1 Introduction

Sorting algorithms. The input is an array of records; each record consists of a key and a satellite; the key is an identifier (typically a number or a string of characters) coming from some linearly ordered set; the satellite is either the information indexed by the key or a pointer to a file that contains the information. The linear order \(<\) on the universe of all keys is specified by a function \(\text{cmp}\) that, given a pair of pointers \(\&R, \&S\) to records \(R, S\), returns a number: with \(K_X\) standing for the key in a record \(X\), we have

- \(\text{cmp}(\&R, \&S) < 0\) whenever \(K_R < K_S\),
- \(\text{cmp}(\&R, \&S) = 0\) whenever \(K_R = K_S\),
- \(\text{cmp}(\&R, \&S) > 0\) whenever \(K_R > K_S\).

The job of a sorting algorithm is to permute (if necessary) the records of the input array

\[ A[1], A[2], \ldots, A[n] \]

so that the resulting output array \(A[1], A[2], \ldots, A[n]\) satisfies

\[ \text{cmp}(\&A[i], \&A[i+1]) < 0\] for all \(i = 1, 2, \ldots, n - 1\).

Quicksort. We are going to consider only one sorting algorithm, namely, one of the many variations on the theme of quicksort. Here it goes.

\[
\text{QUICKSORT}(A, \text{first}, \text{last}):
\]
\[
\text{if}\ \ \ \text{first} < \text{last}
\]
\[
\text{then}\ \ \text{mid} = \text{PARTITION}(A, \text{first}, \text{last});
\]
\[
\text{QUICKSORT}(A, \text{first}, \text{mid} - 1);
\]
\[
\text{QUICKSORT}(A, \text{mid} + 1, \text{last});
\]
\[
\text{end}
\]
\texttt{Partition}(A, \text{first}, \text{last}): \\
\text{left} = \text{first} + 1, \text{right} = \text{last}; \\
\textbf{repeat} \\
\textbf{while} \text{left} \leq \text{last} \textbf{and} \text{cmp}(\&A[\text{left}],\&A[\text{first}]) < 0 \\
\textbf{do} \text{left} = \text{left} + 1; \\
\textbf{end} \\
\textbf{while} \text{cmp}(\&A[\text{right}],\&A[\text{first}]) > 0 \\
\textbf{do} \text{right} = \text{right} - 1; \\
\textbf{end} \\
\textbf{if} \text{right} \leq \text{left} \\
\textbf{then} \text{swap } A[\text{first}] \text{ and } A[\text{right}]; \\
\textbf{return} \text{right}; \\
\textbf{else} \text{swap } A[\text{left}] \text{ and } A[\text{right}]; \\
\text{left} = \text{left} + 1, \text{right} = \text{right} - 1; \\
\textbf{end} \\
\textbf{end} \\

\textbf{Average-case analysis of sorting algorithms.} A typical analysis of a sorting algorithm is restricted to counting the number of calls of \texttt{cmp} and the number of record moves. For this purpose, the satellites are irrelevant and only the relative order of the keys matters: one may just as well assume that each record consists only of its key and that the key is an integer. Following this convention, we may \\
replace the \texttt{cmp}(\&A[\text{left}],\&A[\text{first}]) < 0 in \texttt{Partition} \\
by \text{A[\text{left}] < A[\text{first}]} \\
and we may \\
replace the \texttt{cmp}(\&A[\text{right}],\&A[\text{first}]) > 0 in \texttt{Partition} \\
by \text{A[\text{right}] > A[\text{first}]}.

Furthermore, a common convention in the \textit{average-case} analysis of a sorting algorithm is to assume that \\
\((\star)\) the \(n\) keys are the first \(n\) positive integers and \\
the average is taken over the \(n!\) distinct inputs.

Results of such average-case analysis may and may not describe the typical behavior of the sorting algorithm. In many applications, the input files are files that have been previously sorted and subsequently corrupted (consider the task of updating, every ten years, a list of all the U.S. municipalities sorted by population count); average-case analysis carried out under assumption \((\star)\) is irrelevant to the behavior of sorting algorithms on such \textit{nearly sorted} inputs.
PARTITION preserves randomness. Our average-case analysis of QUICKSORT hinges on a subtle property of our PARTITION. Let us illustrate this property on the 120 inputs with \( n = 6 \) and \( A[1]=3 \):

\[
\begin{align*}
312456 & \rightarrow 213456 & 312465 & \rightarrow 213465 & 312546 & \rightarrow 213546 & 312564 & \rightarrow 213564 \\
312645 & \rightarrow 213645 & 312654 & \rightarrow 213654 & 314256 & \rightarrow 213456 & 314265 & \rightarrow 213465 \\
314526 & \rightarrow 213546 & 314562 & \rightarrow 213564 & 314625 & \rightarrow 213645 & 314652 & \rightarrow 213654 \\
315246 & \rightarrow 213546 & 315264 & \rightarrow 213564 & 315426 & \rightarrow 213456 & 315462 & \rightarrow 213465 \\
315624 & \rightarrow 213654 & 315642 & \rightarrow 213645 & 316245 & \rightarrow 213645 & 316254 & \rightarrow 213654 \\
316425 & \rightarrow 213465 & 316452 & \rightarrow 213456 & 316524 & \rightarrow 213564 & 316542 & \rightarrow 213546 \\
321456 & \rightarrow 123456 & 321465 & \rightarrow 123465 & 321546 & \rightarrow 123546 & 321564 & \rightarrow 123564 \\
321645 & \rightarrow 123645 & 321654 & \rightarrow 123654 & 324156 & \rightarrow 123456 & 324165 & \rightarrow 123465 \\
324516 & \rightarrow 123546 & 324561 & \rightarrow 123564 & 324615 & \rightarrow 123645 & 324651 & \rightarrow 123654 \\
325146 & \rightarrow 123546 & 325164 & \rightarrow 123564 & 325416 & \rightarrow 123456 & 325461 & \rightarrow 123465 \\
325614 & \rightarrow 123654 & 325641 & \rightarrow 123645 & 326145 & \rightarrow 123645 & 326154 & \rightarrow 123654 \\
326415 & \rightarrow 123465 & 326451 & \rightarrow 123456 & 326514 & \rightarrow 123564 & 326541 & \rightarrow 123546 \\
341256 & \rightarrow 123456 & 341265 & \rightarrow 123465 & 341526 & \rightarrow 123546 & 341562 & \rightarrow 123564 \\
341625 & \rightarrow 123645 & 341652 & \rightarrow 123654 & 342156 & \rightarrow 123456 & 342165 & \rightarrow 123465 \\
342516 & \rightarrow 123546 & 342561 & \rightarrow 123564 & 342615 & \rightarrow 123645 & 342651 & \rightarrow 123654 \\
345126 & \rightarrow 123546 & 345162 & \rightarrow 123564 & 345216 & \rightarrow 123546 & 345261 & \rightarrow 123564 \\
345612 & \rightarrow 123654 & 345621 & \rightarrow 123654 & 346125 & \rightarrow 123645 & 346152 & \rightarrow 123654 \\
346215 & \rightarrow 123645 & 346251 & \rightarrow 123654 & 346512 & \rightarrow 123564 & 346621 & \rightarrow 123564 \\
351246 & \rightarrow 123546 & 351264 & \rightarrow 123564 & 351426 & \rightarrow 123456 & 351462 & \rightarrow 123465 \\
351624 & \rightarrow 123654 & 351642 & \rightarrow 123645 & 352146 & \rightarrow 123546 & 352164 & \rightarrow 123564 \\
352416 & \rightarrow 123456 & 352461 & \rightarrow 123465 & 352614 & \rightarrow 123564 & 352641 & \rightarrow 123645 \\
354126 & \rightarrow 123456 & 354162 & \rightarrow 123465 & 354216 & \rightarrow 123546 & 354261 & \rightarrow 123564 \\
354612 & \rightarrow 123645 & 354621 & \rightarrow 123654 & 356124 & \rightarrow 123645 & 356142 & \rightarrow 123654 \\
356214 & \rightarrow 123654 & 356241 & \rightarrow 123645 & 356412 & \rightarrow 123465 & 356421 & \rightarrow 123465 \\
361245 & \rightarrow 123645 & 361254 & \rightarrow 123654 & 361425 & \rightarrow 123456 & 361452 & \rightarrow 123456 \\
361524 & \rightarrow 123564 & 361542 & \rightarrow 123546 & 362145 & \rightarrow 123645 & 362154 & \rightarrow 123654 \\
362415 & \rightarrow 123465 & 362451 & \rightarrow 123456 & 362514 & \rightarrow 123564 & 362541 & \rightarrow 123546 \\
364125 & \rightarrow 123465 & 364152 & \rightarrow 123456 & 364215 & \rightarrow 123465 & 364251 & \rightarrow 123456 \\
364512 & \rightarrow 123546 & 364521 & \rightarrow 123546 & 365124 & \rightarrow 123564 & 365142 & \rightarrow 123546 \\
365214 & \rightarrow 123564 & 365241 & \rightarrow 123546 & 365412 & \rightarrow 123456 & 365421 & \rightarrow 123465 \\
\end{align*}
\]


\[
\]

as required of any variant of PARTITION; among these 120 outputs,


and


More generally, our PARTITION preserves randomness in the sense that, for every choice of positive integers \( n \) and \( k \) such that \( 1 \leq k \leq n \),

\[
3
\]
it produces \((n - 1)!\) outputs such that

each of the \((k - 1)!\) permutations of \(1, 2, \ldots, k - 1\)

and

each of the \((n - k)!\) permutations of \(k + 1, k + 2 \ldots, n\)
appears equally often as \(A[k + 1]A[k + 2] \ldots A[n]\).

We skip a justification of this claim.

2 Average-case analysis of \textsc{Partition}(\(A, 1, n\))

2.1 The average number of key comparisons

Let us write \(k = A[1]\). Cursor “left” moves from 2 to \(k + 1\) and cursor “right” moves from \(n\) to \(k\). During this process, cursor “left” compares \(A[1]\) with all of \(A[2], A[3], \ldots, A[n]\) in case \(k = n\) and with all of \(A[2], A[3], \ldots, A[k+1]\) in case \(k < n\); cursor “right” compares \(A[1]\) with all of \(A[n], A[n - 1], \ldots, A[k]\); the total number of key comparisons comes to \(n\) in case \(k = n\) and to \(n + 1\) in case \(k < n\). Since precisely \((n - 1)!\) of the \(n!\) inputs have \(k = 1\),
we conclude that

\[
\text{\textsc{Partition}}(A, 1, n) \text{ makes } \left( n + 1 - \frac{1}{n} \right) \text{ key comparisons on the average.}
\]

2.2 The average number of record swaps

For every choice of positive integers \(n\) and \(k\) such that \(1 \leq k \leq n\), let \(f(n, k)\) denote average number of times that \textsc{Partition}(\(A, 1, n\)), given an input \(A[1]A[2] \ldots A[n]\) with \(A[1]=k\), executes the instruction

\[
\text{swap } A[\text{left}] \text{ and } A[\text{right}];
\]

In computing \(f(n, k)\), let us call a key \textit{small} if its value is less than \(k\) and let us call a key \textit{large} if its value is more than \(k\); in these terms, \(f(n, k)\) equals the average number of small keys that appear in \(A[k+1]A[k+2] \ldots A[n]\). Now imagine all the inputs \(A[1]A[2] \ldots A[n]\) with \(A[1]=k\) represented as the rows of an \(r \times n\) matrix (of course, we have \(r = (n - 1)!\), but the value of \(r\) is irrelevant to our argument). In each column except the first (which holds \(k\) throughout), all of the \(n - 1\) keys distinct from \(k\) appear equally often; to put it differently, each of these keys appears \(r/(n - 1)\) times in every column except the first. In particular, the number of small entries in the last \(n - k\) columns of the matrix is

\[
(k - 1) \cdot (n - k) \cdot \frac{r}{n - 1}
\]
and so
\[ f(n, k) = \frac{(k - 1)(n - k)}{n - 1} \]

To express
\[ \frac{1}{n} \sum_{k=1}^{n} f(n, k), \]
in a closed form, note that
\[ \sum_{k=1}^{n} (k - 1)(n - k) = \binom{n}{3} : \]
here, the left-hand side counts the number of ways to choose first an integer \( k \) from \( \{1, 2, \ldots n\} \), then an integer from \( \{1, 2, \ldots k-1\} \), and finally an integer from \( \{k+1, k+2, \ldots n\} \), while the right-hand side counts the number of ways to choose three distinct integers from \( \{1, 2, \ldots n\} \).

Hence
\[ \frac{1}{n} \sum_{k=1}^{n} f(n, k) = \frac{n-2}{6}. \]

Recalling the swap of \( A[\text{first}] \) and \( A[\text{right}] \) that we have not accounted for yet, we conclude that
\[ \text{PARTITION}(A, 1, n) \text{ makes } \frac{n + 4}{6} \text{ record swaps on the average.} \]

### 3 Average-case analysis of QUICKSORT\((A, 1, n)\)

Let \( C(n) \) denote the average number of key comparisons made by \( \text{QUICKSORT}(A, 1, n) \) and let \( S(n) \) denote the average number of record swaps made by \( \text{QUICKSORT}(A, 1, n) \). Trivially, we have
\[ C(0) = C(1) = 0 \quad (1) \]
and
\[ S(0) = S(1) = 0; \quad (2) \]
since our \( \text{PARTITION} \) preserves randomness, we have
\[ C(n) = \left( n + 1 - \frac{1}{n} \right) + \frac{1}{n} \sum_{k=1}^{n} C(k - 1) + \frac{1}{n} \sum_{k=1}^{n} C(n - k) \quad \text{whenever } n \geq 2 \quad (3) \]
and
\[ S(n) = \frac{n + 4}{6} + \frac{1}{n} \sum_{k=1}^{n} S(k - 1) + \frac{1}{n} \sum_{k=1}^{n} S(n - k) \quad \text{whenever } n \geq 2. \quad (4) \]
The recurrence relation (3) expresses the \( n \)-th term of the sequence \( C(0), C(1), C(2), C(3), \ldots \) as a function of the preceding terms, and so it, along with the initial conditions (1), determines the entire sequence: using (3) again and again, we find step by step
\[ C(2) = 5/2, \quad C(3) = 16/3, \quad C(4) = 26/3, \quad C(5) = 62/5, \quad C(6) = 247/15, \ldots \]
Similarly, the recurrence relation (4) expresses the \( n \)-th term of the sequence \( S(0), S(1), S(2), S(3), \ldots \) as a function of the preceding terms, and so it, along with the initial conditions (2), determines the entire sequence: using (4) again and again, we find step by step
\[
S(2) = 1, \quad S(3) = 11/6, \quad S(4) = 11/4, \quad S(5) = 56/15, \quad S(6) = 859/180, \quad \ldots
\]

*Solving* a recurrence means expressing the \( n \)-th term of the sequence directly, without any references to the preceding terms. We are going to solve (3) with the initial conditions (1) and we are going to solve (4) with the initial conditions (2). These two procedures begin in similar ways, which we are going to treat as one in a more general setting: we shall solve the recurrence
\[
f(n) = g(n) + \frac{1}{n} \sum_{k=1}^{n} f(k-1) + \frac{1}{n} \sum_{k=1}^{n} f(n-k) \quad \text{whenever } n \geq 2 \quad (5)
\]
with the initial conditions
\[
f(0) = f(1) = 0 \quad (6)
\]
in terms of \( g(2), g(3), g(4), \ldots \).

**A cosmetic change.** Observing that \( \sum_{k=1}^{n} f(k-1) = \sum_{k=1}^{n} f(n-k) \), we record (5) as
\[
f(n) = g(n) + \frac{2}{n} \sum_{k=1}^{n} f(k-1) \quad \text{whenever } n \geq 2. \quad (7)
\]

**A substantial change.** An unpleasant feature of (7) is that it is a *full history* recurrence: each term \( f(n) \) is a function of all the terms \( f(1), f(2), \ldots, f(n-1) \) preceding it. To convert (7) to a form where \( f(n) \) is a function of \( f(n-1) \) alone, let us substitute \( n-1 \) for \( n \) in (7):
\[
f(n-1) = g(n-1) + \frac{2}{n-1} \sum_{i=0}^{n-2} f(i) \quad \text{whenever } n \geq 3. \quad (8)
\]
If only the coefficient in front of the \( \sum f(i) \) in (7) matched the coefficient in front of the \( \sum f(i) \) in (8), we could make progress at once: subtracting (8) from (7) would yield an equation in \( f(n) \) and \( f(n-1) \) that involves no other \( f(i) \). Fortunately, it is easy to make the “if only” come true: all we have to do is multiply (7) by \( n \) and multiply (8) by \( n-1 \). From
\[
nf(n) = ng(n) + \frac{2}{n} \sum_{i=0}^{n-1} f(i)
\]
we subtract
\[
(n-1)f(n-1) = (n-1)g(n-1) + 2 \sum_{i=0}^{n-2} f(i),
\]
obtaining
\[
nf(n) - (n-1)f(n-1) = ng(n) + 2f(n-1) - (n-1)g(n-1);
\]
after simplifications, we conclude that
\[
f(n) = \frac{n+1}{n} f(n-1) + \left( g(n) - \frac{n-1}{n} g(n-1) \right) \quad \text{whenever } n \geq 3. \quad (9)
\]
Telescoping. Recurrence (9) amounts to a sequence of identities

\[
\begin{align*}
  f(3) &= \frac{4}{3} f(2) + \left( g(3) - \frac{2}{3} g(2) \right), \\
  f(4) &= \frac{5}{4} f(3) + \left( g(4) - \frac{3}{4} g(3) \right), \\
  f(5) &= \frac{6}{5} f(4) + \left( g(5) - \frac{4}{5} g(4) \right), \\
  \ldots &= \ldots \\
  f(n-1) &= \frac{n}{n-1} f(n-2) + \left( g(n-1) - \frac{n-2}{n-1} g(n-2) \right), \\
  f(n) &= \frac{n+1}{n} f(n-1) + \left( g(n) - \frac{n-1}{n} g(n-1) \right), \\
  \ldots &= \ldots
\end{align*}
\]

If only the coefficient at \( f(3) \) in (11) matched the coefficient at \( f(3) \) in (10), and the coefficient at \( f(4) \) in (12) matched the coefficient at \( f(4) \) in (11), and so on, we could solve (9) at once: the sum of the first \( n-2 \) identities in this sequence would telescope through cancellations into a formula expressing \( f(n) \) without references to any other \( f(i) \) except \( f(2) \). Fortunately, a simple trick makes the “if only” come true: all we have to do is divide (10) by 4, divide (10) by 5, and so on:

\[
\begin{align*}
  \frac{f(3)}{4} &= \frac{f(2)}{3} + \frac{1}{4} \left( g(3) - \frac{2}{3} g(2) \right), \\
  \frac{f(4)}{5} &= \frac{f(3)}{4} + \frac{1}{5} \left( g(4) - \frac{3}{4} g(3) \right), \\
  \frac{f(5)}{6} &= \frac{f(4)}{5} + \frac{1}{6} \left( g(5) - \frac{4}{5} g(4) \right), \\
  \ldots &= \ldots \\
  \frac{f(n-1)}{n} &= \frac{f(n-2)}{n-1} + \frac{1}{n} \left( g(n-1) - \frac{n-2}{n-1} g(n-2) \right), \\
  \frac{f(n)}{n+1} &= \frac{f(n-1)}{n} + \frac{1}{n+1} \left( g(n) - \frac{n-1}{n} g(n-1) \right), \\
  \ldots &= \ldots
\end{align*}
\]

The sum of the first \( n-2 \) identities in this sequence telescopes into

\[
\frac{f(n)}{n+1} = \frac{f(2)}{3} - \frac{g(2)}{6} + \sum_{i=3}^{n-1} \frac{2g(i)}{(i+1)(i+2)} + \frac{g(n)}{n+1};
\]

observing that \( f(2) = g(2) \) by (6) and (7), we conclude that

\[
f(n) = 2(n+1) \sum_{i=2}^{n-1} \frac{g(i)}{(i+1)(i+2)} + g(n) \quad \text{whenever } n \geq 3.
\]
3.1 The average number of key comparisons

Setting $f(n) = C(n)$ and 
\[ g(n) = \left( n + 1 - \frac{1}{n} \right), \]
we reduce (6),(5) to (1),(3). It follows that 
\[ C(n) = 2(n + 1) \sum_{i=2}^{n-1} \frac{i^2 + i - 1}{i(i+1)(i+2)} + \frac{n^2 + n - 1}{n} \quad \text{whenever } n \geq 3. \quad (14) \]

**Partial fractions.** To simplify the sum 
\[ \sum_{i=2}^{n-1} \frac{i^2 + i - 1}{i(i+1)(i+2)}, \]
we may use a trick from integral calculus, known as “partial fractions”: if $A, B, C$ are constants such that 
\[ \frac{i^2 + i - 1}{i(i+1)(i+2)} = \frac{A}{i} + \frac{B}{i+1} + \frac{C}{i+2} \quad \text{for all } i, \quad (15) \]
then 
\[ \sum_{i=2}^{n-1} \frac{i^2 + i - 1}{i(i+1)(i+2)} = \sum_{i=2}^{n-1} \frac{A}{i} + \sum_{i=2}^{n-1} \frac{B}{i+1} + \sum_{i=2}^{n-1} \frac{C}{i+2} \]
\[ = A \sum_{i=2}^{n-1} \frac{1}{i} + B \sum_{i=3}^{n} \frac{1}{i} + C \sum_{i=4}^{n} \frac{1}{i} \]
\[ = \frac{5A}{6} + \frac{B}{3} + (A + B + C) \sum_{i=4}^{n} \frac{1}{i} + \frac{B}{n} + \frac{C(2n+1)}{n(n+1)} \]

To find the values of $A, B, C$, we write (15) as 
\[ i^2 + i - 1 = (A + B + C)i^2 + (3A + 2B + C)i + 2A \]
and require that $1 = A + B + C$, $1 = 3A + 2B + C$, $-1 = 2A$; this requirement is satisfied by $A = -1/2$, $B = 1$, $C = 1/2$; we conclude that 
\[ \sum_{i=2}^{n-1} \frac{i^2 + i - 1}{i(i+1)(i+2)} = -\frac{1}{12} + \sum_{i=4}^{n} \frac{1}{i} + \frac{2n + 1}{2n(n+1)}. \quad (16) \]
Substituting from (16) into (14) we get, after simplifications,
\[ C(n) = 2(n + 1) \sum_{i=1}^{n} \frac{1}{i} - \frac{17n + 5}{6}. \quad (17) \]
Harmonic numbers. It is not clear from (17) just how fast $C(n)$ grows with $n$. The second term is a simple linear function of $n$; it is the sum in the first term, whose order of magnitude is not obvious. It is customary to write

$$H_n = \sum_{i=1}^{n} \frac{1}{i};$$

these numbers $H_1, H_2, H_3, \ldots$, being partial sums of the harmonic series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots$, are known as the harmonic numbers. In this notation, (17) can be written as

$$C(n) = 2(n + 1)H_n - \frac{17n + 5}{6}.$$  \hspace{1cm} (18)

How fast do the harmonic numbers grow? Since

$$\int_{i}^{i+1} \frac{1}{x} \, dx \leq \frac{1}{i} \leq \int_{i-1}^{i} \frac{1}{x} \, dx,$$

we have

$$H_n = 1 + \sum_{i=2}^{n} \frac{1}{i} \leq 1 + \sum_{i=2}^{n} \int_{i-1}^{i} \frac{1}{x} \, dx = 1 + \int_{1}^{n} \frac{1}{x} \, dx = 1 + \ln n$$

and

$$H_n = \sum_{i=1}^{n-1} \frac{1}{i} + \frac{1}{n} \geq \sum_{i=1}^{n-1} \int_{i}^{i+1} \frac{1}{x} \, dx + \frac{1}{n} = \int_{1}^{n} \frac{1}{x} \, dx + \frac{1}{n} = \ln n + \frac{1}{n};$$

hence

$$\frac{1}{n} \leq H_n - \ln n \leq 1 \quad \text{for all } n.$$  \hspace{1cm} (19)

Substituting into (18), we conclude that

$$C(n) = 2n \ln n + \alpha(n) \text{ with } |\alpha(n)| \in O(n).$$

Euler's constant. By the way, since each of the integrals in the right-hand side of

$$\ln n - (H_n - 1) = \sum_{i=2}^{n} \int_{i-1}^{i} \left( \frac{1}{x} - \frac{1}{i} \right) \, dx,$$

is positive, the differences $\ln n - H_n$ grow with $n$. To put it differently, the differences $H_n - \ln n$ decrease as $n$ grows; since these differences are sandwiched between 0 and 1 by (19), it follows that

$$\lim_{n \to \infty} (H_n - \ln n)$$

exists. This limit is known as Euler's constant and denoted $\gamma$; like $\pi$ and $e$, it is one of the more notorious irrational numbers; its value is about 0.577.
3.2 The average number of record swaps

Setting \( f(n) = S(n) \) and

\[
g(n) = \frac{n + 4}{6},
\]

we reduce (6),(5) to (2),(4). It follows that

\[
S(n) = \frac{n + 1}{3} \sum_{i=2}^{n-1} \frac{i + 4}{(i + 1)(i + 2)} + \frac{n + 4}{6} \quad \text{whenever } n \geq 3. \tag{20}
\]

We have

\[
\sum_{i=2}^{n-1} \frac{i + 4}{(i + 1)(i + 2)} = \sum_{i=2}^{n-1} \frac{3}{i + 1} - \sum_{i=2}^{n-1} \frac{2}{i + 2} = 3 \sum_{i=3}^{n} \frac{1}{i} - 2 \sum_{i=4}^{n+1} \frac{1}{i} = 1 + \sum_{i=4}^{n} \frac{1}{i} - \frac{2}{n + 1} = H_n - \frac{5}{6} - \frac{2}{n + 1};
\]

substituting into (14), we get

\[
S(n) = \frac{n + 1}{3} H_n - \frac{2n + 5}{18} \quad \text{whenever } n \geq 3;
\]

appealing to (19), we conclude that

\[
S(n) = \frac{1}{3} n \ln n + \beta(n) \quad \text{with } |\beta(n)| \in O(n).
\]

——*****——

These notes are based on