A de Bruijn - Erdős theorem in metric spaces?

De Bruijn and Erdős [3] proved that every noncollinear set of n points in the plane determines at least n distinct lines. Chen and Chvátal [1] suggested that this theorem might generalize:

> *True or false? Every finite metric space* (X, d) *where no line consists of the entire ground set* X *determines at least* $|X|$ *distinct lines.*

Here, the *line* $\overline{uv}$ in a metric space is defined as the union of $\{u, v\}$ and all the pins containing $\{u, v\}$, where a *pin* is any three-point set $\{x, y, z\}$ such that $d(x, y) + d(y, z) = d(x, z)$.

It is known that

- in every metric space on n points, there are at least $\lg n$ distinct lines or else some line consists of all n points [1];
- in every metric space on n points, there are $\Omega((n/\rho)^{2/3})$ distinct lines, where ρ is the ratio between the largest distance and the smallest nonzero distance [2];
- in every metric space induced by a connected graph on n vertices, there are $\Omega(n^{2/7})$ distinct lines or else some line consists of all n vertices [2];
- in every metric space on n points where each nonzero distance equals 1 or 2, there are $\Omega(n^{4/3})$ distinct lines and this bound is tight [2].

References

[1] X. Chen and V. Chvátal, "Problems related to a de Bruijn - Erdős theorem", *Discrete Applied Mathematics* **156** (2008), 2101 – 2108.

[2] E. Chiniforooshan and V. Chvátal, "A de Bruijn - Erdős theorem and metric spaces", arXiv:0906.0123v1 [math.CO]

[3] N.G. de Bruijn and P. Erdős, On a combinatorial problem, Indagationes Mathematicae 10 (1948) 421–423.

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