Notes on the maximal Lyapunov exponent of a time series

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1. Discrete-time dynamical systems and their Lyapunov exponents. A *discrete-time dynamical system* on a set X is just a function $\Phi : X \to X$. This function, often called a *map,* may describe deterministic evolution of some physical system: if the system is in state x at time t_0, then it will be in state $\Phi(x)$ at time t_0+1. Study of discrete-time dynamical systems is concerned with iterates of the map: the sequence

$$x, \Phi(x), \Phi^2(x), \ldots$$

is called the *trajectory of* x and the set of its points is called the *orbit of* x. We will restrict our attention to maps $\Phi : X \to X$ such that X is a subset of $\mathbf{R}^d$.

For every such map Φ, for every point x in the interior of X, and for every y in $\mathbf{R}^d$, the *local Lyapunov exponent* of Φ at x with respect to direction y is defined as the limit (if it exists)

$$\lim_{\delta \to 0} \ln \frac{\|\Phi(x+\delta y) - \Phi(x)\|}{\|\delta y\|};$$

the *global Lyapunov exponent* of Φ at x with respect to y is defined as the limit (if it exists)

$$\lim_{t \to \infty} \frac{1}{t} \lim_{\delta \to 0} \ln \frac{\|\Phi^t(x+\delta y) - \Phi^t(x)\|}{\|\delta y\|}.$$

If the global Lyapunov exponent of Φ at x with respect to y equals λ then, for all sufficiently large values of t for all sufficiently small (with respect to the value of t) values of δ, we have

$$\frac{\|\Phi^t(x+\delta y) - \Phi^t(x)\|}{\|\delta y\|} \approx e^{\lambda t}.$$

2. Lyapunov exponents of differentiable maps. Given any differentiable function $F : X \to X$ such that X is a subset of $\mathbf{R}^d$ and given any point x in X, let $J(F,x)$ denote the Jacobian matrix of F evaluated at x: the entry in the i-th row and the j-th column of $J(F,x)$ is the value of $\partial y_i/\partial x_j$ at x, where $(y_1, y_2, \ldots, y_d) = F(x_1, x_2, \ldots, x_d)$. Since

$$\|F(x+\delta y) - F(x)\| \;=\; \|J(F,x)\delta y\| + o(\|\delta y\|) \quad \text{as } \delta \to 0,$$

the local Lyapunov exponent of a differentiable map Φ at a point x with respect to a direction y equals

$$\ln \frac{\|J(\Phi,x)y\|}{\|y\|}$$

and the global Lyapunov exponent of Φ at x with respect to y equals

$$\lim_{n\to\infty} \frac{1}{n} \ln \frac{\|J(\Phi^n, x)y\|}{\|y\|}.$$

For future reference, let us note that

$$\begin{aligned}
\frac{1}{n} \ln \frac{\|J(\Phi^n, x)y\|}{\|y\|} &= \frac{1}{n} \sum_{t=0}^{n-1} \ln \frac{\|J(\Phi^{t+1}, x)y\|}{\|J(\Phi^t, x)y\|} \\
&= \frac{1}{n} \sum_{t=0}^{n-1} \ln \frac{\|J(\Phi, \Phi^t(x))J(\Phi^t, x)y\|}{\|J(\Phi^t, x)y\|}.
\end{aligned}$$

This means that the global Lyapunov exponent of Φ at point x with respect to direction y is the average of the local Lyapunov exponents of Φ at points $\Phi^0(x)$, $\Phi^1(x)$, $\Phi^2(x)$, ... of the trajectory of x with respect to direction $J(\Phi^t, x)y$ at each point $\Phi^t(x)$.

3. Computing global Lyapunov exponents. How can we compute the global Lyapunov exponent of a prescribed differentiable map $\Phi : X \to X$ (such that $X \subseteq \mathbf{R}^d$) at a prescribed point x^0 in the interior of X and with respect to a prescribed direction y^0? Answers to this question depend on the way Φ is prescribed; let us assume that it is prescribed by an oracle that, given any x in X, returns $\Phi(x)$.

In this situation, we can compute iteratively

$$x^t = \Phi(x^{t-1}), \;\; y^t = \delta_t \left(\Phi(x^{t-1} + y^{t-1}) - \Phi(x^{t-1})\right),$$

with each δ_t a positive number small enough to ensure that

$$\Phi(x^t + y^t) - \Phi(x^t) \approx J(\Phi, x^t)y^t,$$

until the sequence of averages

$$\frac{1}{n} \sum_{t=0}^{n-1} \ln \frac{\|\Phi(x^t + y^t) - \Phi(x^t)\|}{\|y^t\|}$$

shows signs of convergence, at which time we return an estimate of its limit as an estimate of the Lyapunov exponent.

To justify this policy, we use induction on t to show that

$$y^t \approx \delta_1 \delta_2 \cdots \delta_t J(\Phi^t, x^0)y^0 : \tag{1}$$

in the induction step, we argue that

$$\begin{aligned} y^{t+1} &= \delta_{t+1}\left(\Phi(x^t+y^t)-\Phi(x^t)\right) \\ &\approx \delta_{t+1}J(\Phi,x^t)y^t \\ &\approx \delta_{t+1}J(\Phi,x^t)\delta_1\delta_2\cdots\delta_t J(\Phi^t,x^0)y^0 \\ &= \delta_1\delta_2\cdots\delta_{t+1}J(\Phi,\Phi^t(x^0))J(\Phi^t,x^0)y^0 \\ &= \delta_1\delta_2\cdots\delta_{t+1}J(\Phi^{t+1},x^0)y^0. \end{aligned}$$

From (1), it follows that

$$\frac{\|\Phi(x^t+y^t)-\Phi(x^t)\|}{\|y^t\|} = \frac{\|y^{t+1}\|}{\delta_{t+1}\|y^t\|} \approx \frac{\|J(\Phi^{t+1},x^0)y^0\|}{\|J(\Phi^t,x^0)y^0\|},$$

and so

$$\frac{1}{n}\sum_{t=0}^{n-1}\ln\frac{\|\Phi(x^t+y^t)-\Phi(x^t)\|}{\|y^t\|} \approx \frac{1}{n}\sum_{t=0}^{n-1}\ln\frac{\|J(\Phi^{t+1},x^0)y^0\|}{\|J(\Phi^t,x^0)y^0\|}; \qquad (2)$$

as n tends to infinity, the right-hand side of (2) converges to the global Lyapunov exponent of Φ at point x^0 with respect to direction y^0.

4. The spectrum of Lyapunov exponents. Consider a specific differentiable map $\Phi : X \to X$ with $X \subseteq \mathbf{R}^d$ and a specific point x in X; given an arbitrary direction y in $\mathbf{R}^d$, let $\lambda(y)$ denote the global Lyapunov exponent of Φ at x with respect to y. In Section 2, we have noted that

$$\lambda(y) = \lim_{n\to\infty}\frac{1}{n}\ \ln\!\left(\frac{\|J(\Phi^n,x)y\|}{\|y\|}\right);$$

this identity can be presented as

$$\lambda(y) = \lim_{n\to\infty}\frac{1}{n}\ \ln\|J(\Phi^n,x)y\| = \lim_{n\to\infty}\frac{1}{2n}\ \ln\left(y^TJ(\Phi^n,x)^TJ(\Phi^n,x)y\right).$$

Oseledets ([1]; see also [2] and Section 9.1 of [4]) proved that under certain conditions on Φ, which are only mildly restrictive, there is a $d\times d$ real symmetric matrix A such that

$$\lim_{n\to\infty}\frac{1}{2n}\ \ln\left(y^TJ(\Phi^n,x)^TJ(\Phi^n,x)y\right) = \lim_{n\to\infty}\frac{1}{2n}\ \ln\left(y^TA^{2n}y\right) \text{ for all } y \text{ in } \mathbf{R}^d.$$

The Principal Axis Theorem guarantees that there are real numbers $\lambda_1,\ldots,\lambda_d$ and an orthonormal basis $y^1,\ldots,y^d$ of $\mathbf{R}^d$ such that $Ay^j = \lambda_j y^j$ for all j. Writing y as $\sum_{i=1}^d c_iy^i$ (with $c_i = y^Ty^i$), we find that

$$y^TA^{2n}y = \left(\sum_{i=1}^d c_iy^i\right)^TA^{2n}\left(\sum_{j=1}^d c_jy^j\right) = \sum_{i=1}^d c_i(y^i)^T\sum_{j=1}^d c_j\lambda_j^{2n}y^j = \sum_{i=1}^d c_i^2\lambda_i^{2n},$$

and so

$$\lambda(y) = \ln \max\{|\lambda_i| : c_i \neq 0\}.$$

In particular, $\lambda(y)$ ranges through only at most d distinct values as y ranges through $\mathbf{R}^d$. The largest of these values is called the *maximal Lyapunov exponent* of Φ at x.

For future reference, let us make note of two facts. First, a randomly chosen direction y in $\mathbf{R}^d$ will almost surely have $\lambda(y)$ equal to the maximal Lyapunov exponent: all exceptions lie in the $(d-1)$-dimensional space of vectors orthogonal to any of the vectors y^k, for which the eigenvalue λ_k of A has the largest absolute value. Second, the maximal Lyapunov exponent of Φ at x equals the maximal Lyapunov exponent of Φ at every $\Phi^t(x)$, and so it is a function of a trajectory rather than a function of a point.

5. From a trajectory to its maximal Lyapunov exponent. Given only the trajectory $x^0, x^1, x^2, \ldots$ of a point in $\mathbf{R}^d$ under some otherwise unknown differentiable map $\Phi : \mathbf{R}^d \to \mathbf{R}^d$, we can estimate the maximal Lyapunov exponent of Φ at x^0 by choosing iteratively suitable nonnegative integers $s(0)$, $s(1)$, $s(2)$, ... until the sequence of averages

$$\frac{1}{n} \sum_{t=0}^{n-1} \ln \frac{\|x^{s(t)+1} - x^{t+1}\|}{\|x^{s(t)} - x^t\|}$$

shows signs of convergence, at which time we return an estimate of its limit as an estimate of the maximal Lyapunov exponent of the trajectory. (What we are actually going to estimate is the global Lyapunov exponent of Φ at x^0 with respect to direction $x^{s(0)} - x^0$; as noted at the end of the preceding section, this Lyapunov exponent is likely to be the maximal Lyapunov exponent of the trajectory.)

This procedure is a close relative of the procedure described in Section 3: with y^t standing for $x^{s(t)} - x^t$, we have

$$\frac{\|x^{s(t)+1} - x^{t+1}\|}{\|x^{s(t)} - x^t\|} = \frac{\|\Phi(x^t + y^t) - \Phi(x^t)\|}{\|y^t\|}.$$

We choose $s(0)$ to make $\|x^{s(0)} - x^0\|$ as small as possible above a prescribed level of noise. The policy of Section 3 would guide us to choose each subsequent $s(t)$ so that

$$x^{s(t)} = x^t + \delta_t \left(x^{s(t-1)+1} - x^t \right)$$

with each δ_t a positive number small enough to ensure that

$$\Phi(x^{s(t)}) - \Phi(x^t) \approx J(\Phi, x^t)(x^{s(t)} - x^t).$$

Whenever $\|x^{s(t-1)+1} - x^t\|$ is small enough to let us settle for $\delta_t = 1$, we can simply set $s(t) = s(t-1) + 1$; when $\|x^{s(t-1)+1} - x^t\|$ is too large, we consider additional candidates for $s(t)$, too. We want to choose $s(t)$ in such a way that $x^{s(t)} - x^t$ approximates a small multiple of $x^{s(t-1)+1} - x^t$. Consequently, candidates for $s(t)$ are ranked by a partial order $\succeq$: with $\alpha(k)$ standing for the angle between $x^k - x^t$ and $x^{s(t-1)+1} - x^t$, we have $i \succeq j$ whenever $\|x^i - x^t\| \leq \|x^j - x^t\|$ and $\alpha(i) \leq \alpha(j)$. The selected $s(t)$ is a maximal element in this partial order.

The choice made in a seminal paper of Wolf et al. [5] goes as follows. There are four angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that

$$\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$$

and four distances z_1, z_2, z_3, z_4 such that

$$z_1 < z_2 < z_3 < z_4.$$

If there is no k with $\alpha(k) \leq \alpha_4$ and $\|x^k - x^t\| \leq z_4$, then we set

$$s(t) = s(t-1) + 1. \tag{3}$$

Otherwise we find
first the smallest m for which there is a k with $\alpha(k) \leq \alpha_m$ and $\|x^k - x^t\| \leq z_4$,
then the smallest n for which there is a k with $\alpha(k) \leq \alpha_m$ and $\|x^k - x^t\| \leq z_n$,
and finally a k that minimizes $\alpha(k)$ subject to $\|x^k - x^t\| \leq z_n$;
then we set $s(t) = k$. (The "fixed evolution time" program of Wolf et al. [5] resorts to this policy only when t is a positive integer multiple of a prescribed parameter EVOLV and uses the deafult (3) for all other values of t.)

6. The maximal Lyapunov exponent of a time series. Given positive integers d, τ and a sequence $\xi_0, \xi_1, \xi_2, \ldots$ of real numbers, let us write

$$x^t = (\xi_t, \xi_{t+\tau}, \xi_{t+2\tau}, \ldots, \xi_{t+(d-1)\tau})$$

for all nonnegative integers t. If the resulting sequence $x^0, x^1, x^2, \ldots$ of points in $\mathbf{R}^d$ includes no point more than once, then it can be thought of as the trajectory of x^0 under some map, in which case its maximal Lyapunov exponent (computed as in the preceding section) is referred to as *the maximal Lyapunov exponent of* $\xi_0, \xi_1, \xi_2, \ldots$ *(in dimension d and with time delay τ)*.

Here are a few comments on this notion: When a map $\Phi : X \to X$ describes deterministic evolution of some physical system, measurements in this system define functions $\pi : X \to \mathbf{R}$. The trajectory $x^0, x^1, x^2, \ldots$ of a state x^0 under Φ can be often reconstructed from the measured values $\pi(x^0), \pi(x^1), \pi(x^2), \ldots$. In particular, Ruelle ([4], Chapter 6; see also [3], p. 713 and footnote 8) suggested that, when $X \subseteq \mathbf{R}^d$, each state x^t on the trajectory can be reconstructed from the d measured values

$$\pi(x^t), \pi(x^{t+\tau}), \pi(x^{t+2\tau}), \ldots, \pi(x^{t+(d-1)\tau})$$

with τ a fixed positive integer. This means that there is a mapping $F : \mathbf{R}^d \to \mathbf{R}^d$ such that, with the notation

$$\widetilde{x}^t = (\pi(x^t), \pi(x^{t+\tau}), \pi(x^{t+2\tau}), \ldots, \pi(x^{t+(d-1)\tau})),$$

we have $x^t = F(\widetilde{x}^t)$ for all t. In this sense, the trajectory $\widetilde{x}^0, \widetilde{x}^1, \widetilde{x}^2, \ldots$ is a model of the trajectory $x^0, x^1, x^2, \ldots$; the maximal Lyapunov exponent of the trajectory $x^0, x^1, x^2, \ \ldots$ is a quantity that has a physical meaning; the maximal Lyapunov exponent of the trajectory $\widetilde{x}^0, \widetilde{x}^1, \widetilde{x}^2, \ldots$ (also known as the maximal Lyapunov exponent of the time series $\pi(x^0), \pi(x^1), \pi(x^2), \ \ldots$) is a quantity that we can compute.

References

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