$ *true*. Thus our task reduces to proving the following statements:

(i) If G contains an odd hole, then one of $G_1, \ldots, G_k, H$ contains an odd hole or else one of $G_1, \ldots, G_k$ contains a bat with head z.

(ii) If some G_i contains a bat with head z, then it contains an odd hole or a fragile bat with head z.

(iii) Every antihole of length at least seven in G is contained in one of $G_1, \ldots, G_k$.

(iv) If H contains an odd hole, then G contains an odd hole.

(v) H contains no antihole of length at least seven.

(vi) If some G_i contains a bat with head z, then G contains an odd hole.

Proof of (i). Let G contain an odd hole. If this hole contains at most one z-edge, then it is contained in one of $G_1, \ldots, G_k$. Thus we may assume that the hole has the form $e_1P_1e_2P_2 \ldots e_tP_t$ where $e_1, e_2, \ldots, e_t$ are z-edges, $t \geq 2$, and each P_i is a chordless path in G that contains no z-edge; since P_i does not pass through z, all its edges come from some $F_{j(i)}$, and so P_i is a chordless path in $G_{j(i)} - z$. If some P_i has

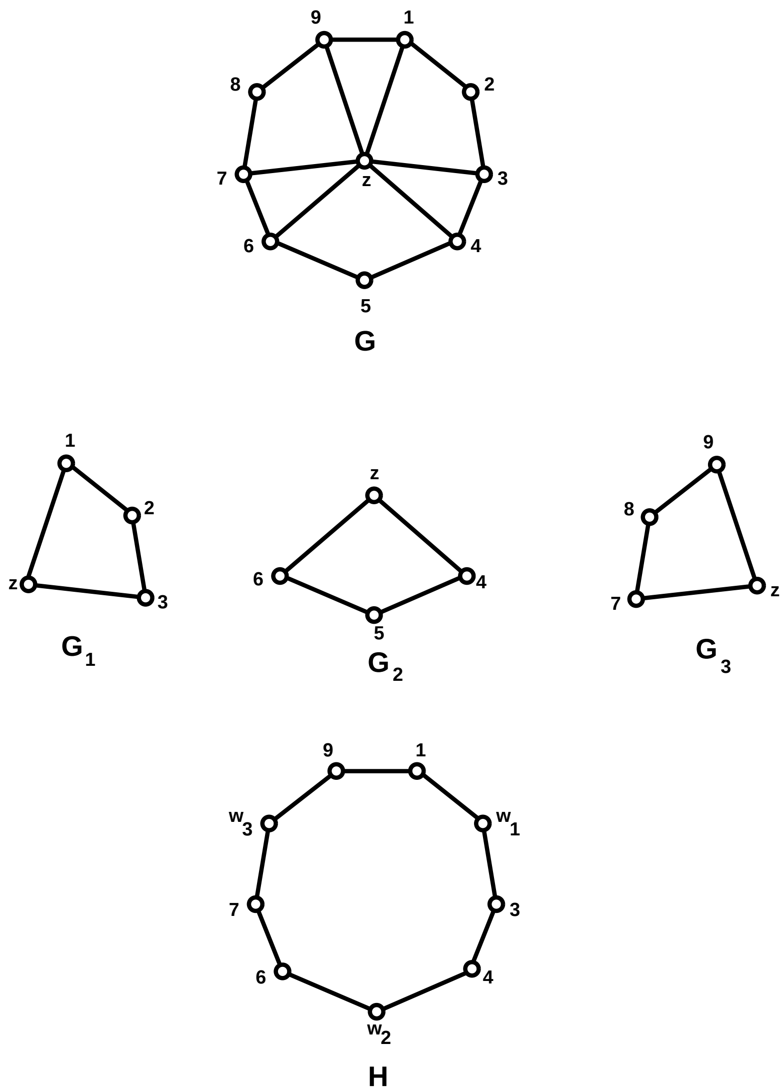

FIG. 4.1. DECOMP3 *illustrated.*

an odd number of edges, then the subgraph of G induced by $P_i \cup \{z\}$ is not bipartite; since it contains no triangle, it contains an odd hole and we are done. Thus we may assume that each P_i has an even number of edges. Now t is odd, and so the closed walk $e_1 w_{j(1)} e_2 w_{j(2)} \dots e_t w_{j(t)}$ in H has an odd length, $3t$; hence the subgraph H_0 of H induced by all the vertices of this walk is not bipartite. If H_0 contains no triangle, then it contains an odd hole and we are done. Thus we may assume that H_0 contains a triangle. It is easy to see that the triangle consists of some e_i and some w_j. However, then both endpoints of e_i belong to F_j, and so both P_{i-1} (with $P_0 = P_t$) and P_i are fully contained in F_j; the subgraph of G_j induced by $P_{i-1} \cup P_i \cup \{z\}$ contains a bat.

Proof of (ii). If some G_i contains a bat with head z, then—since $G_i * z$ is connected—Theorem 2.4 guarantees that G_i contains an odd hole (in which case we are done), or G_i contains a fragile bat with head z (in which case we are done

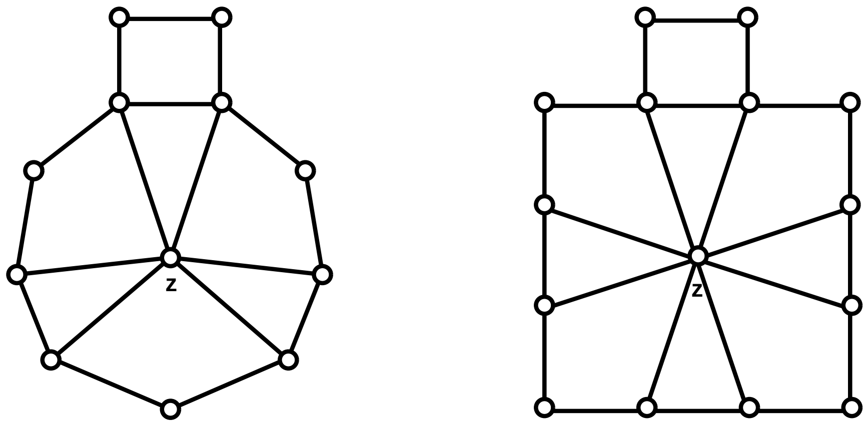

FIG. 4.2. *Clique-cutsets create counterexamples.*

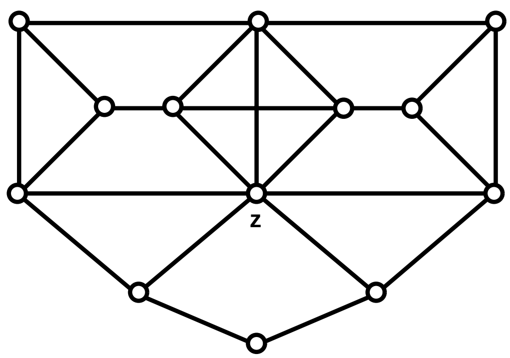

FIG. 4.3. *The significance of fragile bats.*

again), or G_i contains a clique-cutset C such that $z \in C$ and some component of G_i includes no neighbor of z (in which case C is a clique-cutset in G, contradicting our assumption that BERGE(G) does not return in Step 1).

Proof of (iii). Let A be an antihole of length at least seven in G. Observe that $z \notin A$ (since the neighborhood of z consists of vertex-disjoint cliques). If A is contained in one of $G_1, \ldots, G_k$, then we are done; else symmetry allows us to assume that A includes at least one vertex of G_1 and at least one vertex of G_2. Since A is connected, it follows that A contains an edge x_1x_2 with $x_1 \in G_1, x_2 \in G_2$. Note that x_1x_2 is a z-edge. In every antihole of length at least six, each edge is contained in a hole of length four; in particular, x_1x_2 is contained in a hole $x_1x_2y_2y_1$. Since the neighborhood of z consists of vertex-disjoint cliques and x_1, x_2 are adjacent to z, neither y_1 nor y_2 is adjacent to z. Hence $x_1y_1y_2x_2$ is a path in $G * z$ and yet its endpoints belong to distinct components of $G * z$, a contradiction.

Proof of (iv). Let H contain an odd hole. Since $w_1, w_2, \ldots, w_k$ have pairwise disjoint neighborhoods, this hole has the form

$$u_1v_1w_{j(1)}u_2v_2w_{j(2)} \ldots u_tv_tw_{j(t)}$$

such that u_1v_1, u_2v_2, $\ldots$, u_tv_t are z-edges in G and t is odd. Replacing each path $v_iw_{j(i)}u_{i+1}$ (here, $u_{t+1} = u_1$) by a chordless path P_i from v_i to u_{i+1} in $F_{j(i)}$, we obtain a cycle C in $G - z$.

Note that each P_i, although chordless in $G * z$, may have chords in G. However, choosing C with as few edges as possible, we claim that each P_i is chordless in G or else G contains an odd hole (in which case we are done). To justify this claim, consider a chord xy of P_i in G. The assumption that C has as few edges as possible guarantees that neither x nor y equals v_i or u_{i+1} (if $x = v_i$, then we may replace v_i and P_i by y and $P[y, u_{i+1}]$); first choosing x as close to v_i as possible and then y as close to u_{i+1} as possible, we conclude that G_i contains a bat with head z. Since BERGE(G) did not return in Step 1, G has no clique-cutset; hence Theorem 2.4 guarantees that G contains an odd hole.

Now we may assume that each P_i is chordless in G. Let F denote the subgraph of G induced by the vertices of C; note that the edge-set of F partitions into edge-sets of $P_1, P_2, \ldots, P_t$ and edge-sets of pairwise vertex-disjoint cliques formed by z-edges.

If some P_i has an odd number of edges, then the subgraph of G induced by $P_i \cup \{z\}$ is not bipartite; since it contains no triangle, it contains an odd hole and we are done. Thus we may assume that each P_i has an even number of edges. Now C has an odd number of edges, and so F is not bipartite. If F contains no triangle, then it contains an odd hole and we are done. Thus we may assume that F contains a triangle.

Observe that at least one vertex of this triangle is an interior vertex of some P_i. Let u denote this vertex and let v, w denote the remaining two vertices of the triangle so that, proceeding from u along C in some cyclic order, we encounter first v and then w; let a denote the immediate predecessor of u in this order and let b denote the immediate successor of u.

Observe that uv, uw, vw are z-edges and au, ub are not; it follows that the five vertices u, v, w, a, b are distinct and (since the path P_i is chordless) the subgraph of G induced by them has no edges other than uv, uw, vw, au, ub. Hence Lemma 3.4 (with F in place of G) guarantees that F contains an odd hole (in which case we are done) or a bat. Thus we may assume that F contains a bat. The bat consists of a head x and a chordless path $a_1 a_2 \ldots a_m$ such that x is adjacent to a_1, a_i, a_{i+1}, a_m for some i with $3 \le i \le m - 3$ (and to no other a_j). Again, all three edges of the triangle $x a_i a_{i+1}$ must be z-edges. Since all the paths $P_1, P_2, \ldots, P_t$ are chordless in G, x and a_i cannot belong to the same P_j; hence the path $x a_1 a_2 \ldots a_i$ must involve at least one z-edge; similarly, the path $a_{i+1} a_{i+2} \ldots a_m x$ must involve at least one z-edge. Thus a_1 and a_m belong to the same component of $G * x$; since BERGE(G) did not return in Step 1, G has no clique-cutset; hence Theorem 2.4 guarantees that G contains an odd hole.

Proof of (v). Each vertex in an antihole of length at least seven has the property that its neighborhood contains a chordless path with three edges; no vertex of H has this property.

Proof of (vi). Let G_i contain a bat with head z. The two wing-tips of this bat belong to the same component of $G * z$ (this component is F_i). Now Theorem 2.4 guarantees that G contains an odd hole (in which case we are done) or G has a clique-cutset (contradicting our assumption that BERGE(G) does not return in Step 1).

CASE 3. BERGE(G) *returns in Step* 3.

Now G has neither a clique-cutset nor a rosette; hence Theorem 2.2 guarantees that G is bat-free or else G contains an odd hole; in turn, Lemma 4.1 guarantees that TEST(G) = *true* if and only if G is a Berge graph. □

THEOREM 4.4. BERGE(G) *runs in polynomial time.*

Proof. A clique-cutset in G can be found (or its absence established) in polynomial time by an algorithm designed by Whitesides [13]; evidently, a rosette in G can be found (or its absence established) in polynomial time; $\text{Decomp2}(G;C)$, $\text{Decomp3}(G;z)$ and $\text{NoFB}(G_i;z)$ can be executed in polynomial time; by Lemma 4.2, $\text{Test}(H)$ in Step 2 and $\text{Test}(G)$ in Step 3 can be evaluated in polynomial time. Thus our task reduces to showing that the number $t(G)$ of nodes in the recursion tree of $\text{Berge}(G)$ does not exceed some polynomial in n, the number of vertices of the input graph G.

We are going to show that $t(G) \le n^2$. More precisely, with $\overline{e}(G)$ standing for the number of edges in the complement of G, we claim that $t(G) \le 2\overline{e}(G) + 1$. Justifying this claim by induction on $\overline{e}(G)$ amounts to proving that

$$2(\overline{e}(G_1) + \overline{e}(G_2) + \cdots + \overline{e}(G_k)) + k \le 2\overline{e}(G)$$

whenever $G_1, \ldots, G_k$ are returned by $\text{Decomp2}(G;C)$ or $G_1, \ldots, G_k, H$ are returned by $\text{Decomp3}(G;z)$. A stronger inequality,

$$\overline{e}(G_1) + \overline{e}(G_2) + \cdots + \overline{e}(G_k) + \binom{k}{2} \le \overline{e}(G),$$

follows from observing that (i) there is a clique C such that $G_i \cap G_j = C$ whenever $i \ne j$ and that (ii) for every choice of distinct i and j, there are nonadjacent vertices x_i and x_j with $x_i \in G_i - C$, $x_j \in G_j - C$. The first observation is trivial. (We have $C = \{z\}$ in case of Decomp3.) To make the second observation in case of Decomp2, choose any x_j in $G_j - C$. To make the second observation in case of Decomp3, first recall that the subgraph N of G induced by the neighborhood of z consists of vertex-disjoint cliques and then note that (since G has no clique-cutset) each G_i meets at least two of these cliques; hence x_i and x_j can be chosen from two distinct cliques of N. ∎

Finally, Theorem 2.1 reduces the task of recognizing dart-free Berge graphs in polynomial time to the task of recognizing friendly Berge graphs in polynomial time: given any dart-free graph H, we can find in polynomial time a family $\mathbf{F}$ of pairwise vertex-disjoint friendly induced subgraphs of H such that H is a Berge graph if and only if all the members of $\mathbf{F}$ are Berge graphs. The procedure is obvious. To initialize, we set $\mathbf{F} = \{H\}$. While some member G of $\mathbf{F}$ has one of the properties (i), (ii), (iii) of Theorem 2.1, we replace it in $\mathbf{F}$ by its connected components (in case G is disconnected) or by graphs $G_1, G_2, \ldots, G_k$ such that $\overline{G}_1, \overline{G}_2, \ldots, \overline{G}_k$ are connected components of $\overline{G}$ (in case $\overline{G}$ is disconnected) or by $G - w$ such that w and some other vertex of G are adjacent twins. Upon termination, Theorem 2.1 guarantees that all the members of $\mathbf{F}$ are friendly.

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