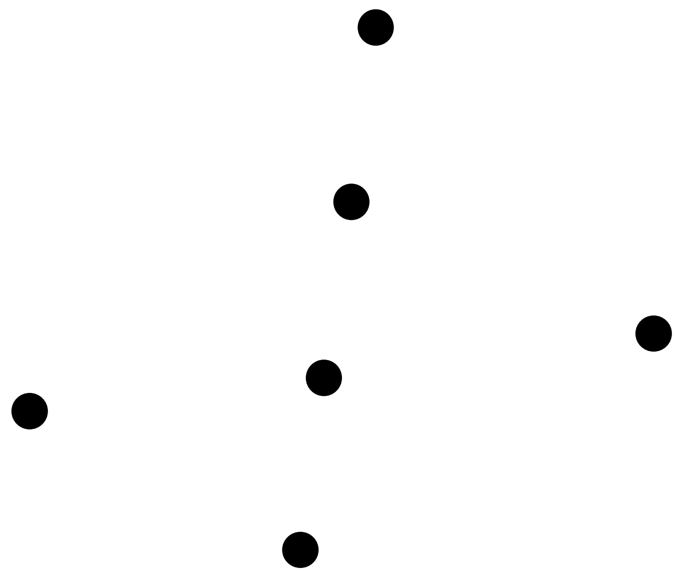
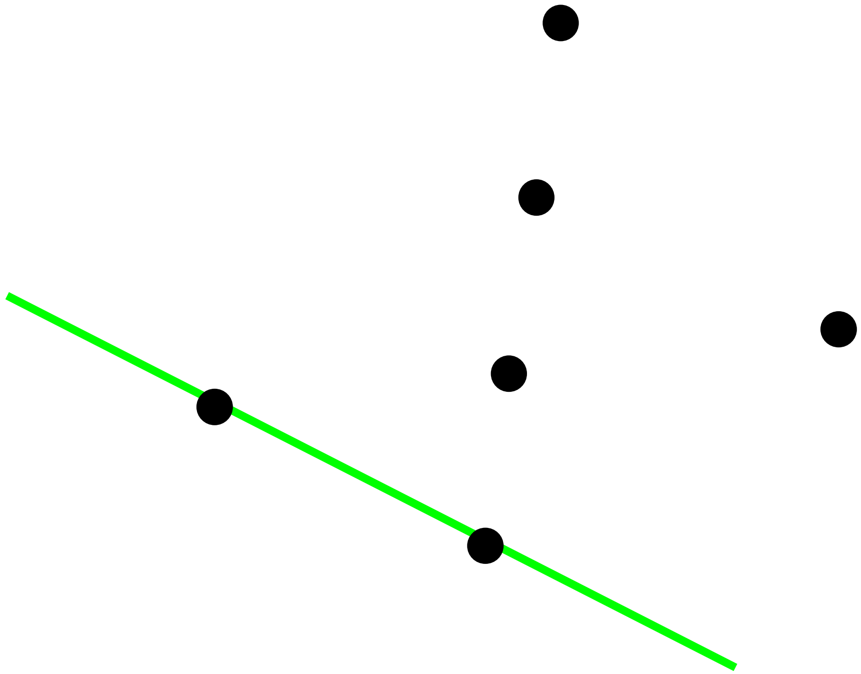
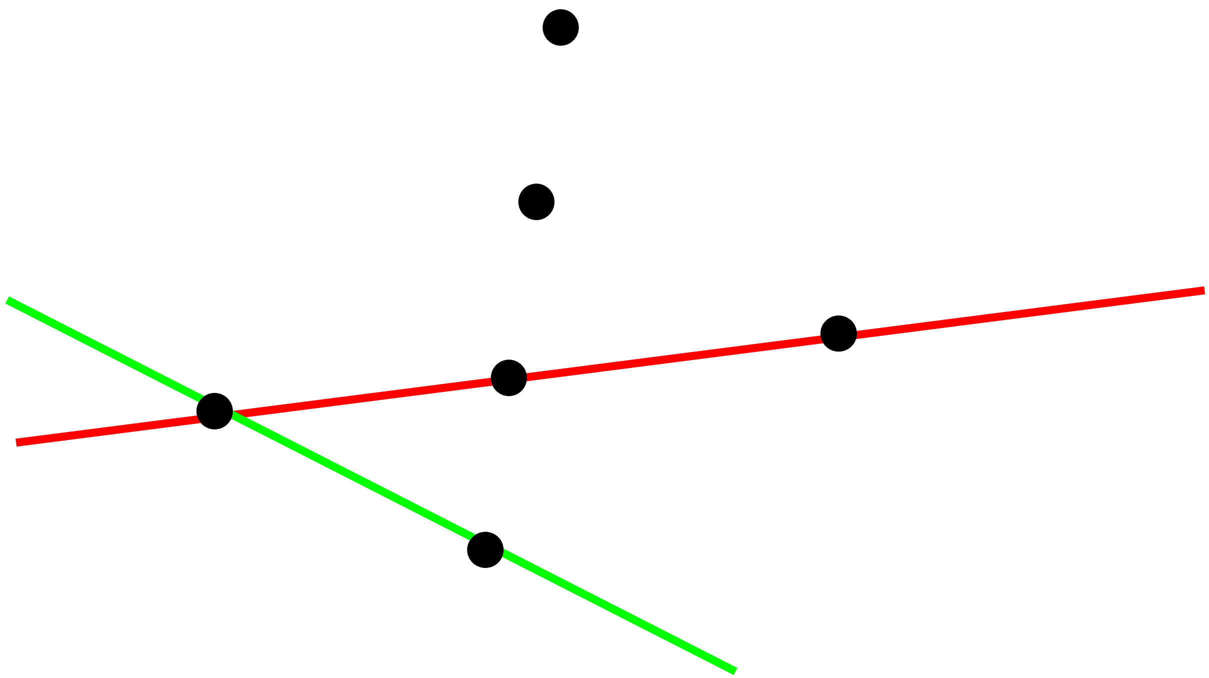

NO
PARKING
BETWEEN
SIGNS

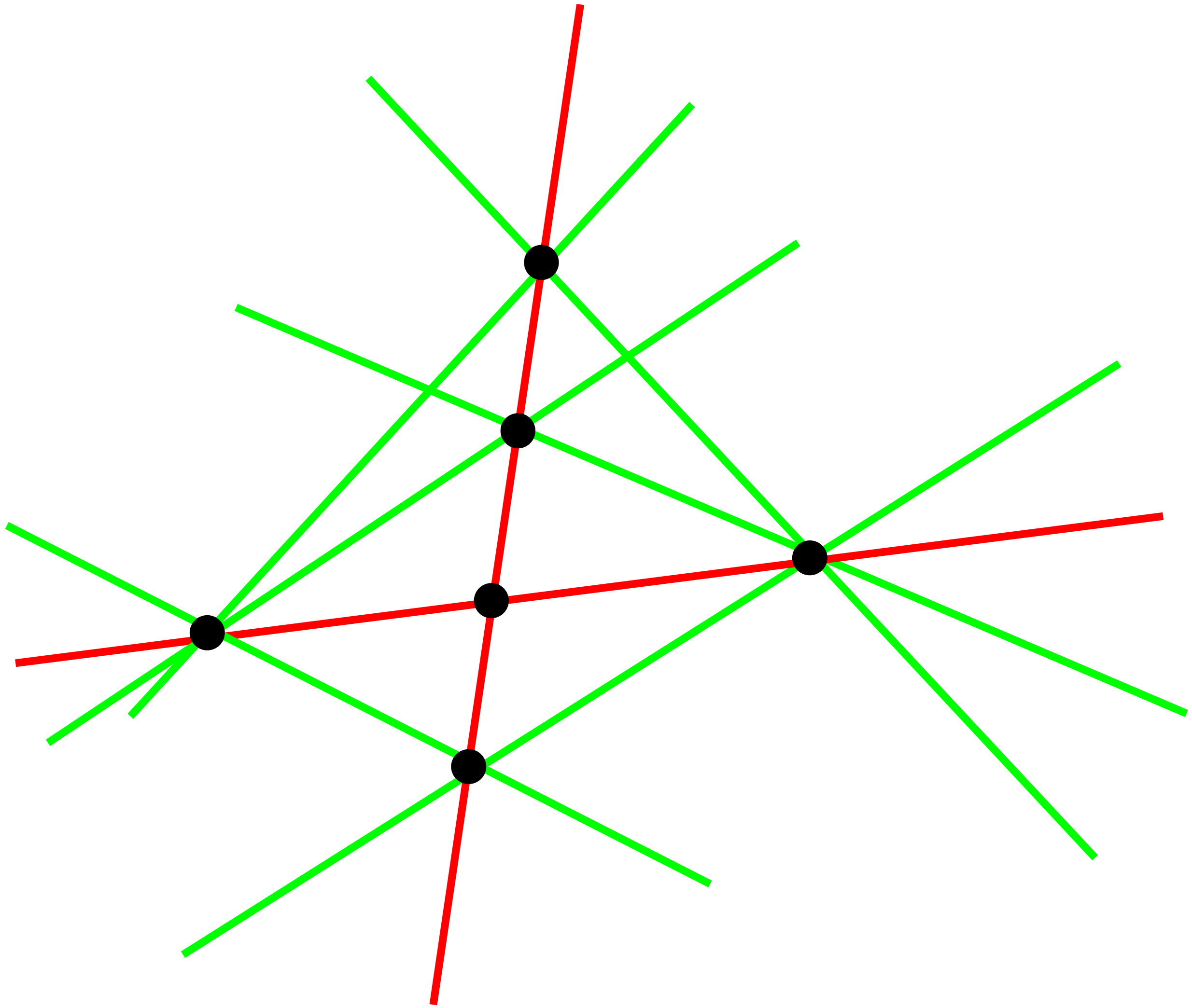

NO
PARKING
BETWEEN
SIGNS

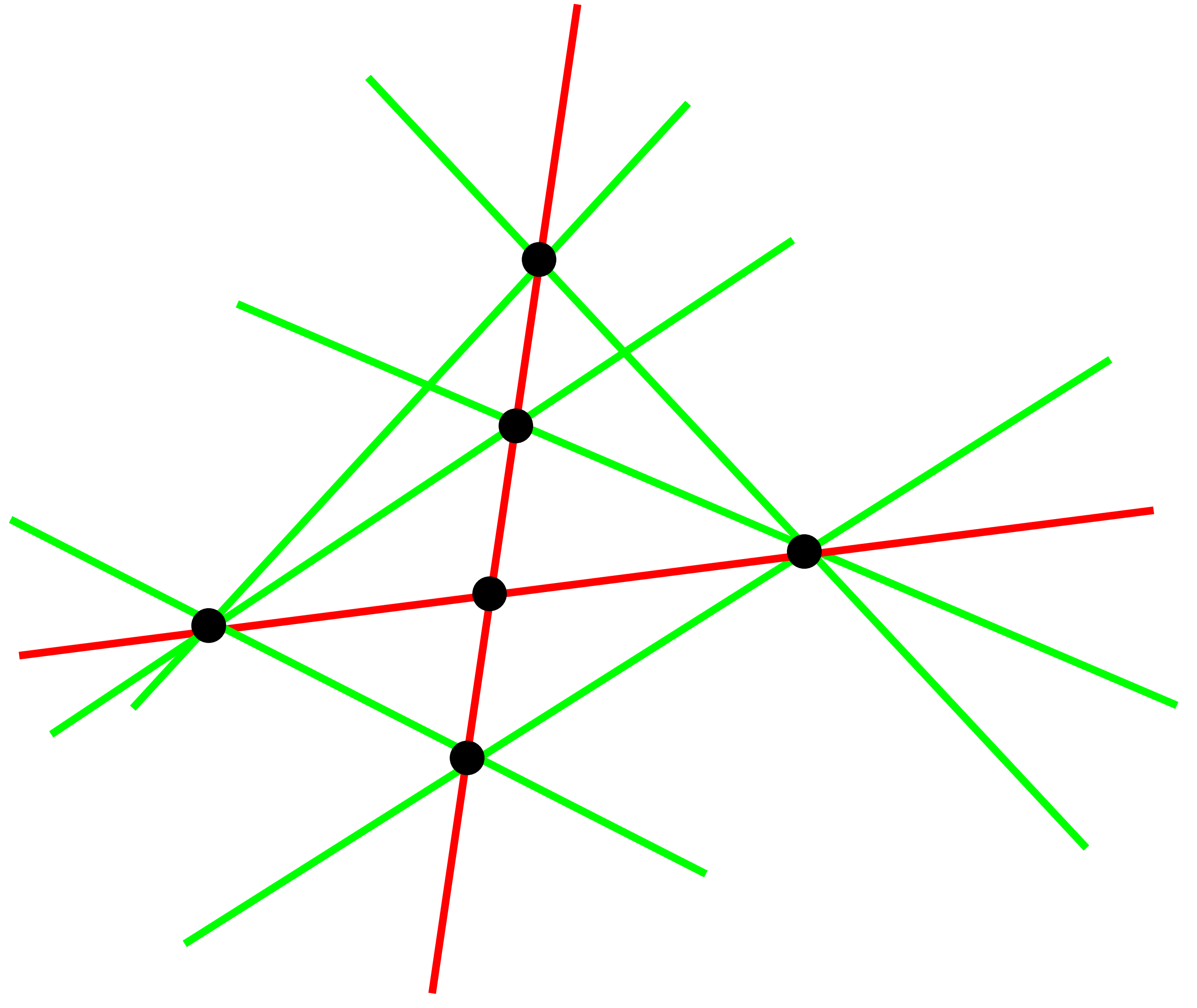
1. **Points and lines**
2. **Convexity**
3. **Causality**



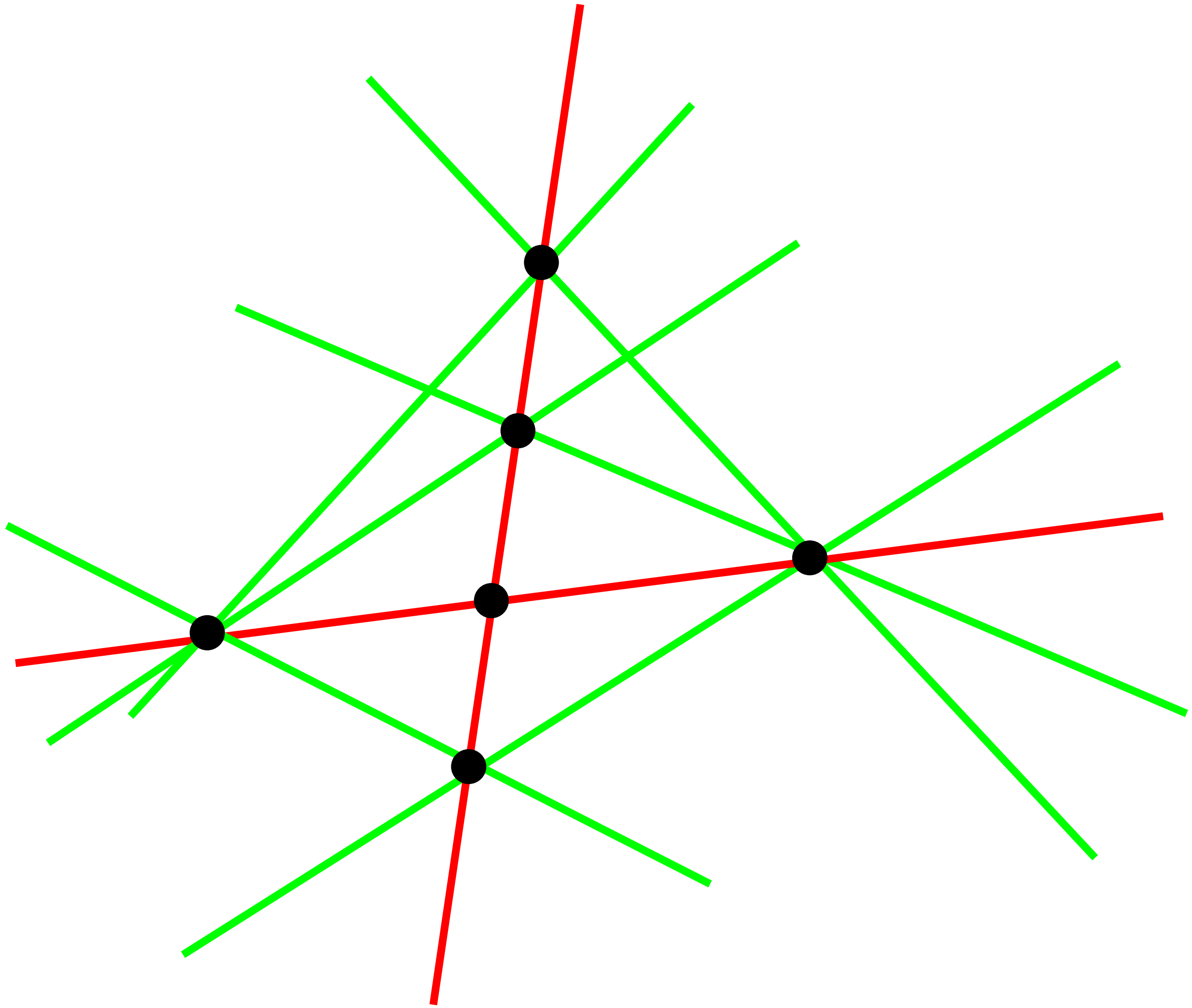








ORDINARY LINES



ORDINARY LINES

EXTRAORDINARY LINES?

Educational Times, March 1893

Educational Times, March 1893

James Joseph Sylvester

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Prove that it is not possible to arrange any finite number of real points so that a right line through every two of them shall pass through a third, unless they all lie in the same right line.

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A four-line solution

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A four-line solution … containing two distinct flaws

First proof: T.Gallai (1933)

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L.M. Kelly's proof:

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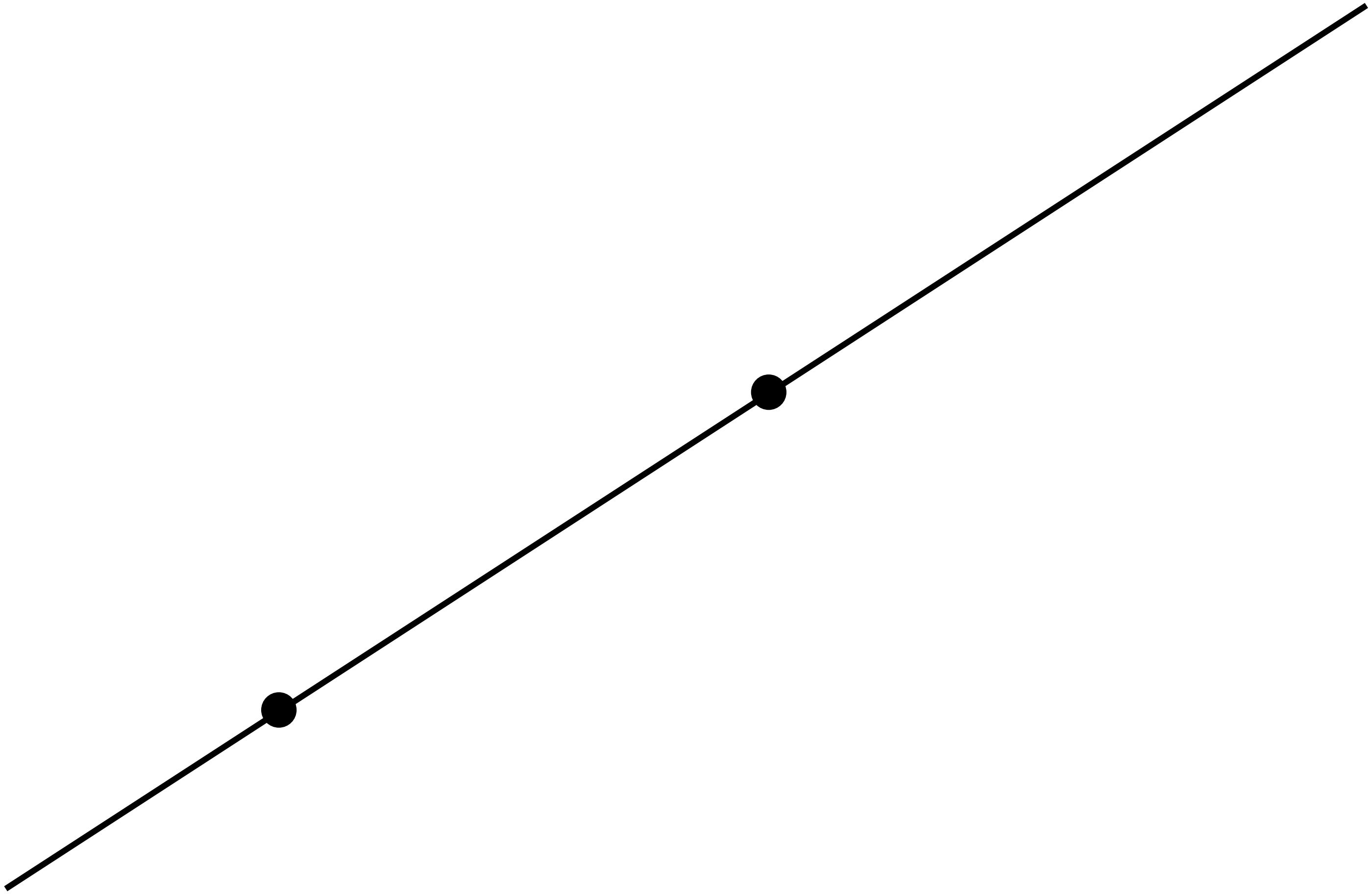

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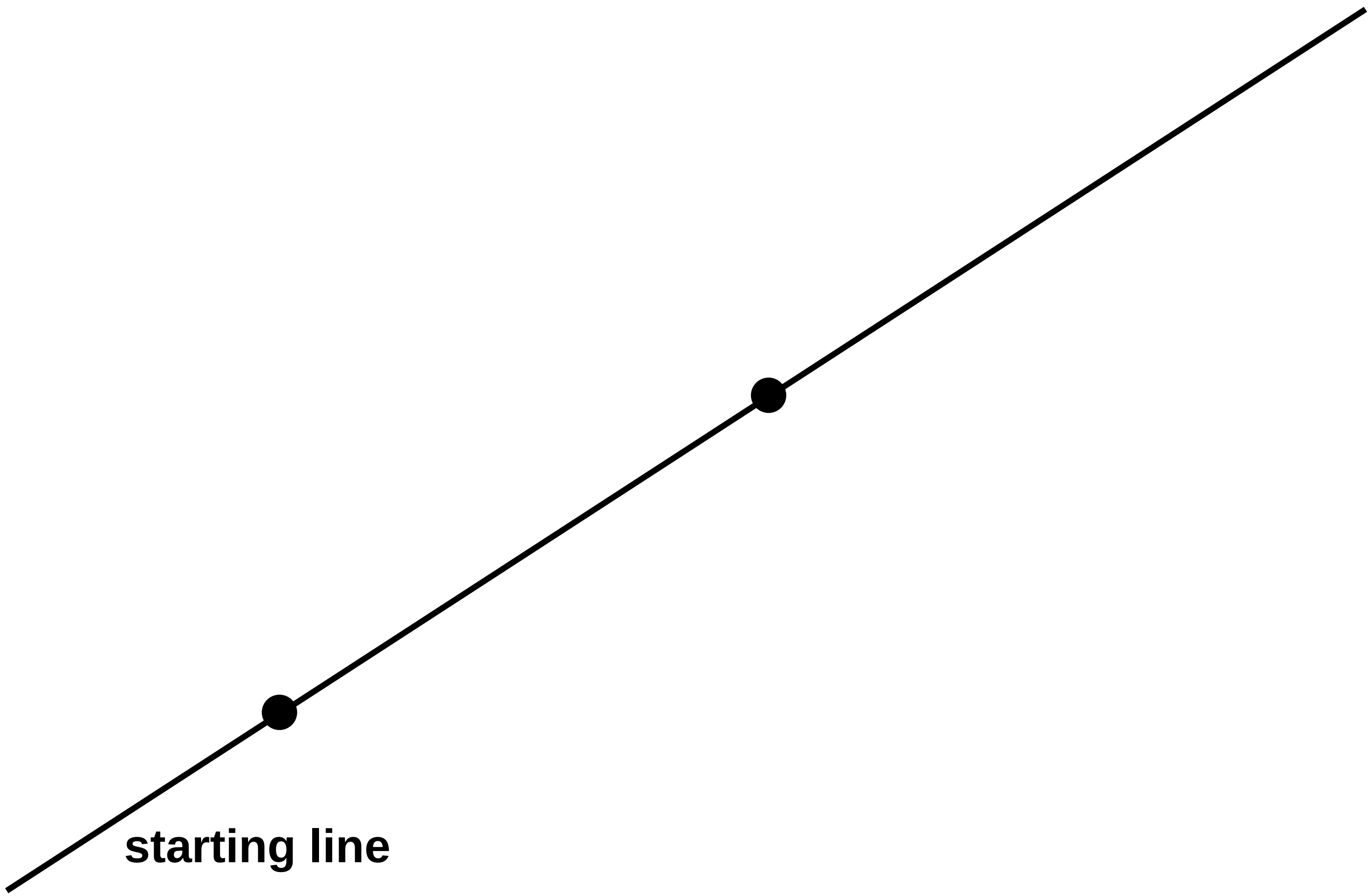

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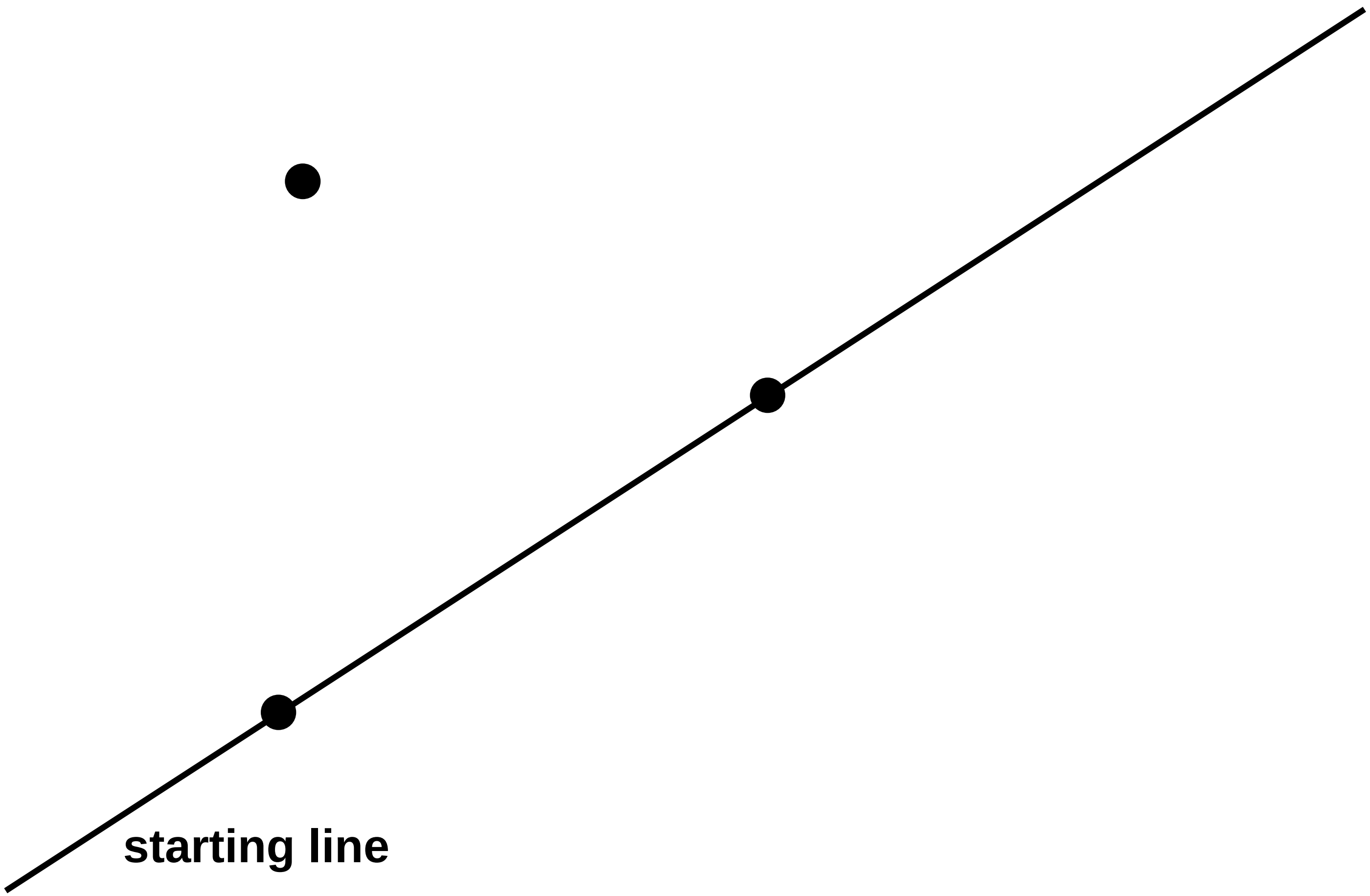

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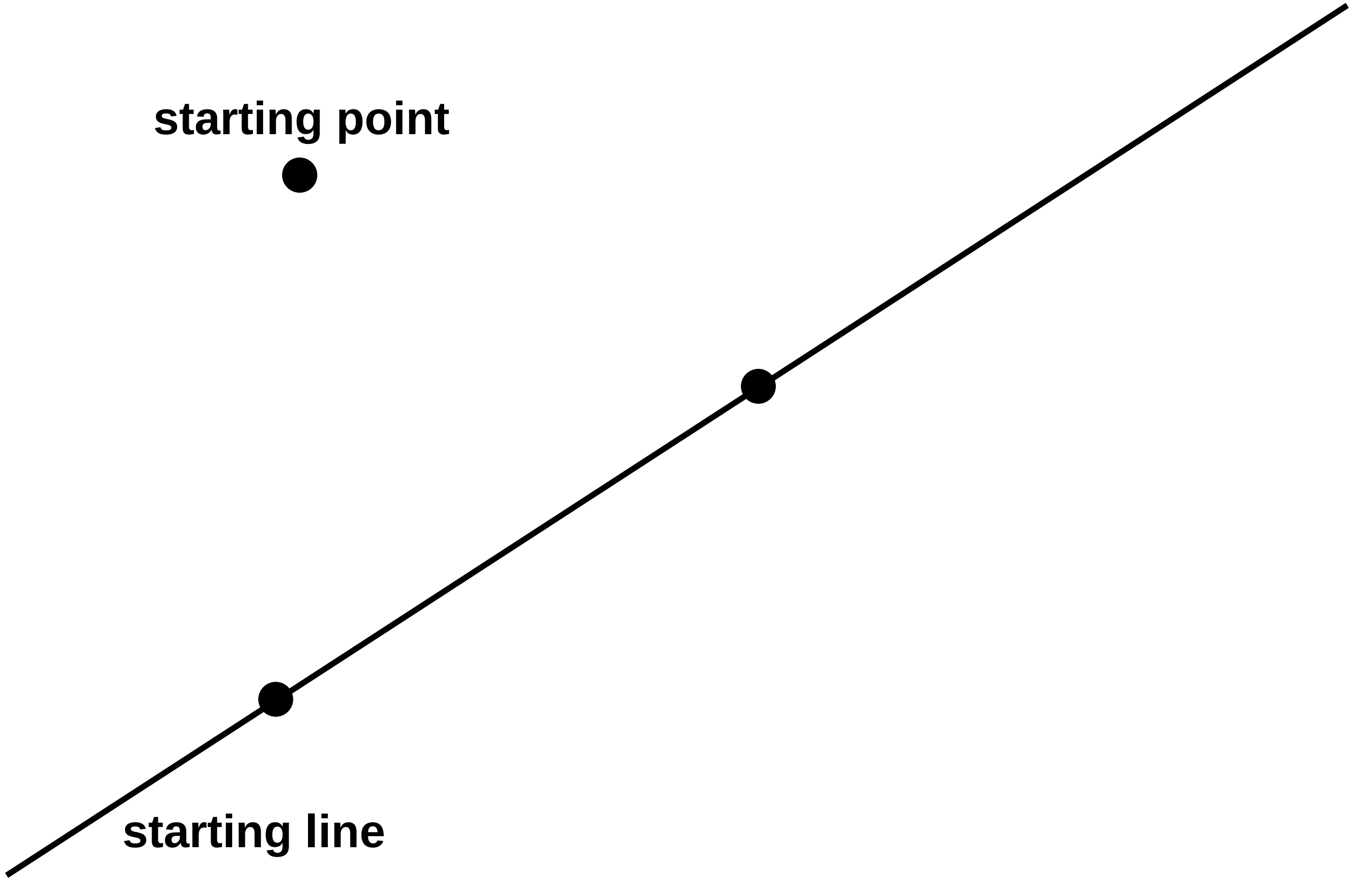

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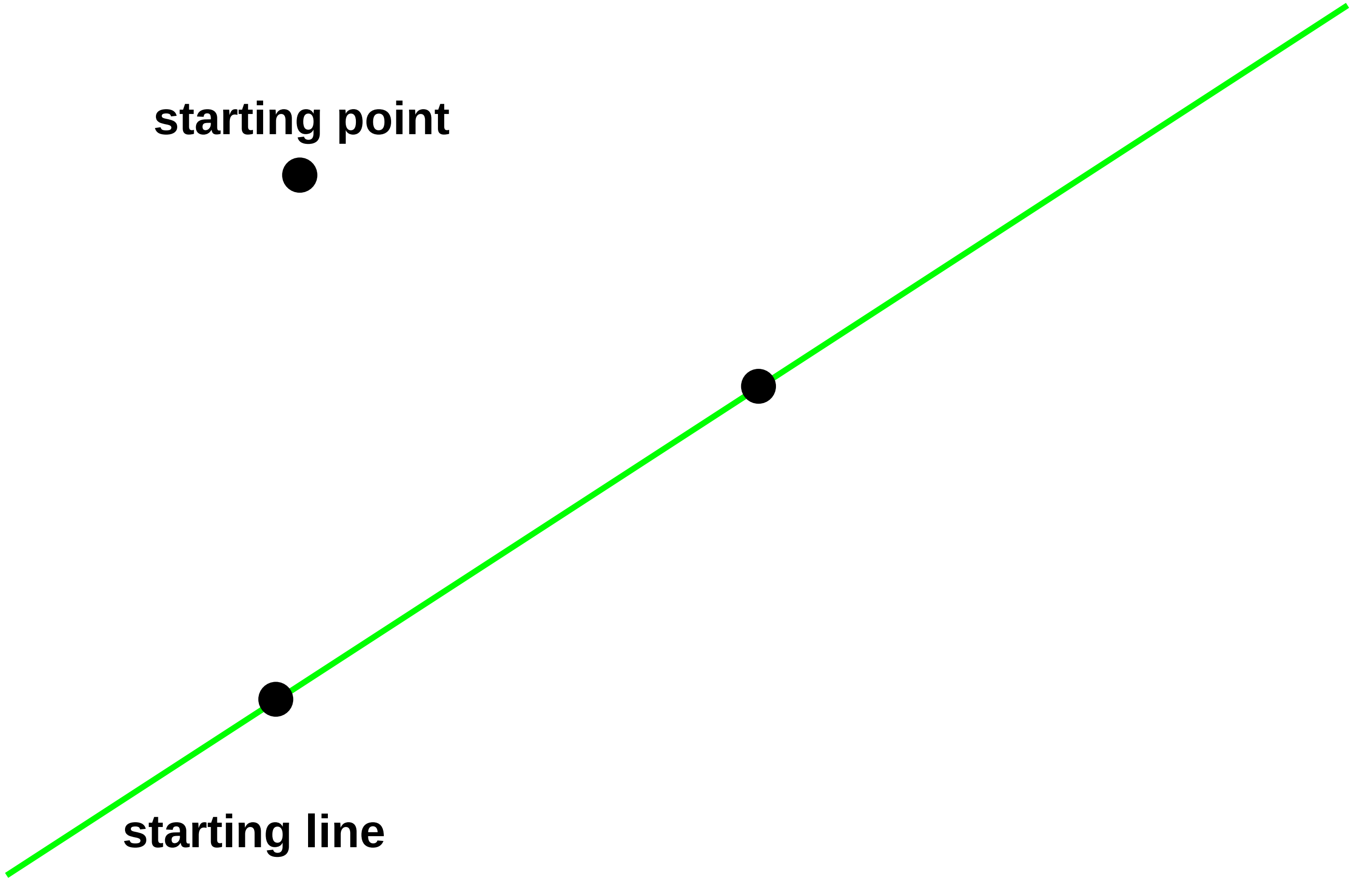

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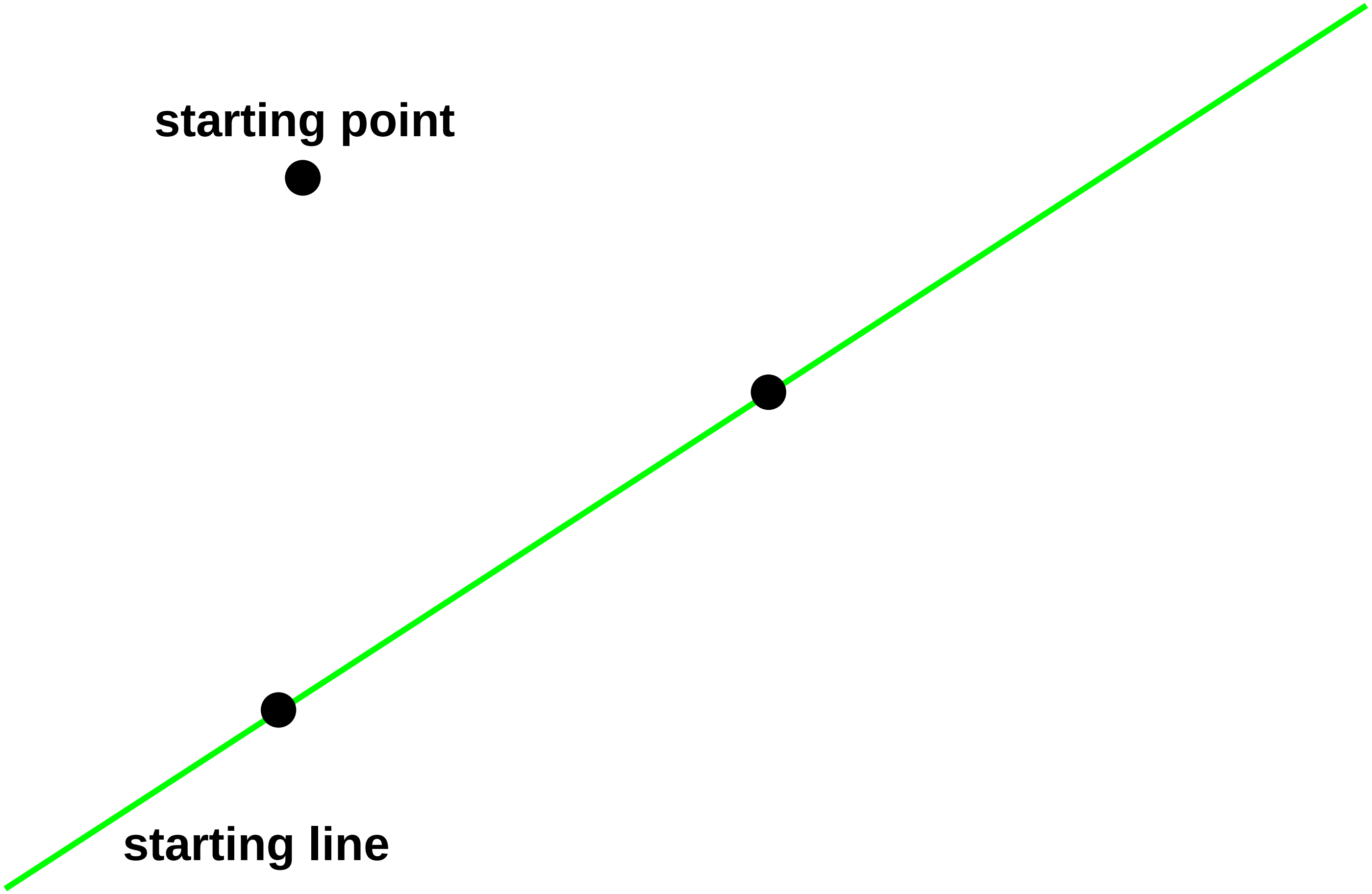

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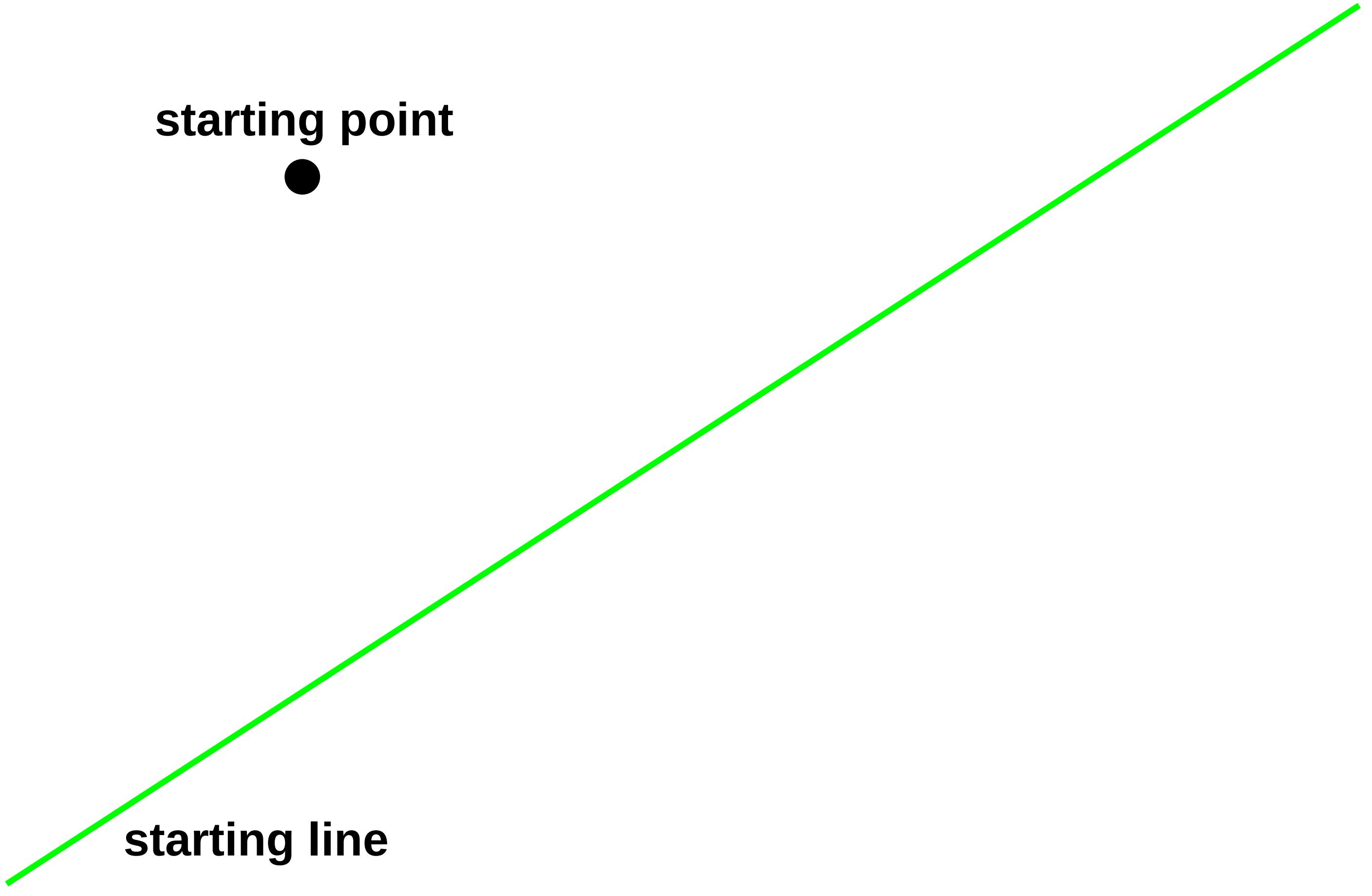

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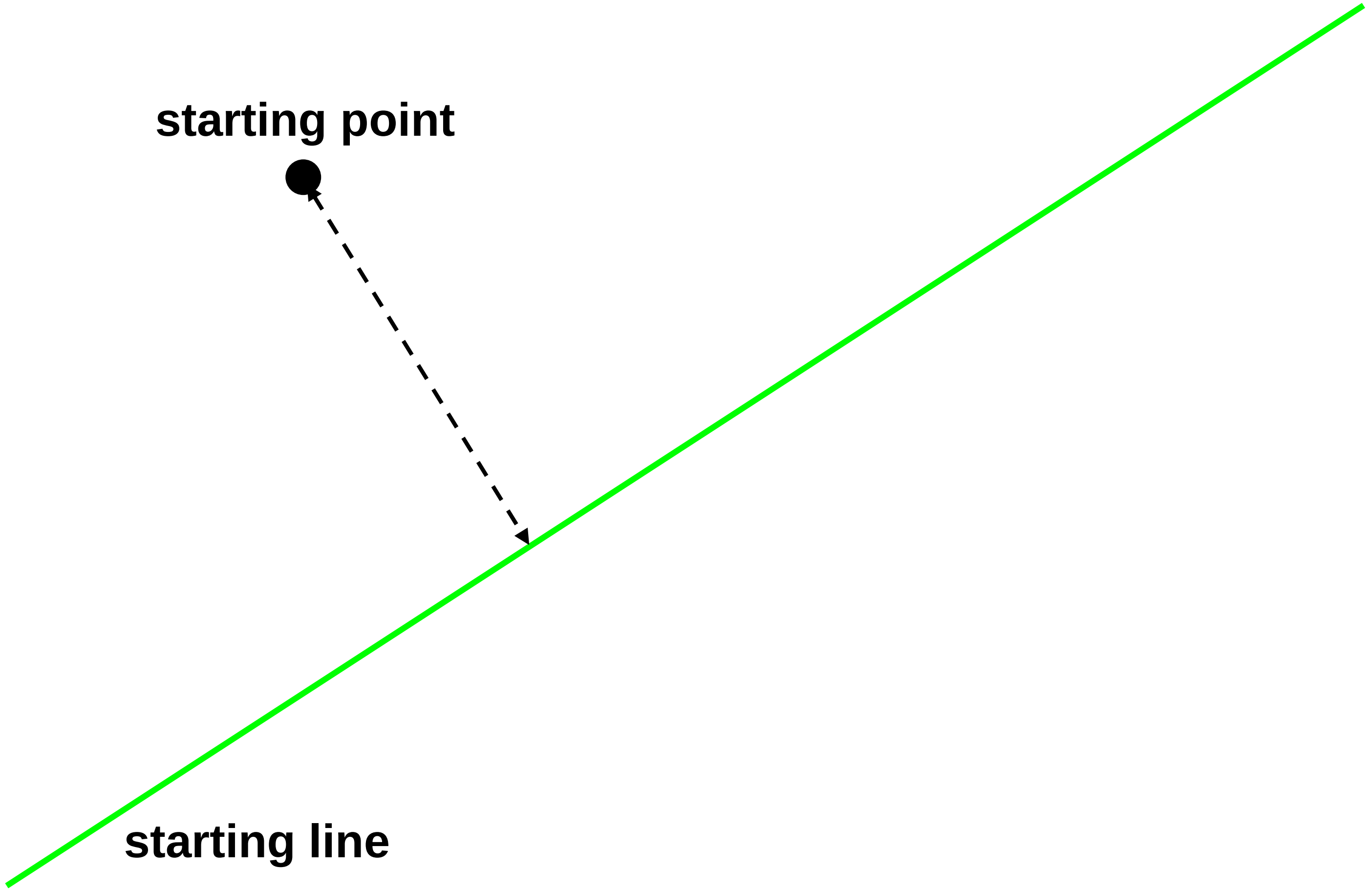

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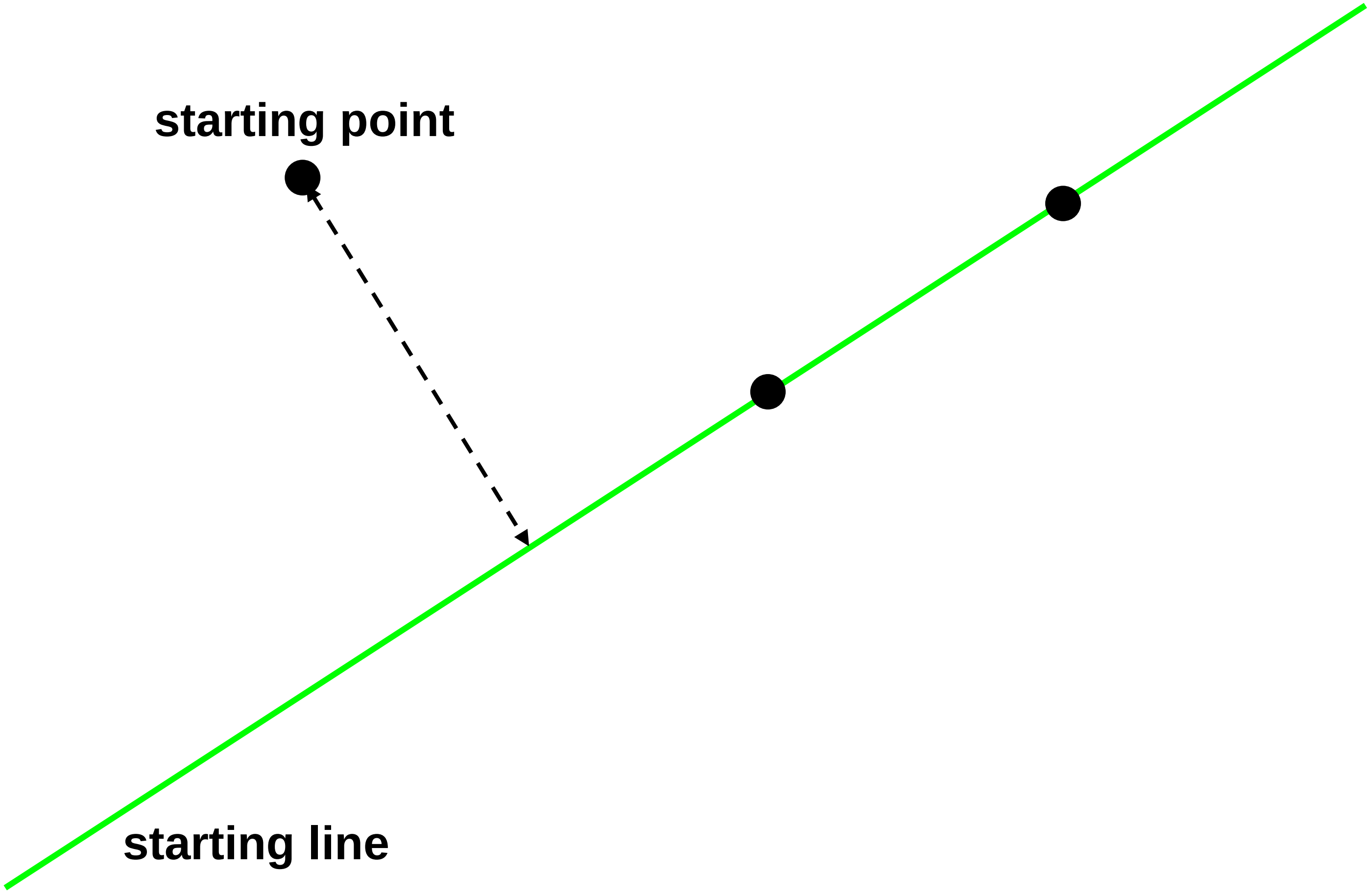

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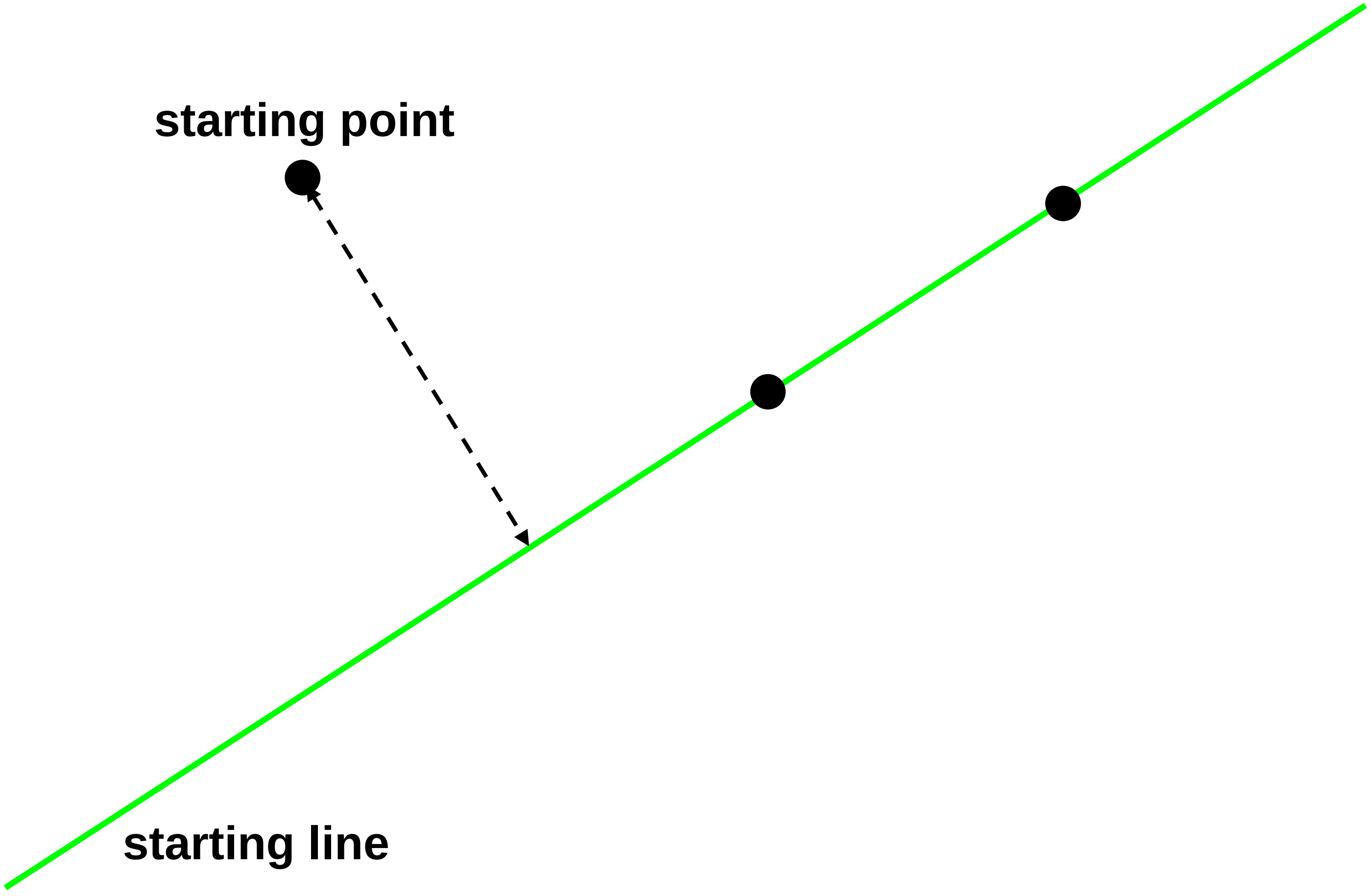

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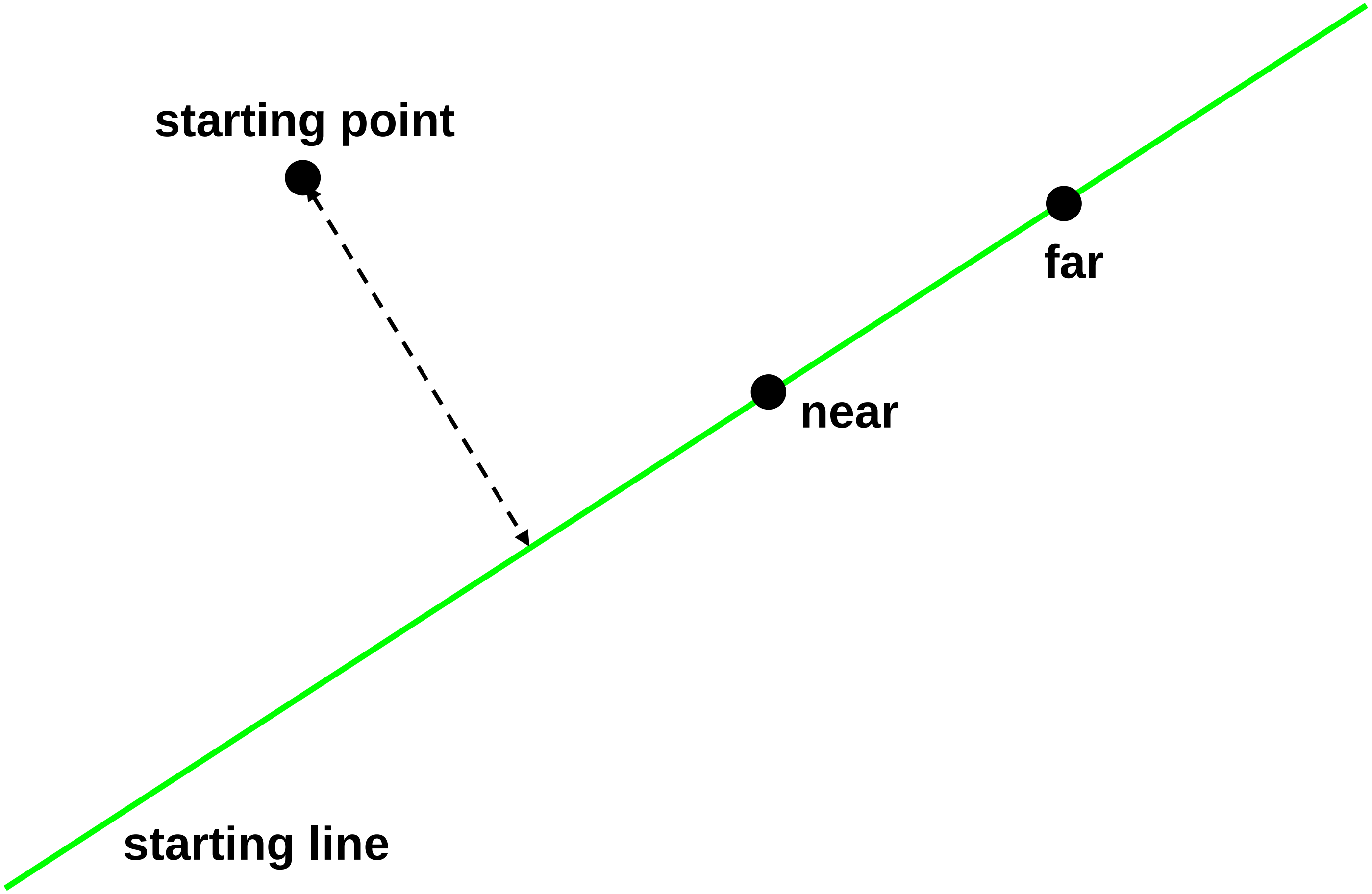

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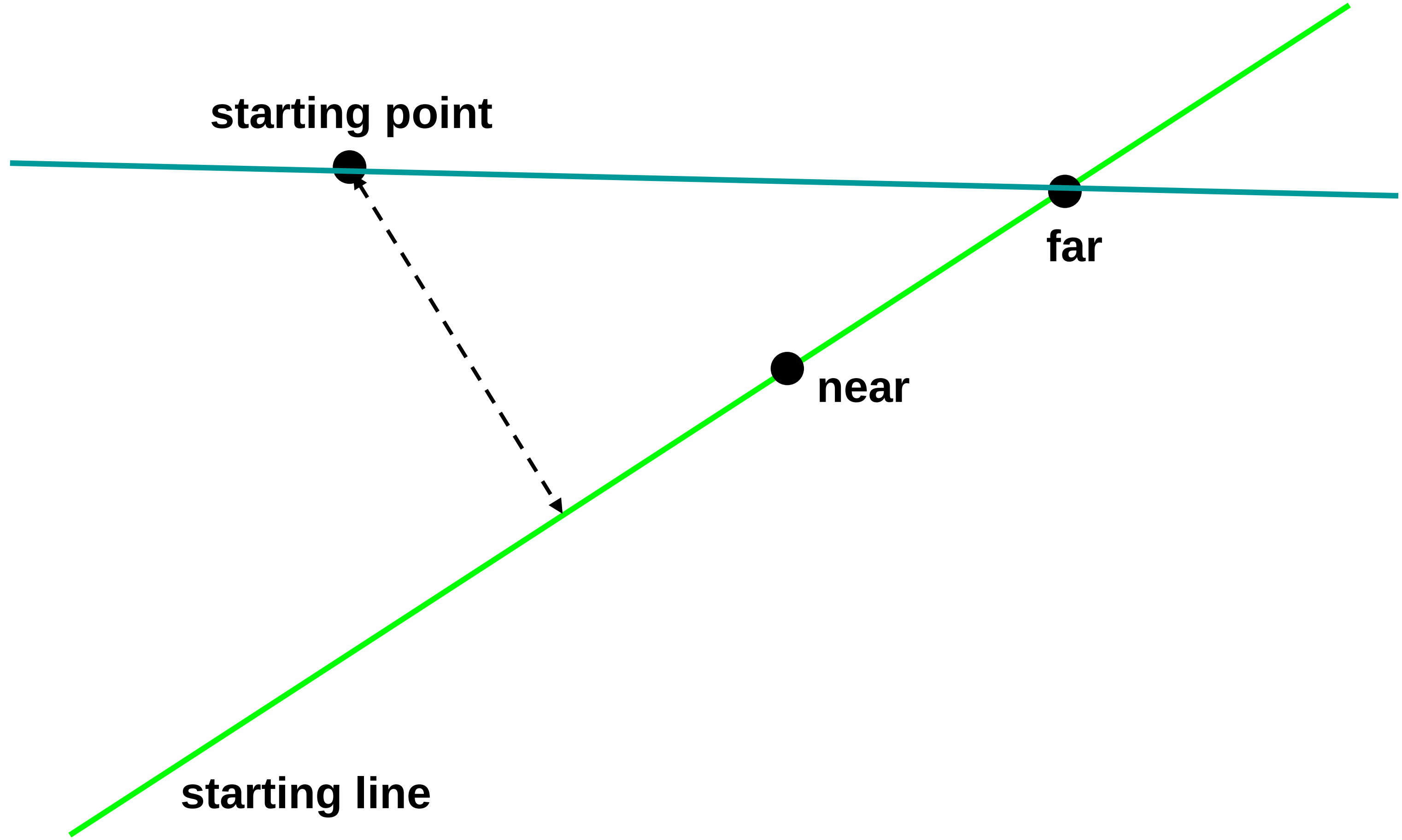

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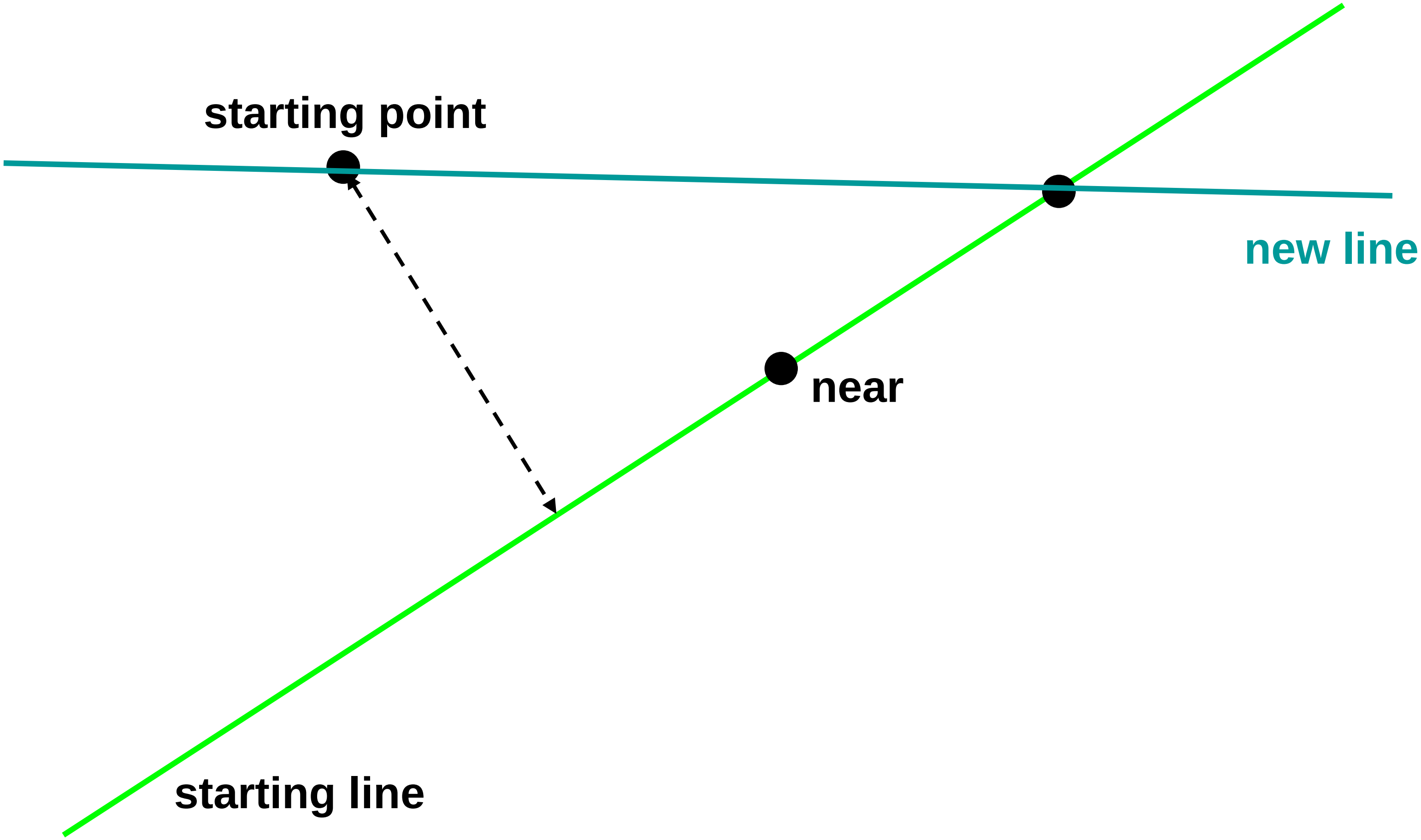

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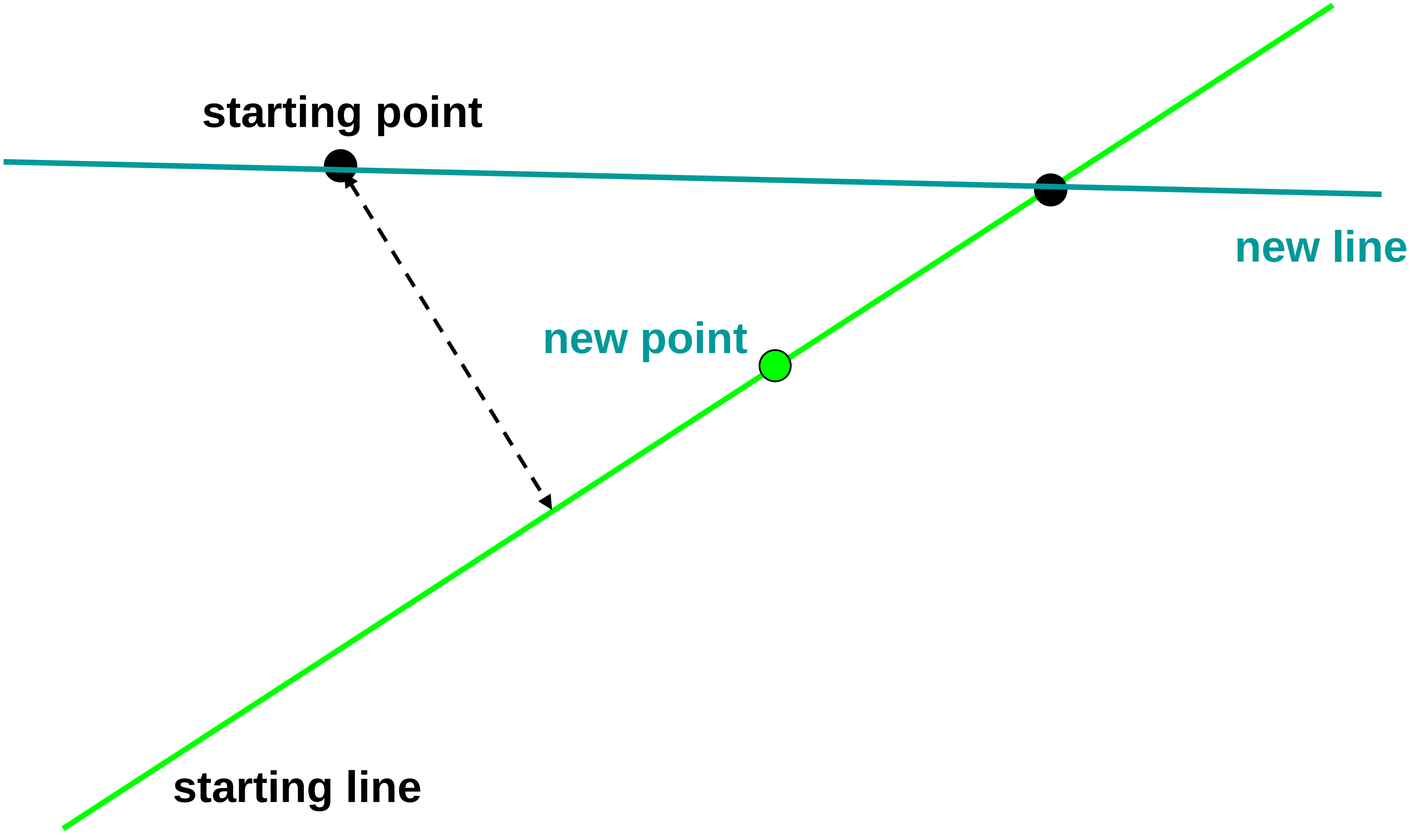

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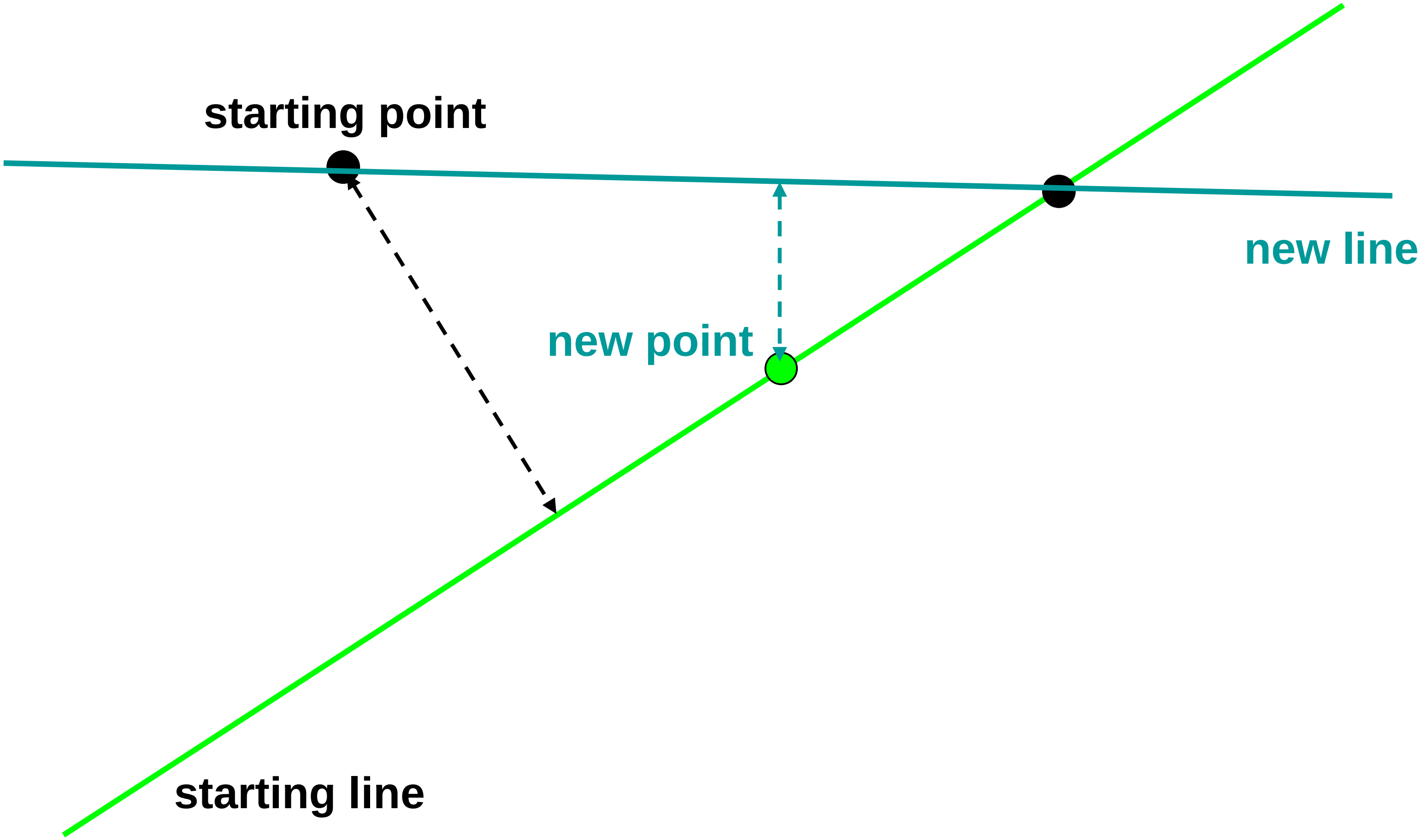

Be wise: Generalize!

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or

Be wise: Generalize!

or

What iceberg

is the Sylvester-Gallai theorem a tip of?

Definition. A *metric* on a set S is a function

$$\mathrm{dist} : S \times S \to [0, +\infty)$$

such that

$$\begin{aligned}
\mathrm{dist}(x, y) = 0 \quad &\Leftrightarrow \quad x = y, \\
\mathrm{dist}(x, y) \quad &= \quad \mathrm{dist}(y, x), \\
\mathrm{dist}(x, y) + \mathrm{dist}(y, z) \quad &\geq \quad \mathrm{dist}(x, z).
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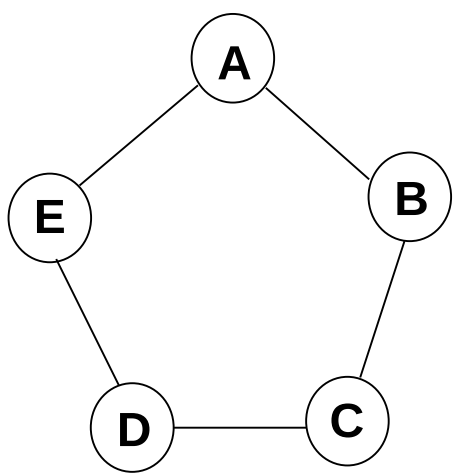

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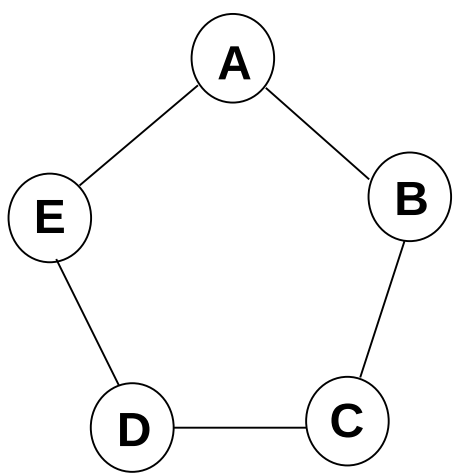

dist(A,B) = 1,

dist(A,C) = 2,

etc.

Definition. In a metric space, point y is said to lie *between* points x and z if, and only if,

$$\operatorname{dist}(x, y) + \operatorname{dist}(y, z) = \operatorname{dist}(x, z).$$

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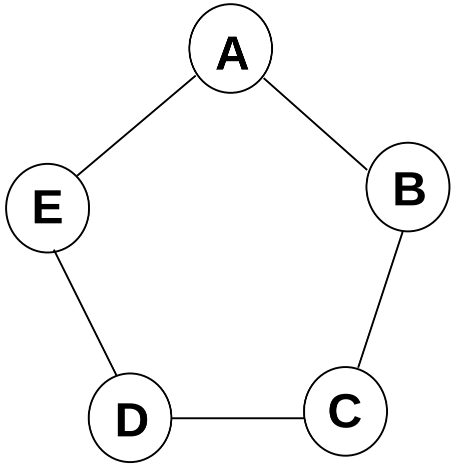

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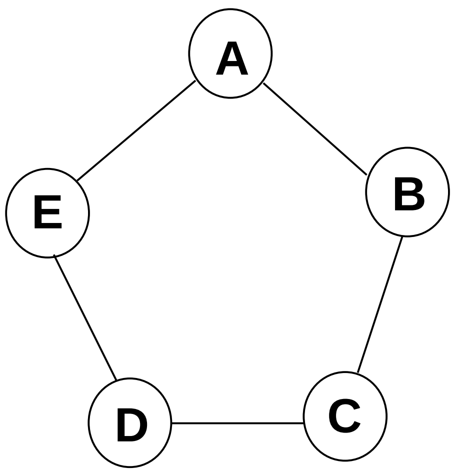


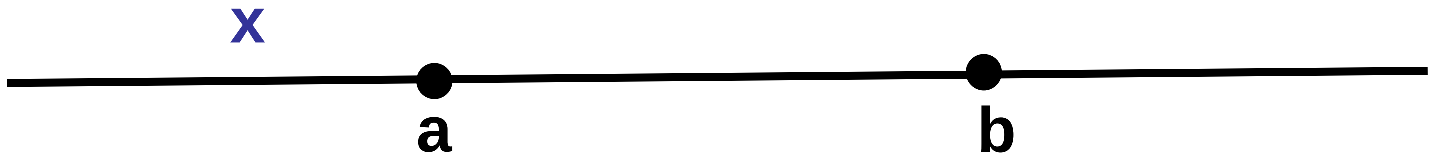
B lies between A and C,

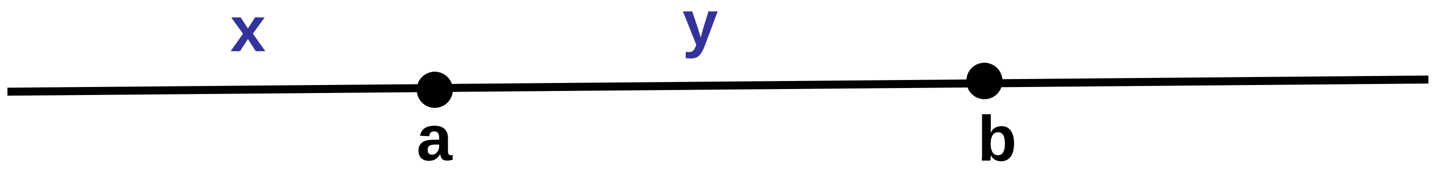
C lies between B and D,

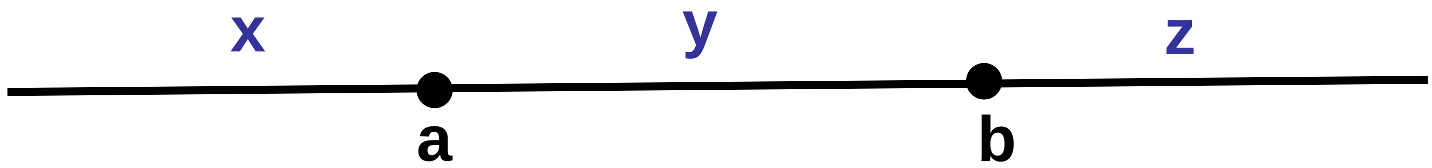
etc.

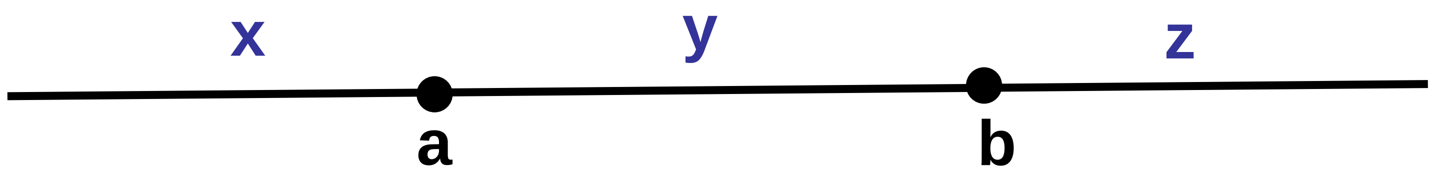

a
b


a
b


x
a
b

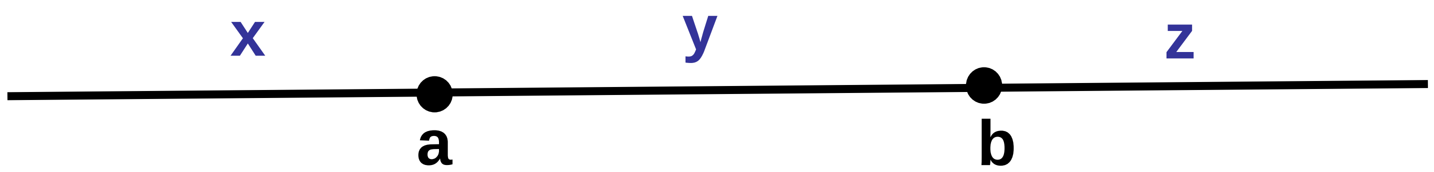

x
y
a
b


x
y
z
a
b



Definition. The *line ab* consists of points a, b and

- all points x such that a lies between x and b,
- all points y such that y lies between a and b,
- all points z such that b lies between a and z.



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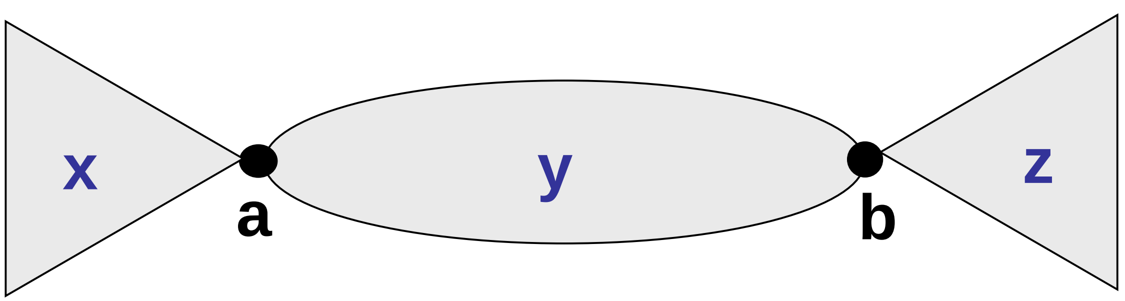
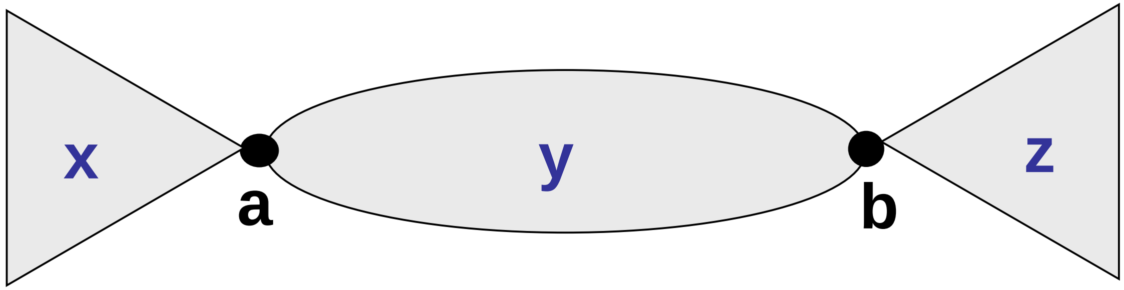
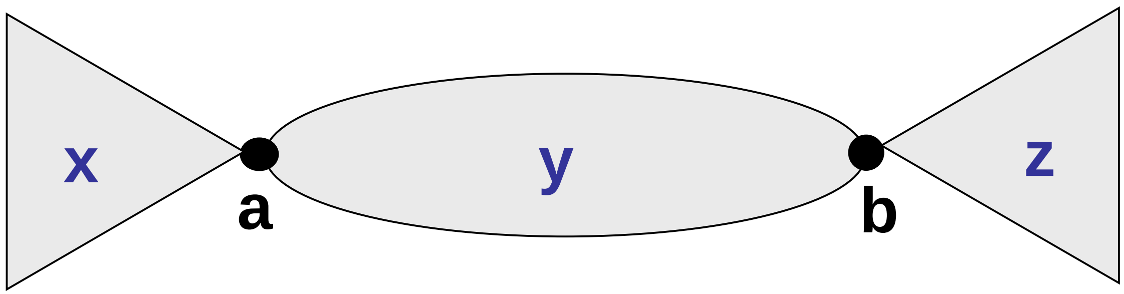
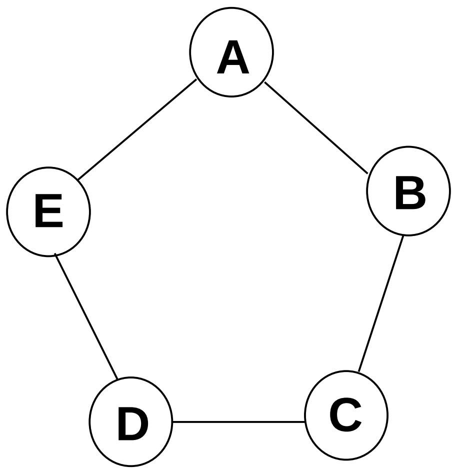
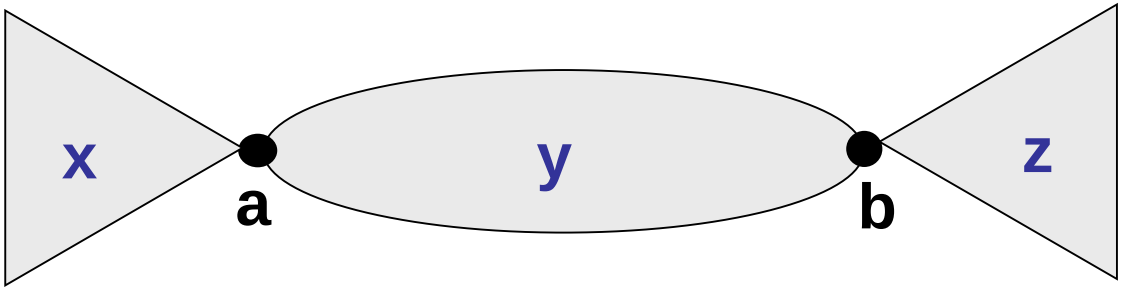
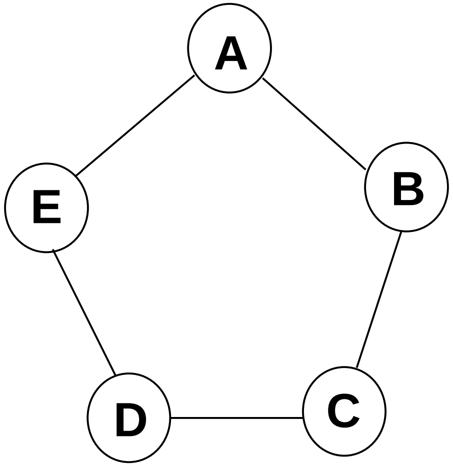



x
y
z
a
b

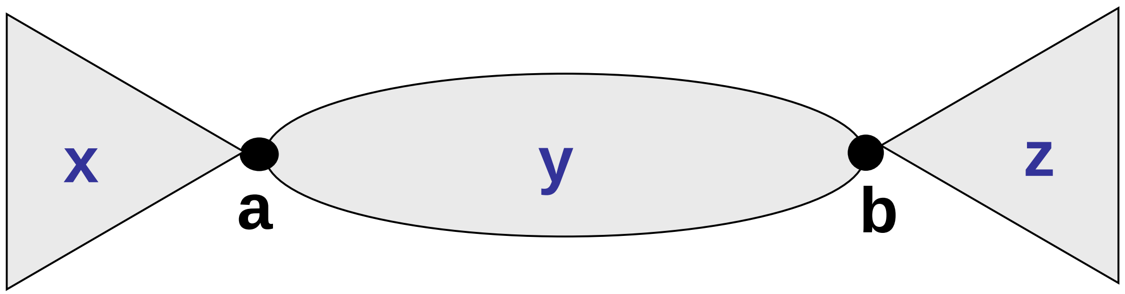

x
a
y
b
z

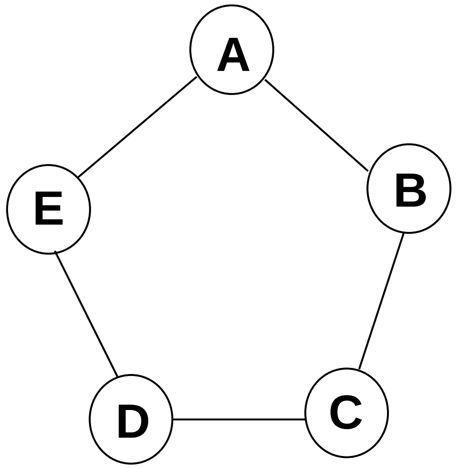

A
B
C
D
E





line AB consists of E,A,B,C

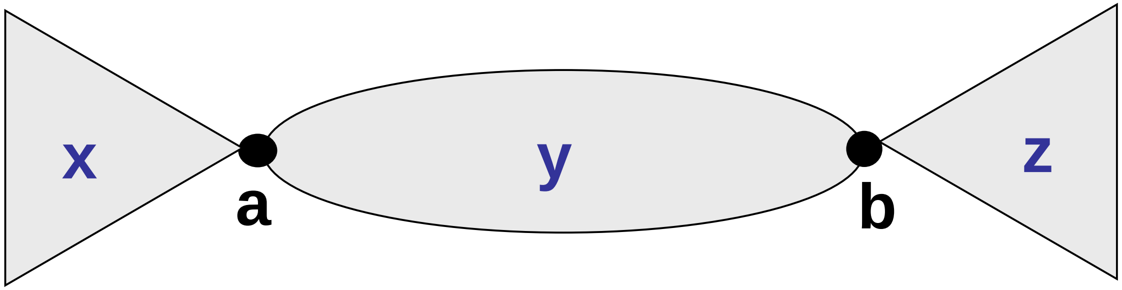


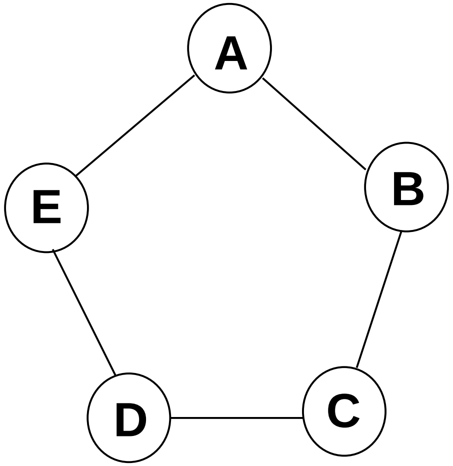


line AB consists of E,A,B,C

line AC consists of A,B,C


UN TRAIN
PEUT EN CACHER
UN AUTRE





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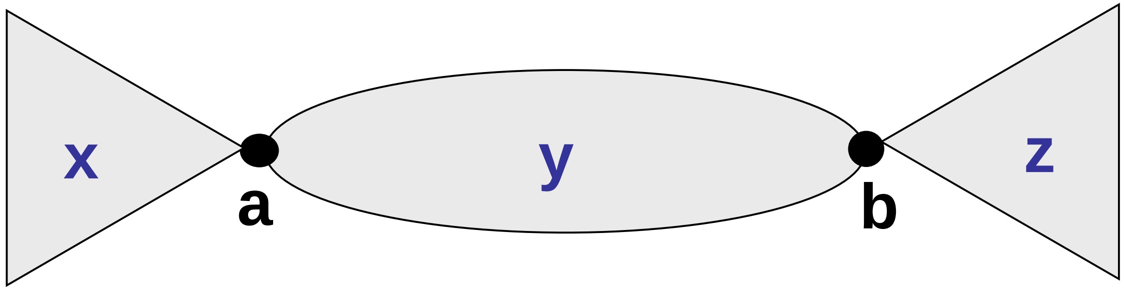
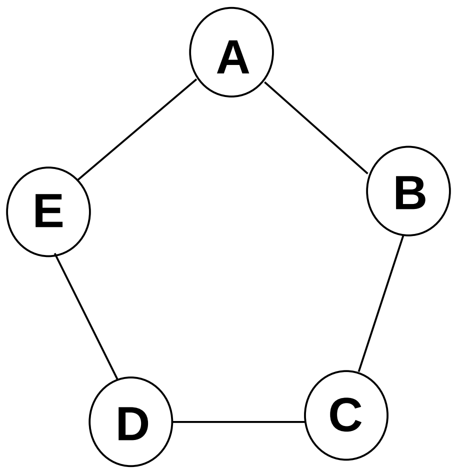
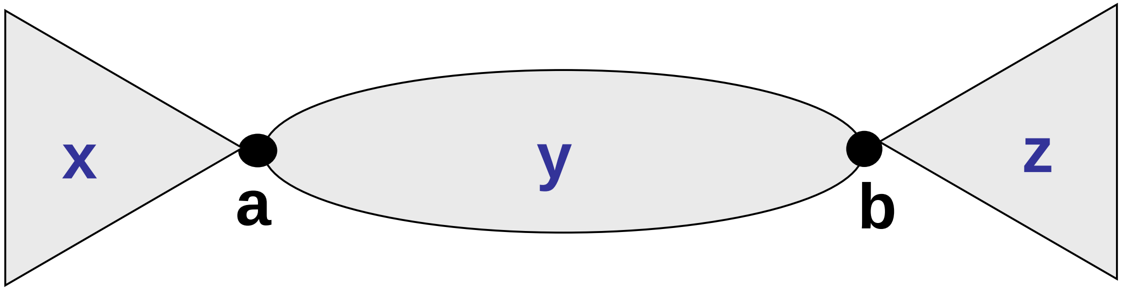


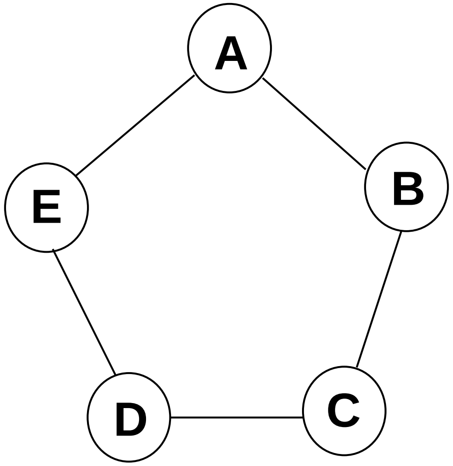


line AB consists of E,A,B,C

line AC consists of A,B,C

One line can hide another!





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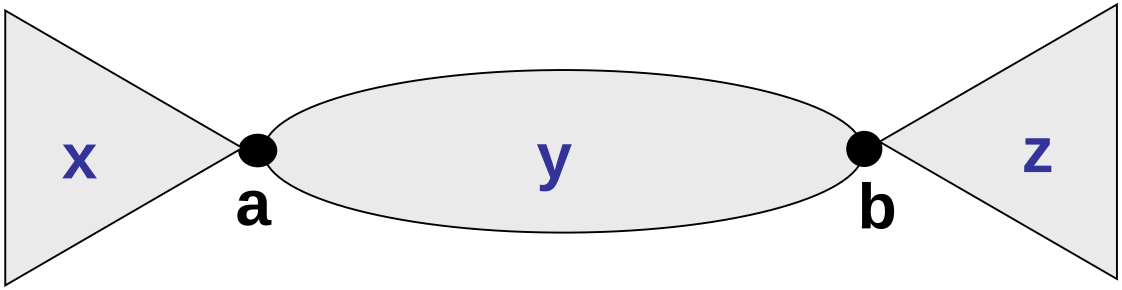
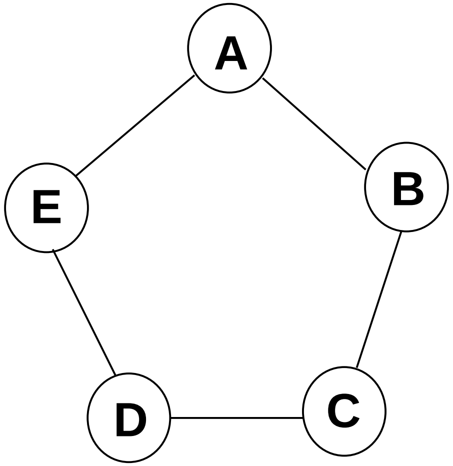
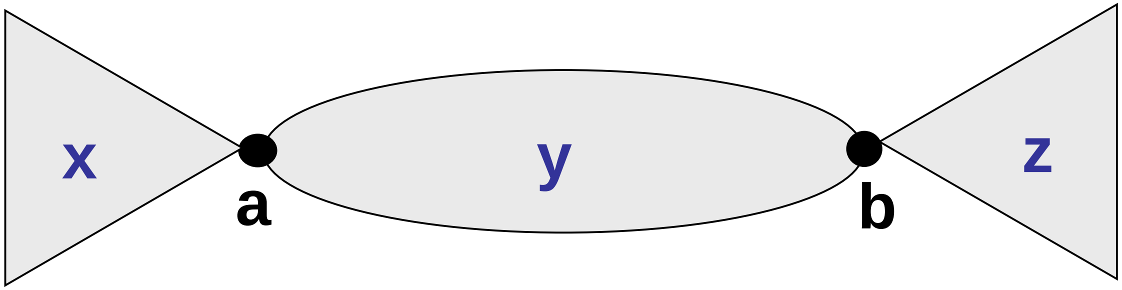


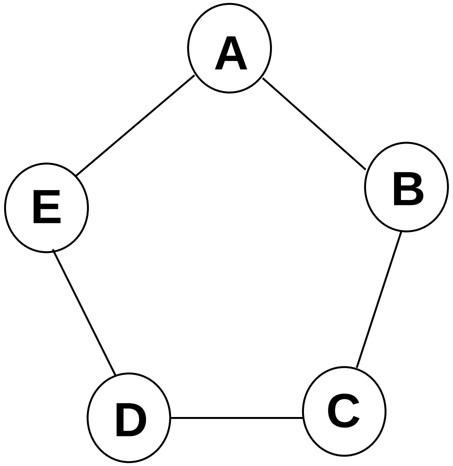


line AB consists of E,A,B,C

line AC consists of A,B,C

no line consists of all points





line AB consists of E,A,B,C

line AC consists of A,B,C

no line consists of all points

no line consists of two points

Remedy: An alternative definition of a line

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Definition: A *pin* is a set of three points such that one of these three points lies between the other two.

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Observation: Intersection of affine sets is affine.

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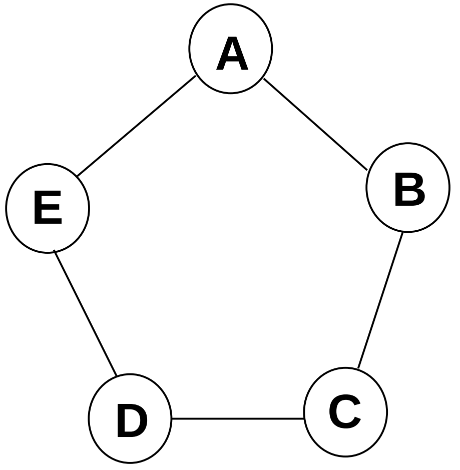
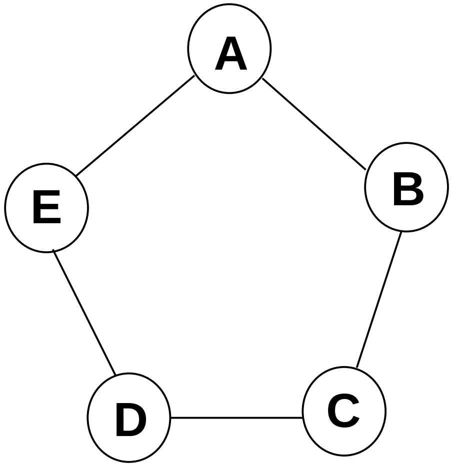
Definition: The ***closure line*** AB is the affine hull of the set consisting of the two points A and B.


A
B
C
D
E



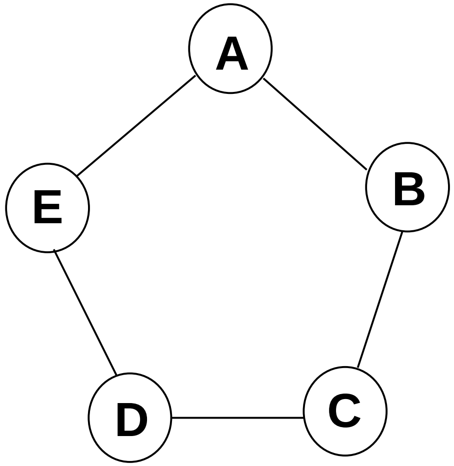
Pins: {A,B,C},

{B,C,D},

{C,D,E},

{D,E,A},

{E,A,B}.



Pins: {A,B,C},

{B,C,D},

{C,D,E},

{D,E,A},

{E,A,B}.

Every closure line here

consists of all five points A,B,C,D,E

Conjecture (V.C. 1998)

In every finite metric space,

some closure line consists of two points or else

some closure line consists of all the points.

~~Conjecture (V.C. 1998)~~

Theorem (Xiaomin Chen 2003)

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Ordered geometry

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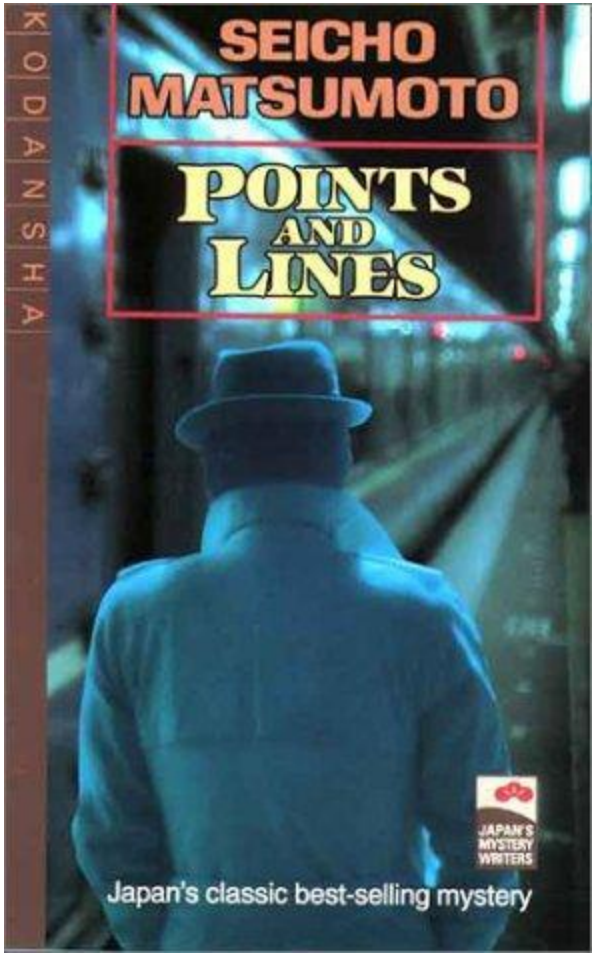

Corollary of Sylvester-Gallai

(generalized by de Bruijn and Erdős in 1948)

Every set of n points in the plane determines at least n distinct lines unless all these n points lie on a single line.

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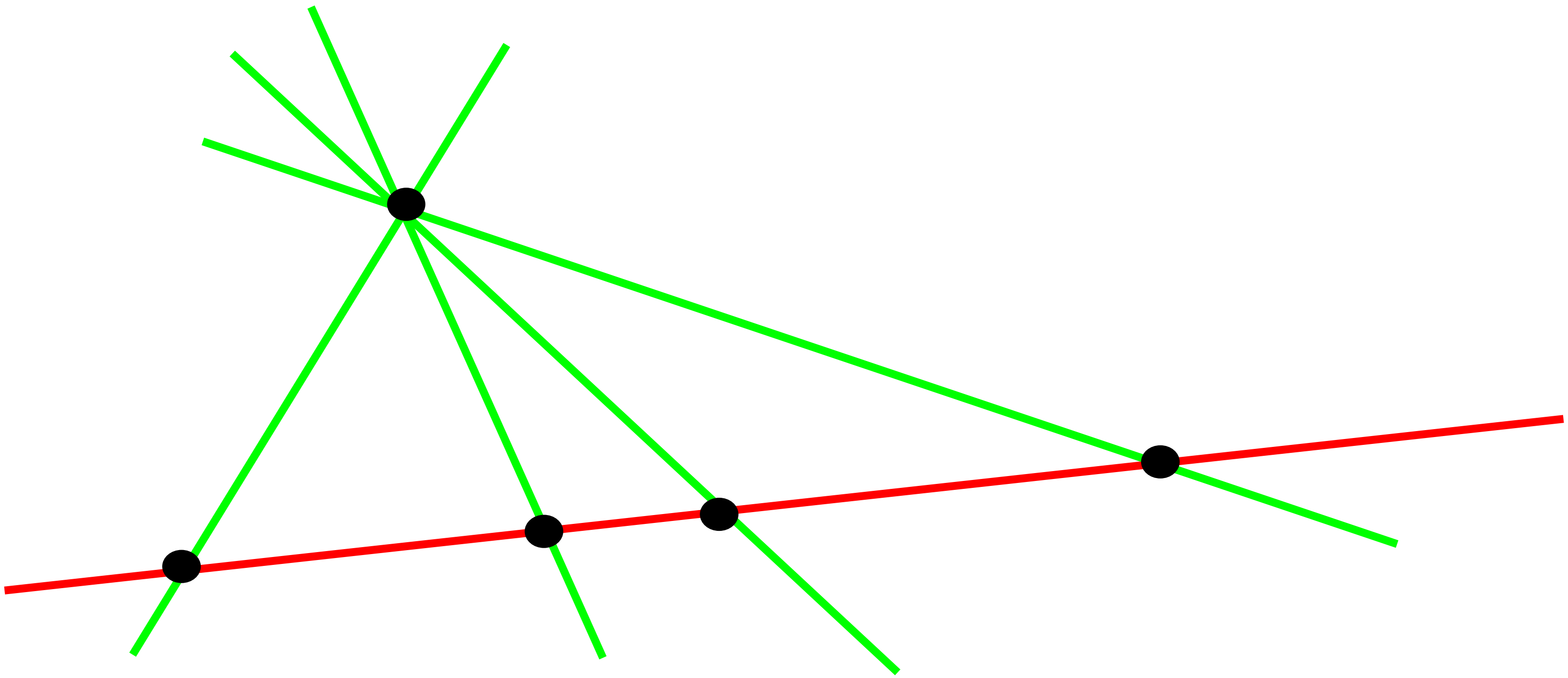
(generalized by de Bruijn and Erdős in 1948)

Every set of n points in the plane determines **at least n** distinct lines unless all these n points lie on a single line.



Conjecture (Chen and C. 2006):

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“Closure lines” in place of “lines” do not work here:

For arbitrarily large *n*, there are metric spaces on *n* points, where there are precisely seven distinct closure lines and none of them consist of all the *n* points.

Manhattan distance

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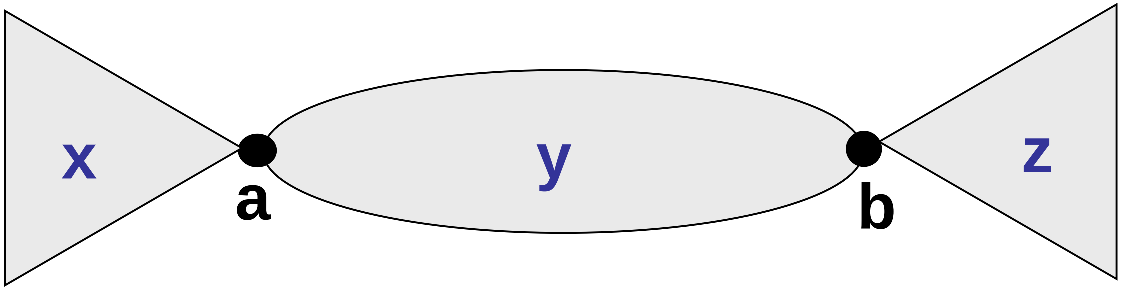

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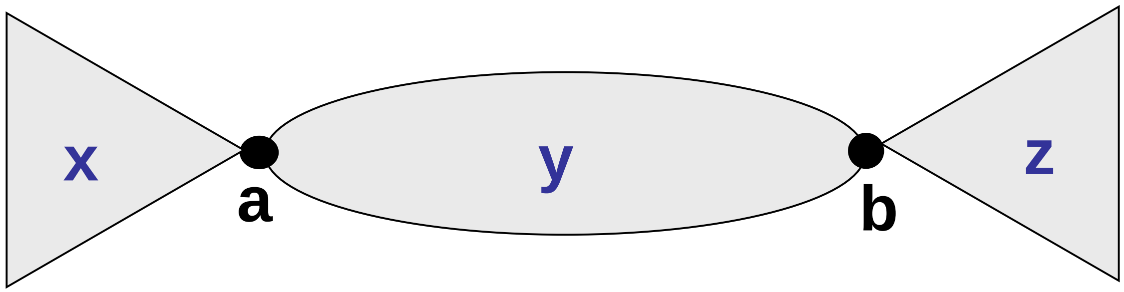

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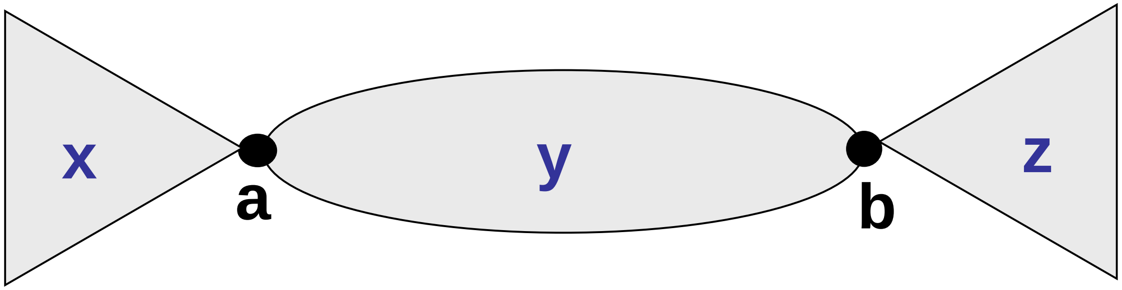

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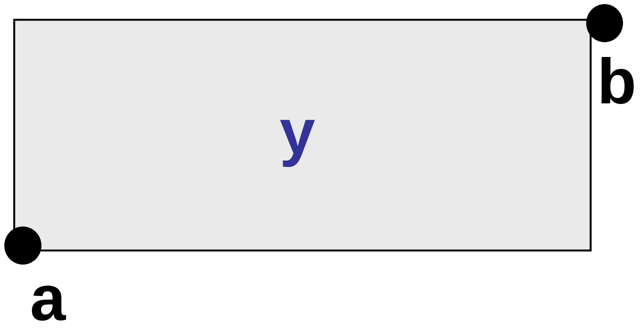

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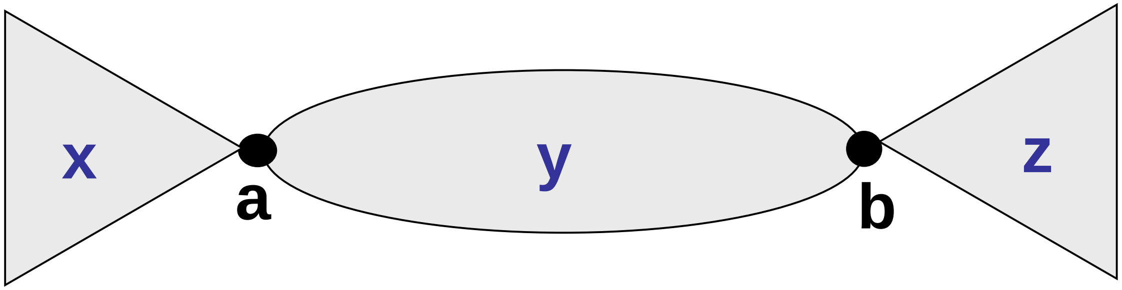

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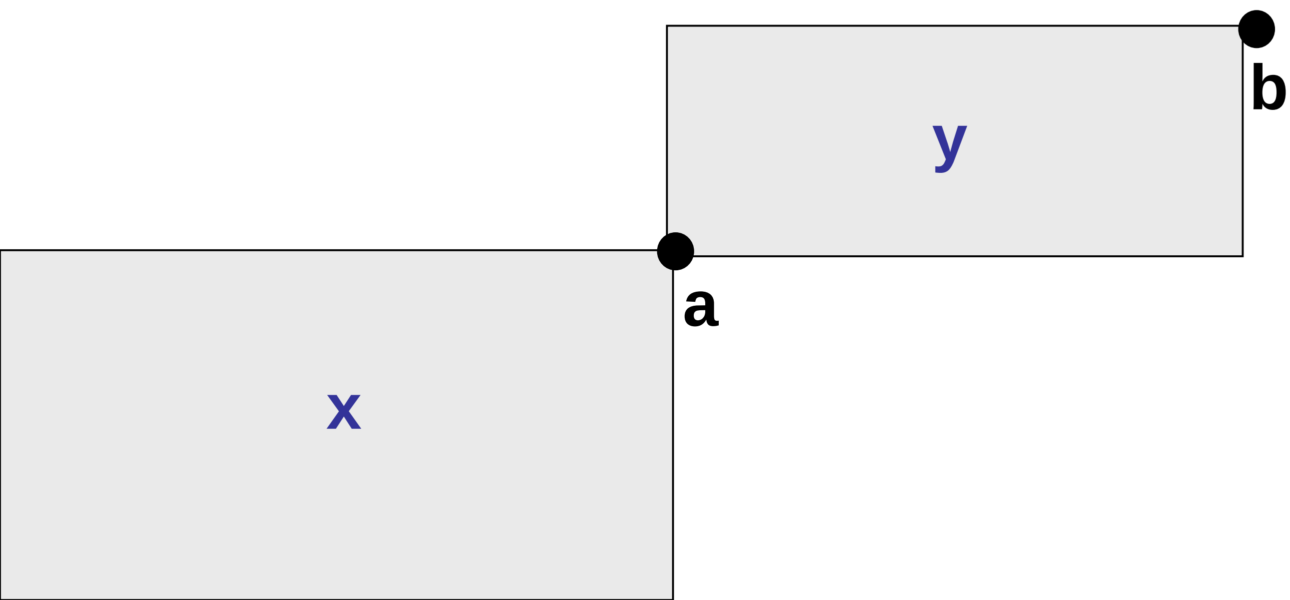

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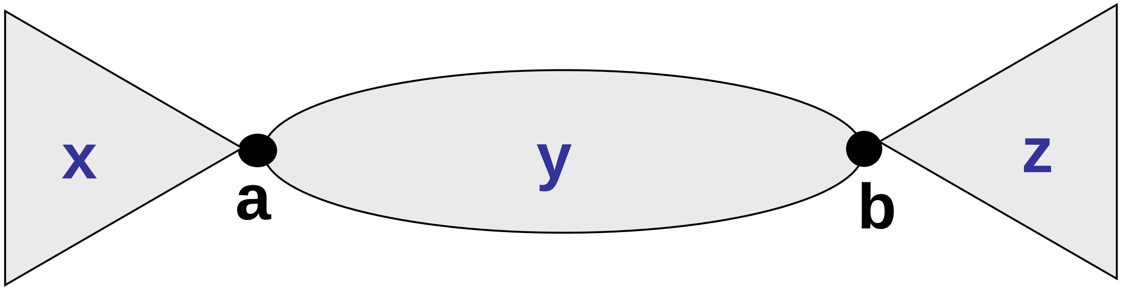

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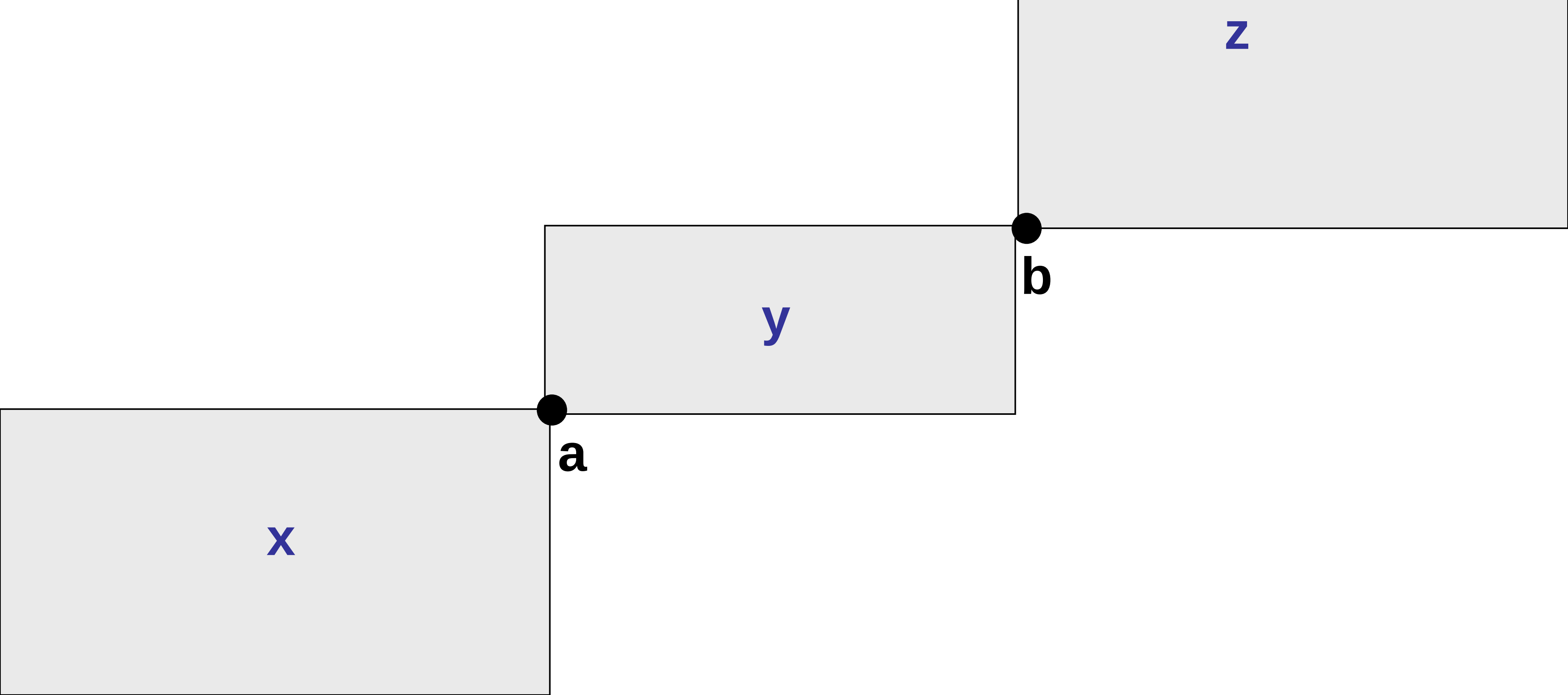

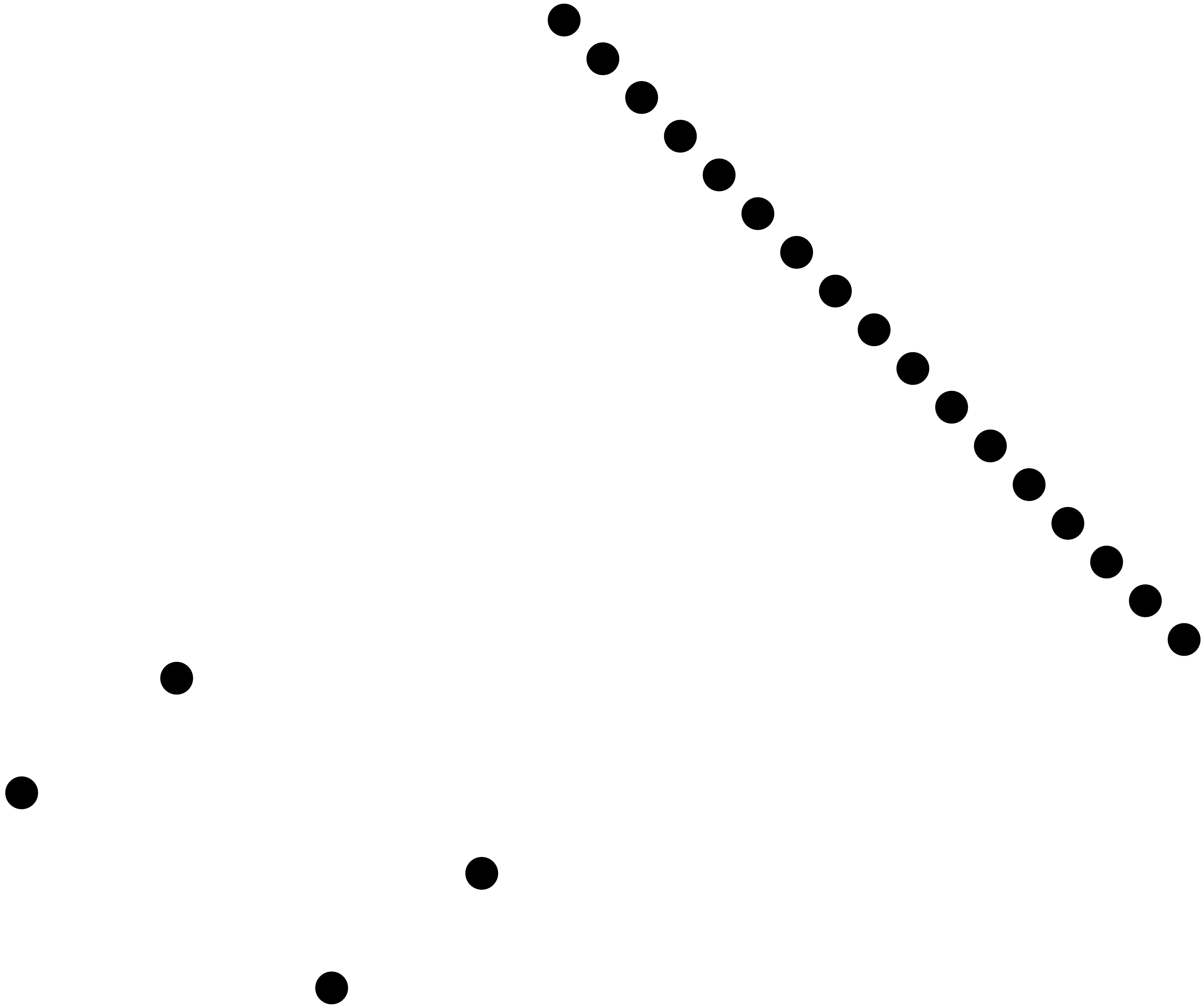

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- if all distances are 0, 1, or 2, then there are $\Omega(n^{4/3})$ distinct lines
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GRAHAM CHAPMAN · JOHN CLEESE · TERRY GILLIAM · ERIC IDLE · TERRY JONES · MICHAEL PALIN
MONTY PYTHON'S
AND NOW FOR SOMETHING COMPLETELY DIFFERENT
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12

Abstract convexity

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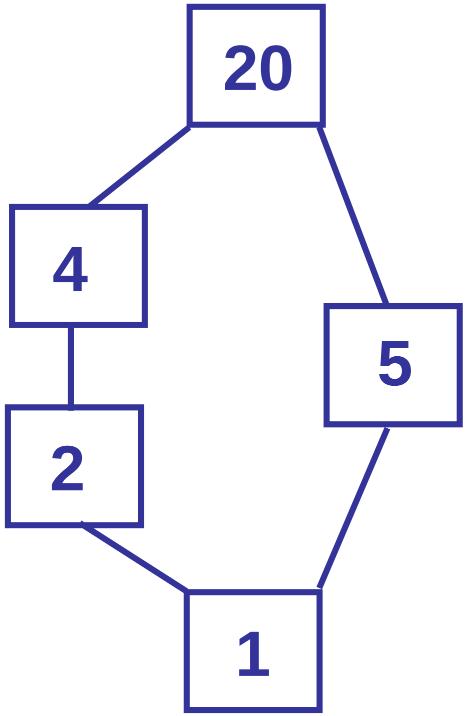

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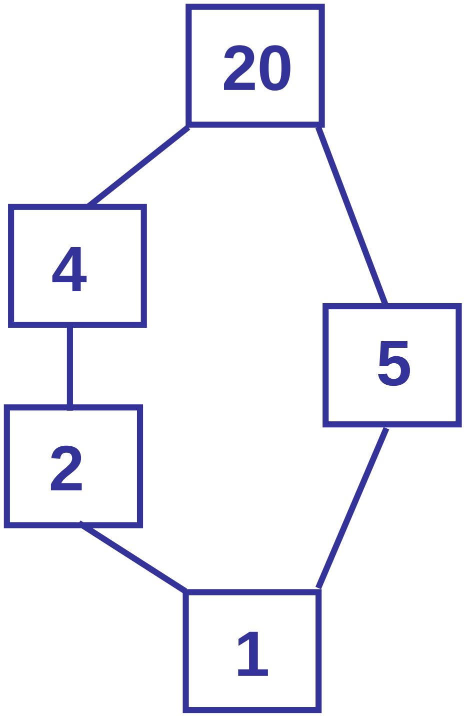

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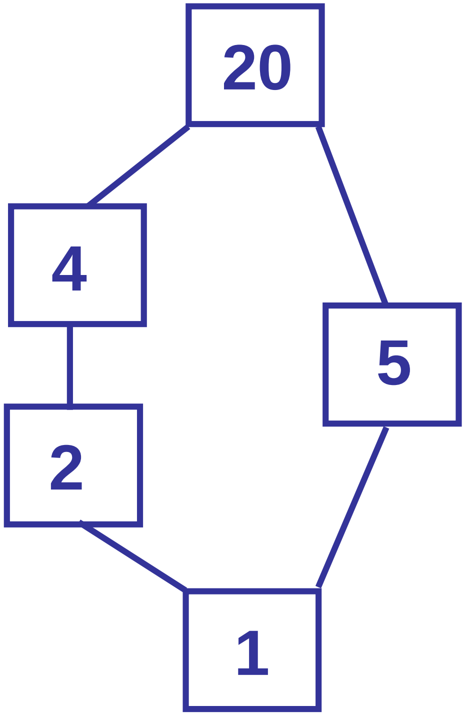

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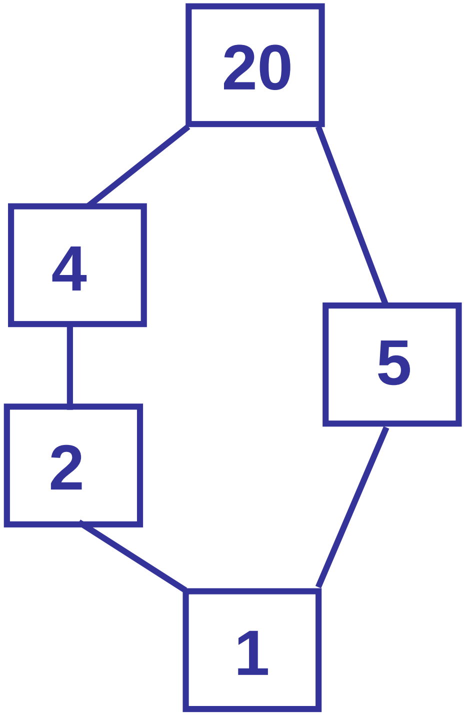

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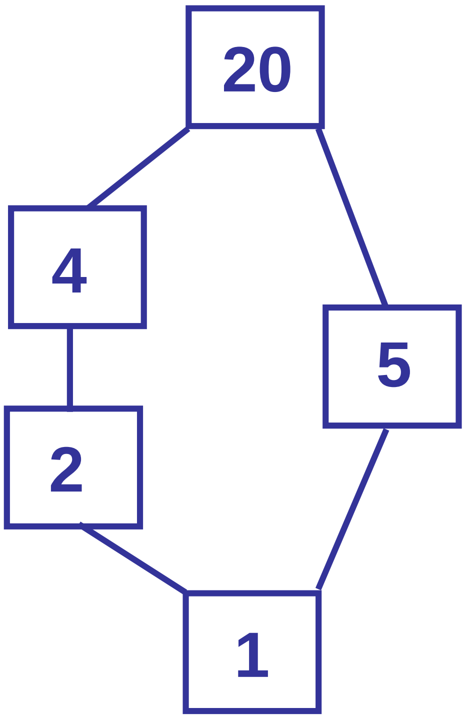

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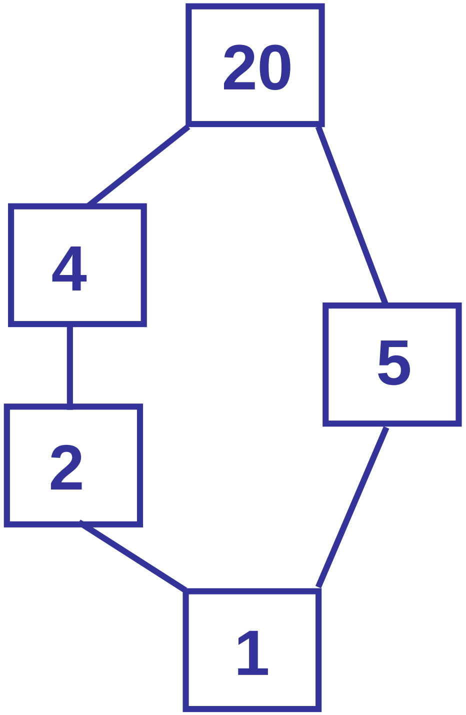

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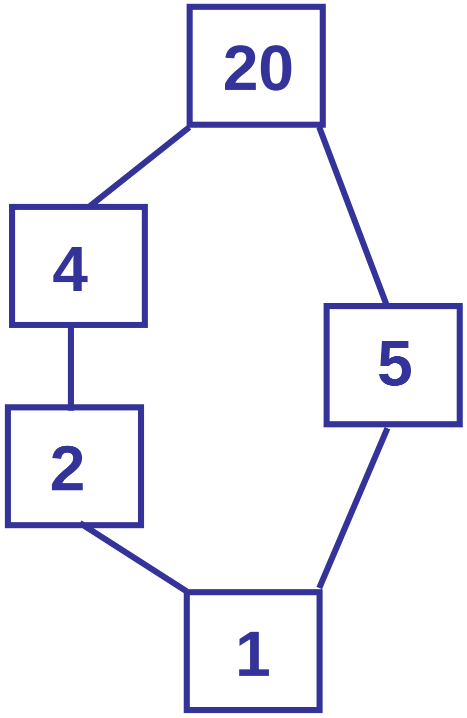

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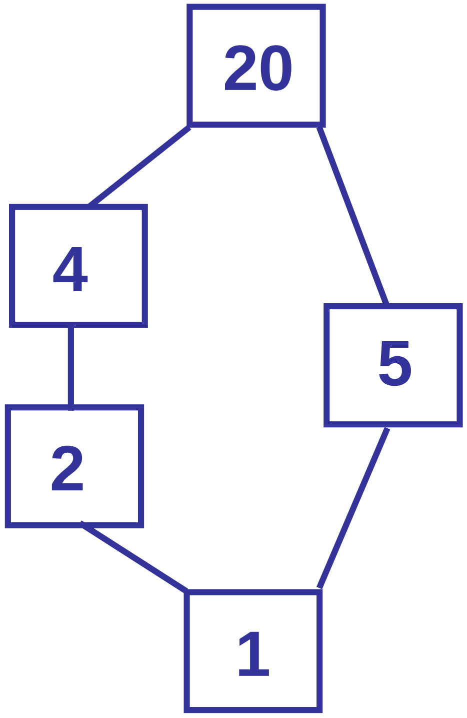

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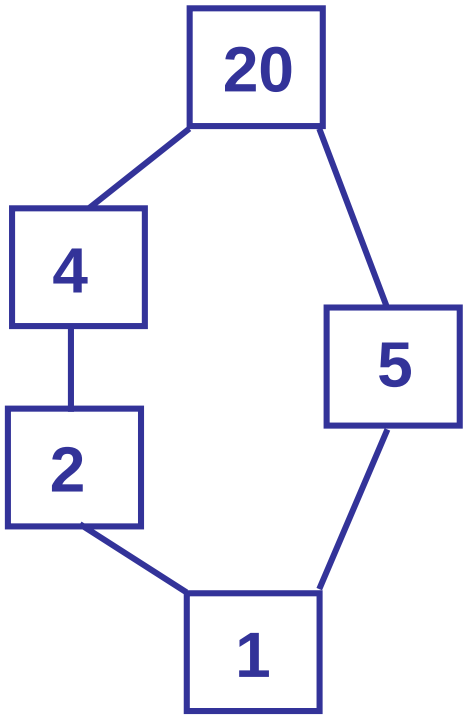

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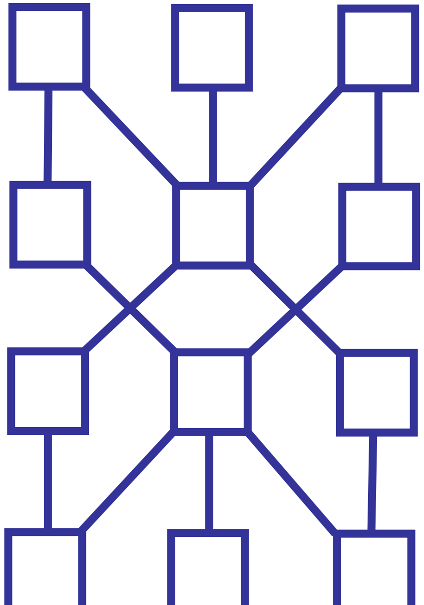

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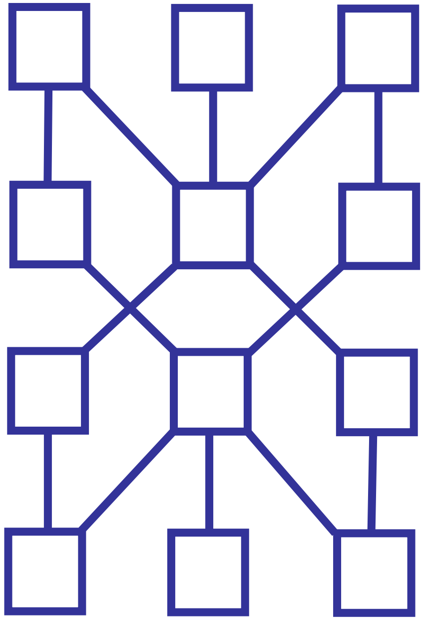

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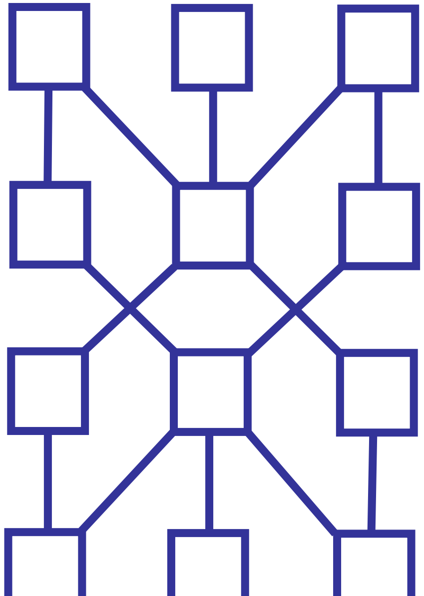

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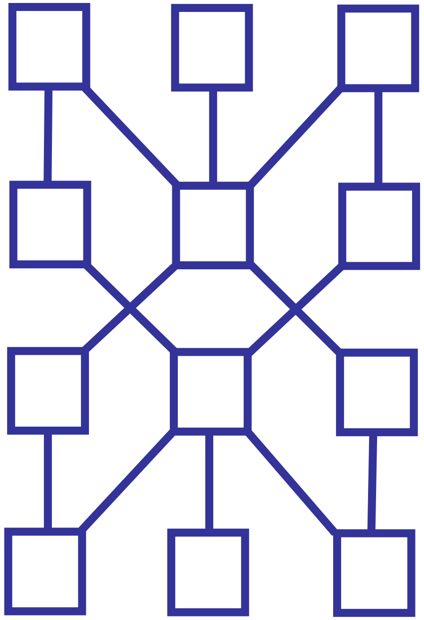

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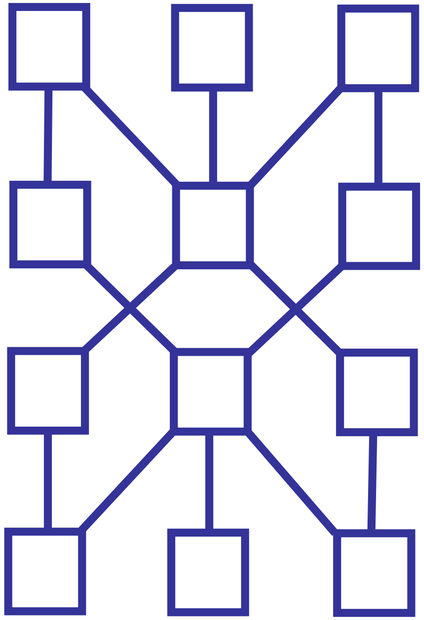

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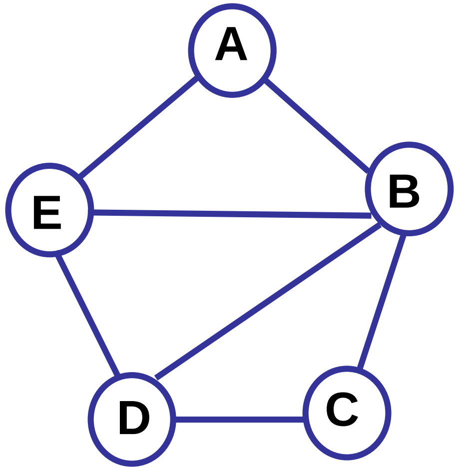

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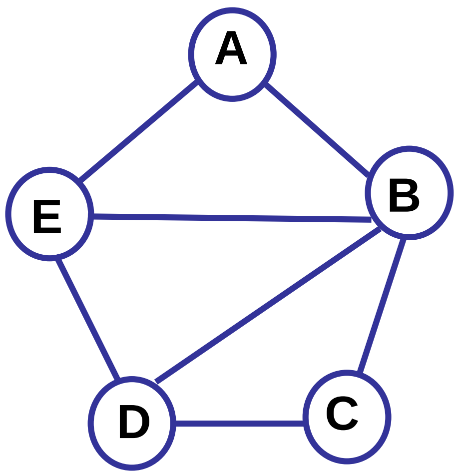

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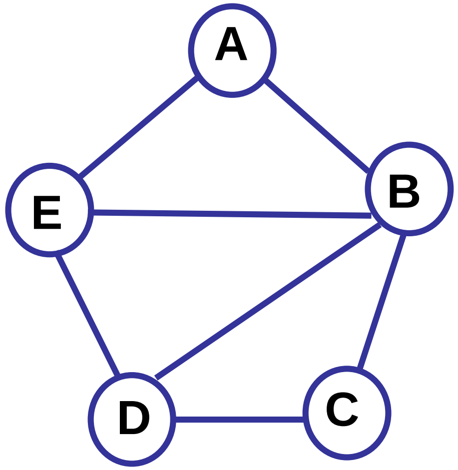

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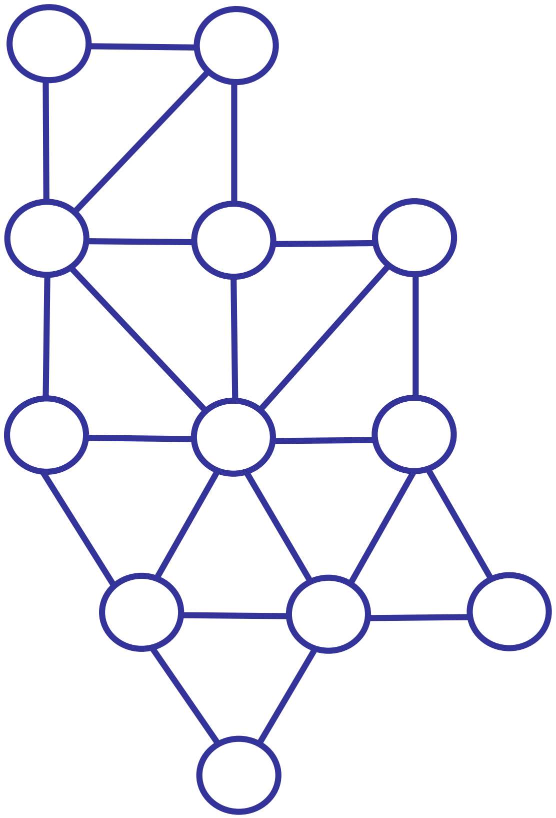

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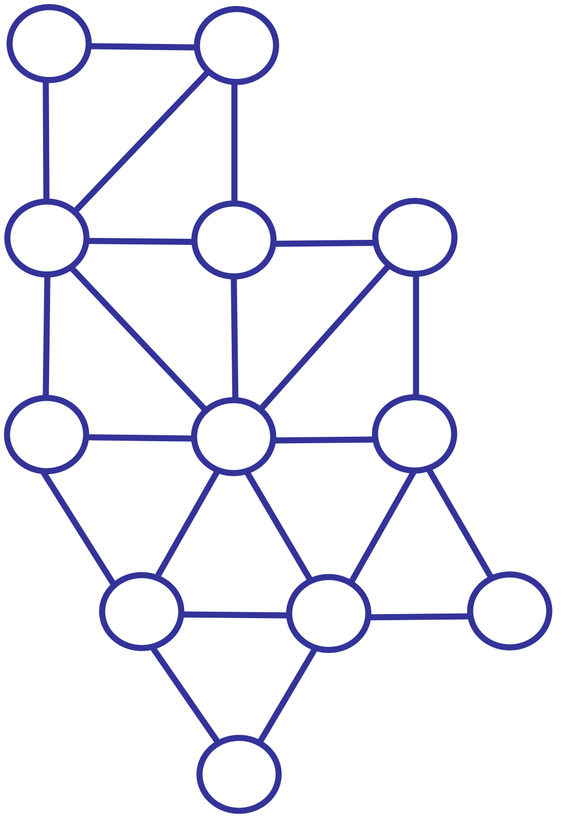

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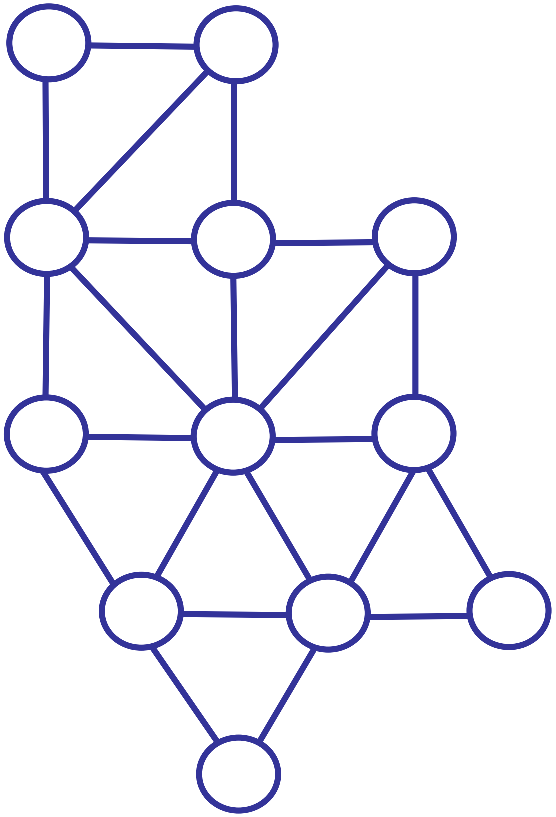

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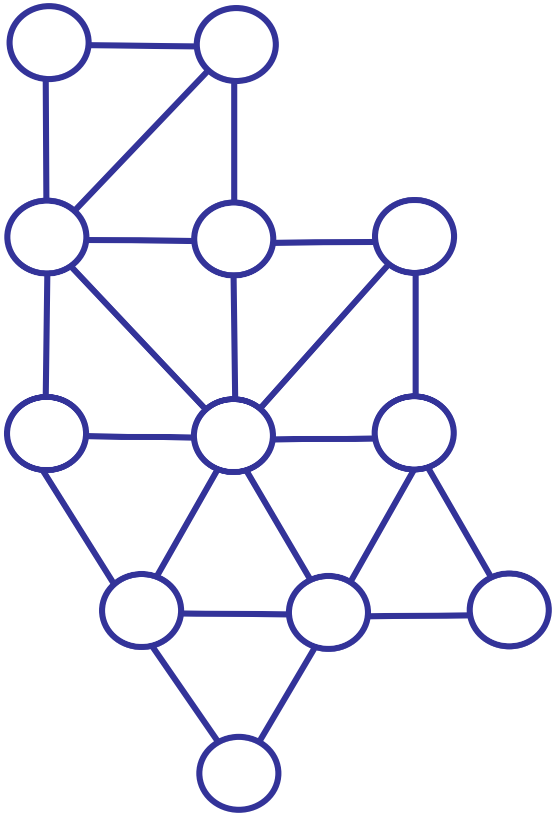

Observation: A point in a convex set C is not extreme if and only if it has two nonadjacent neighbours in C

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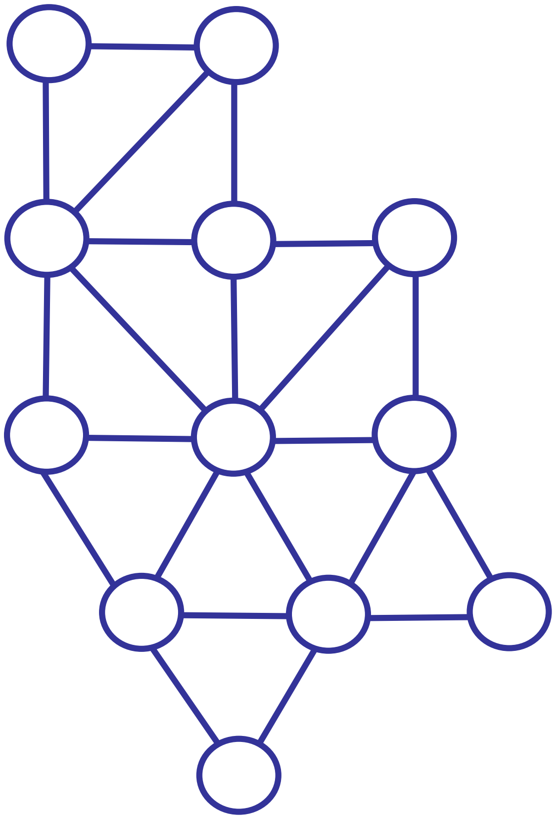

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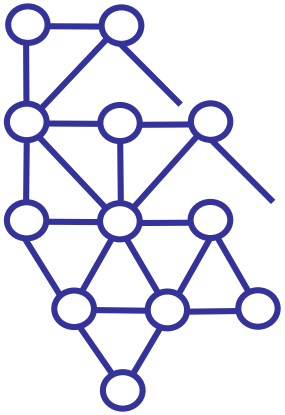

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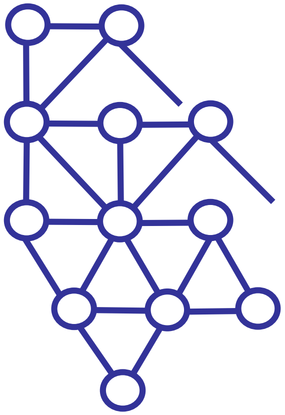

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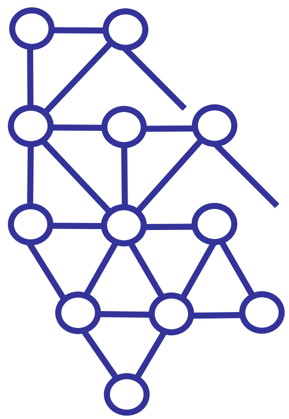
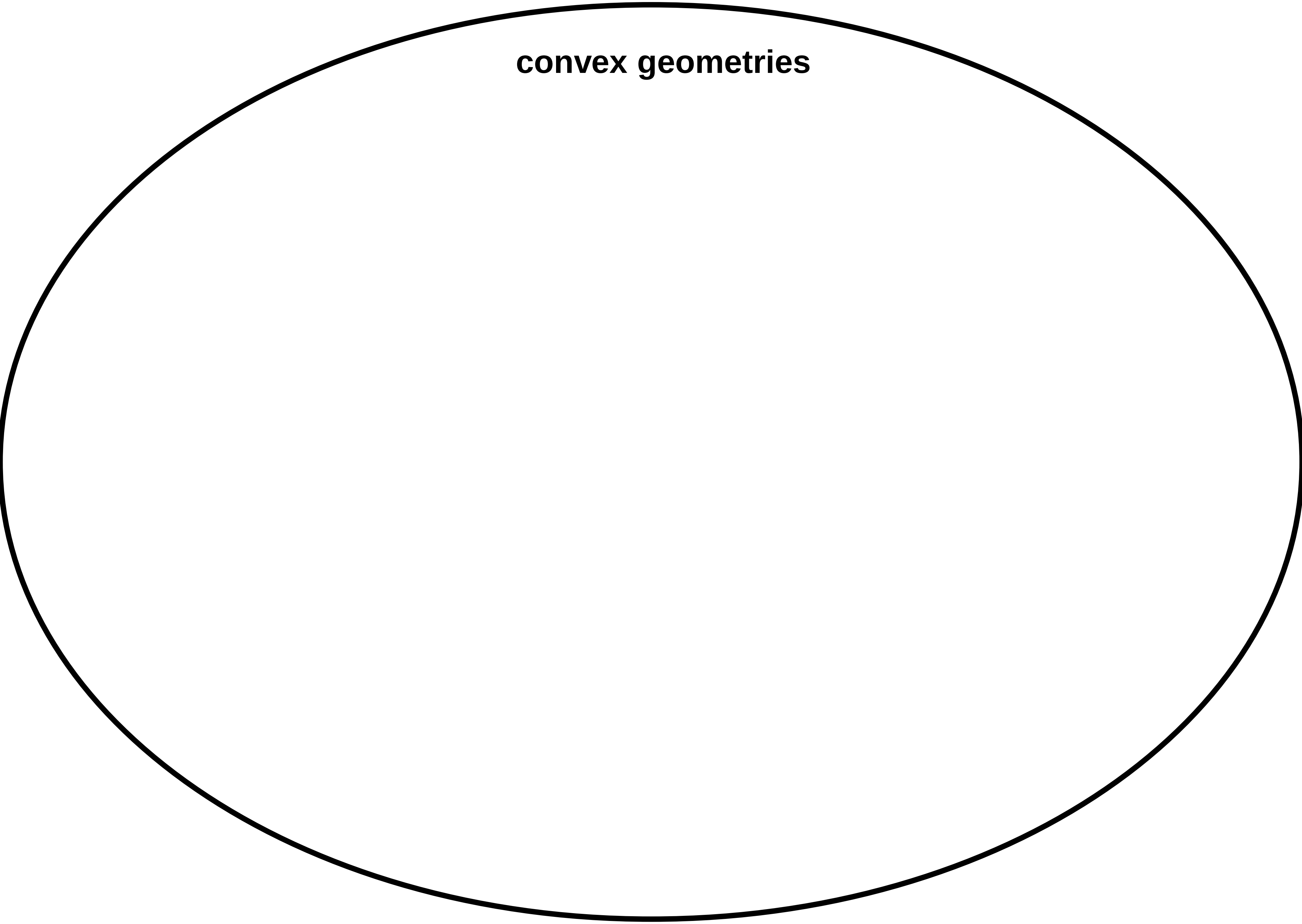


Theorem (Farber & Jamison 1986):

Monophonic convexity in triangulated graphs

is a convex geometry.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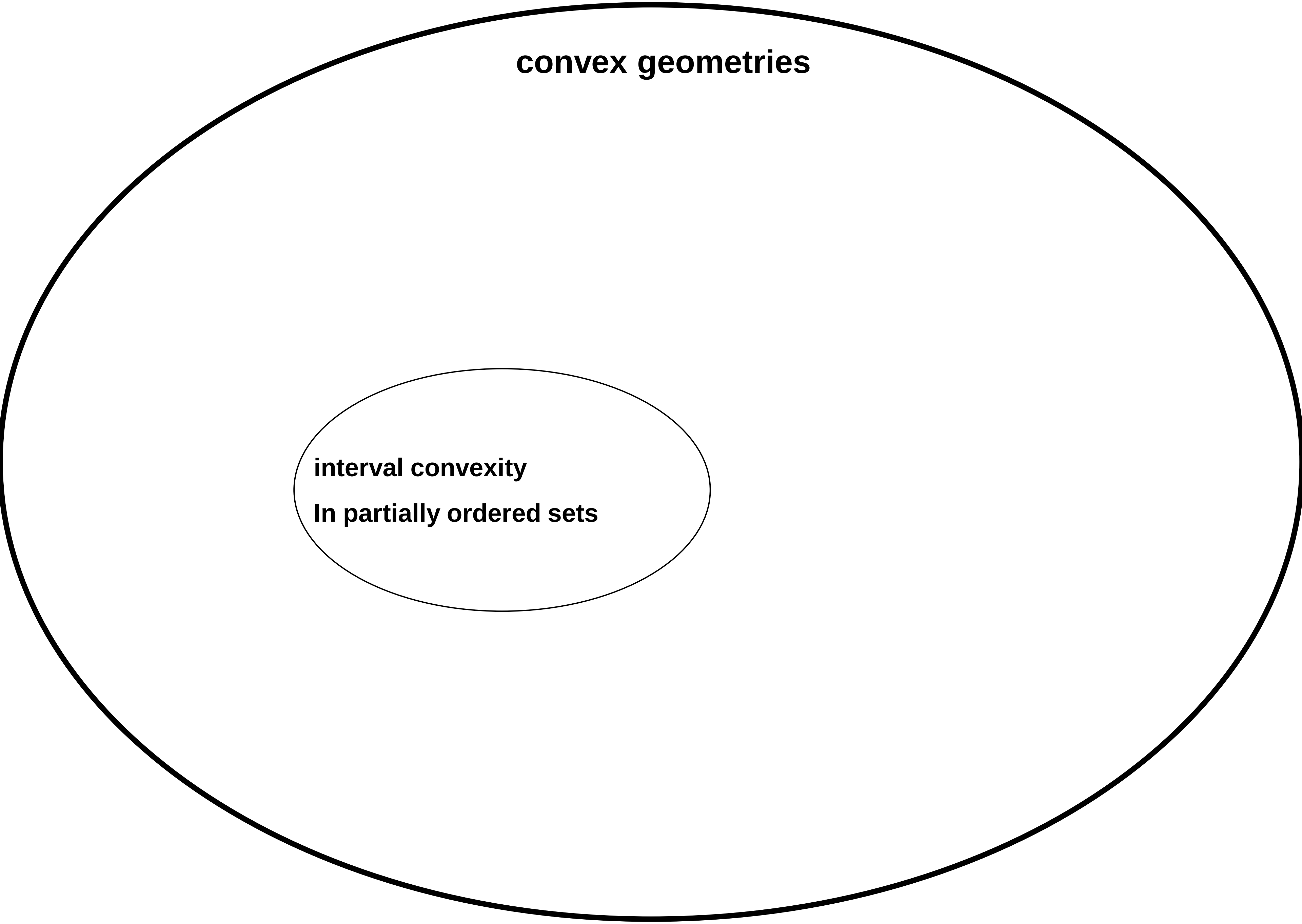
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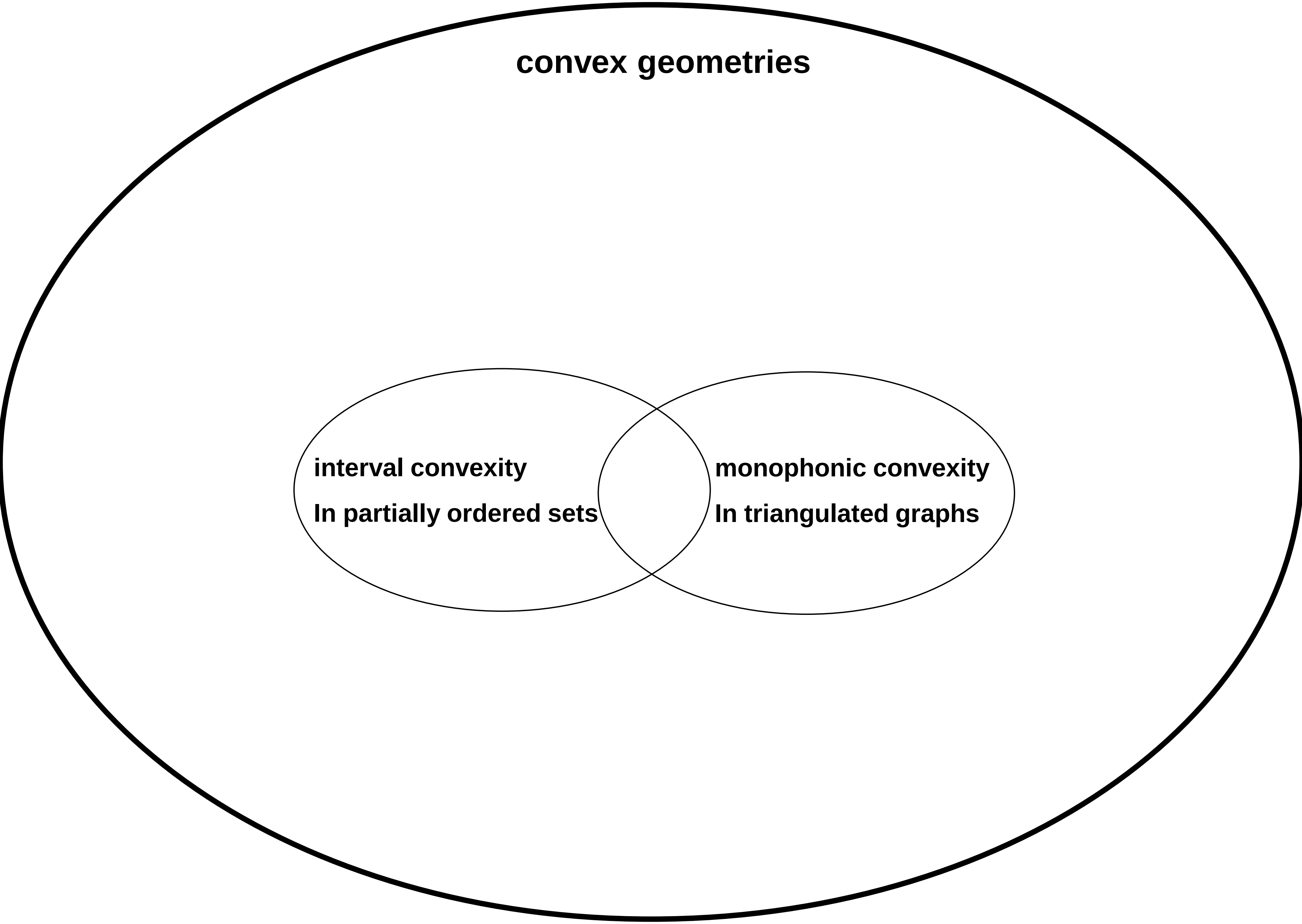
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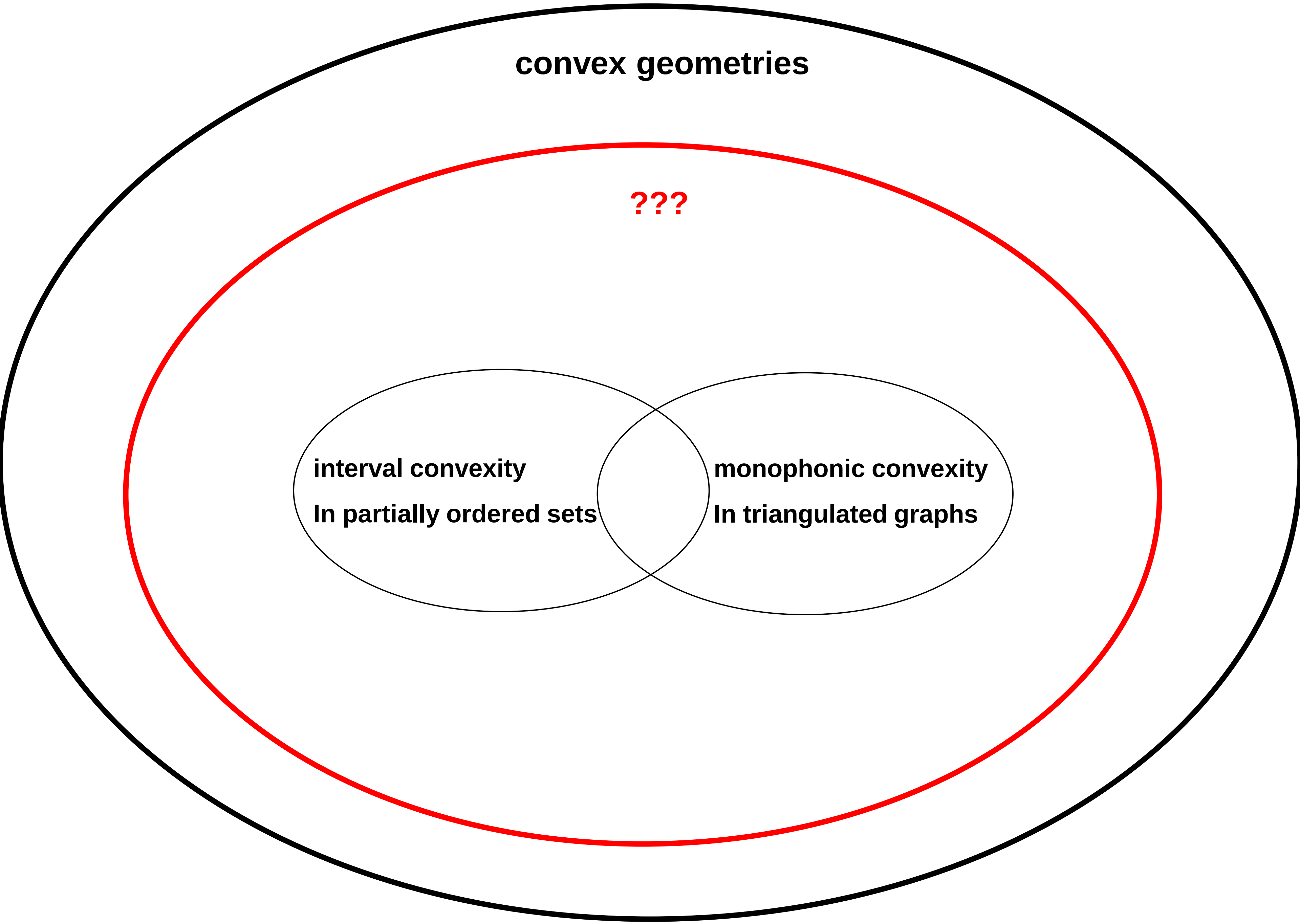
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What do we mean by *betweenness*?

Any ternary relation $\mathscr{B}$ such that

$$
\begin{aligned}
(A,B,C)\in\mathcal{B} &\Rightarrow A,B,C \text{ are all distinct},\\
(A,B,C)\in\mathcal{B} &\Rightarrow (C,B,A)\in\mathcal{B},\\
(A,B,C)\in\mathcal{B} &\Rightarrow (C,A,B)\notin\mathcal{B}.
\end{aligned}
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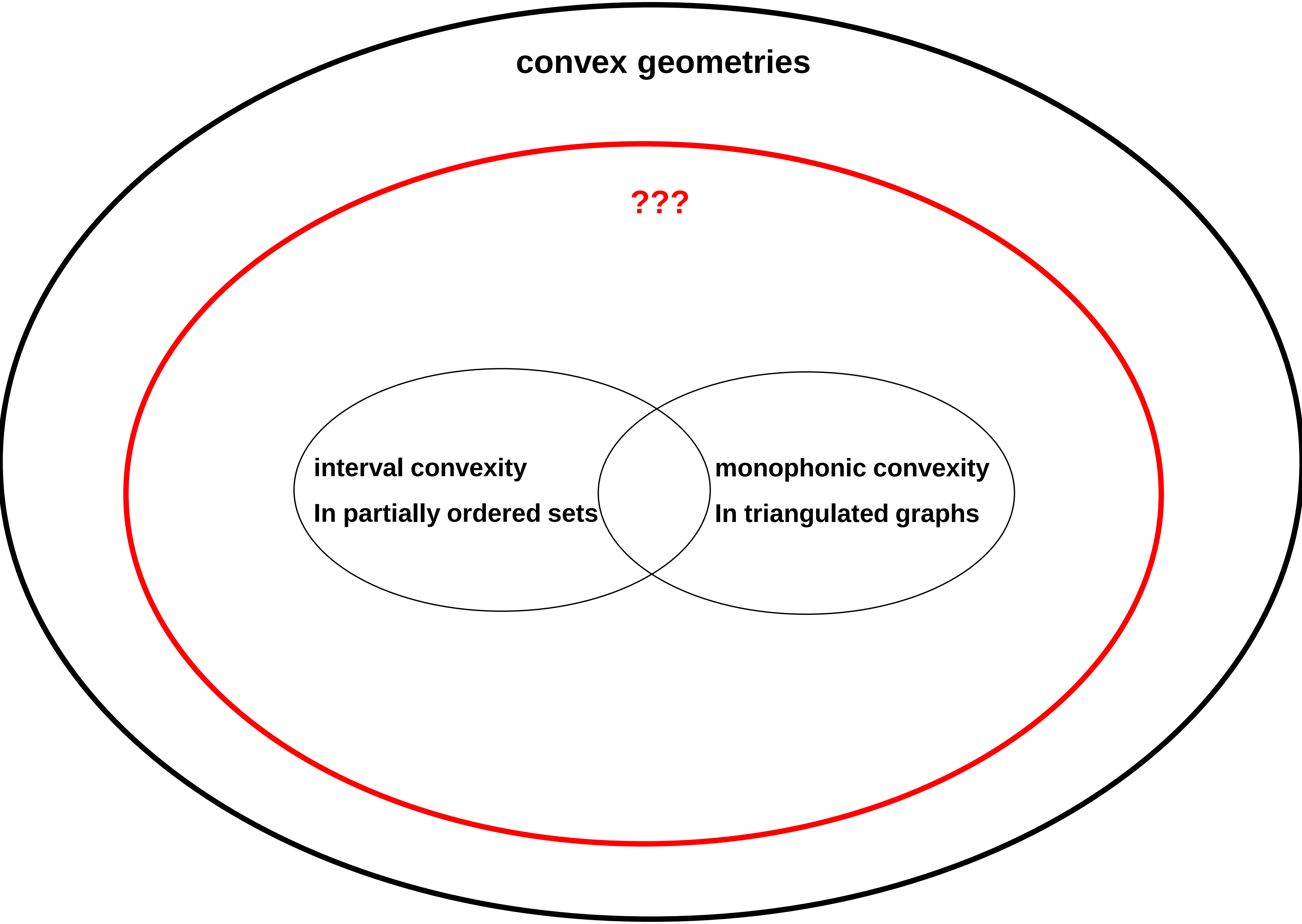
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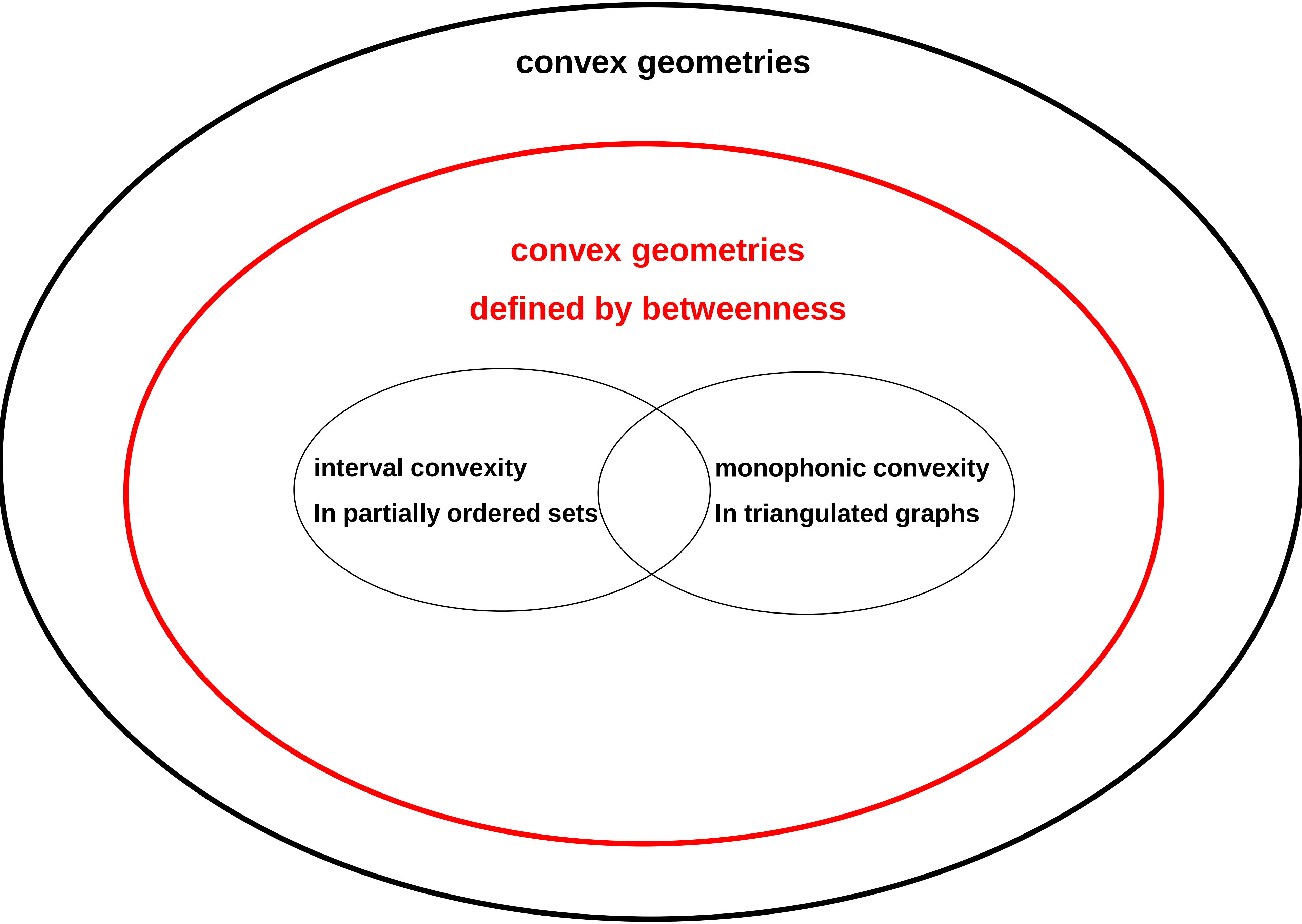
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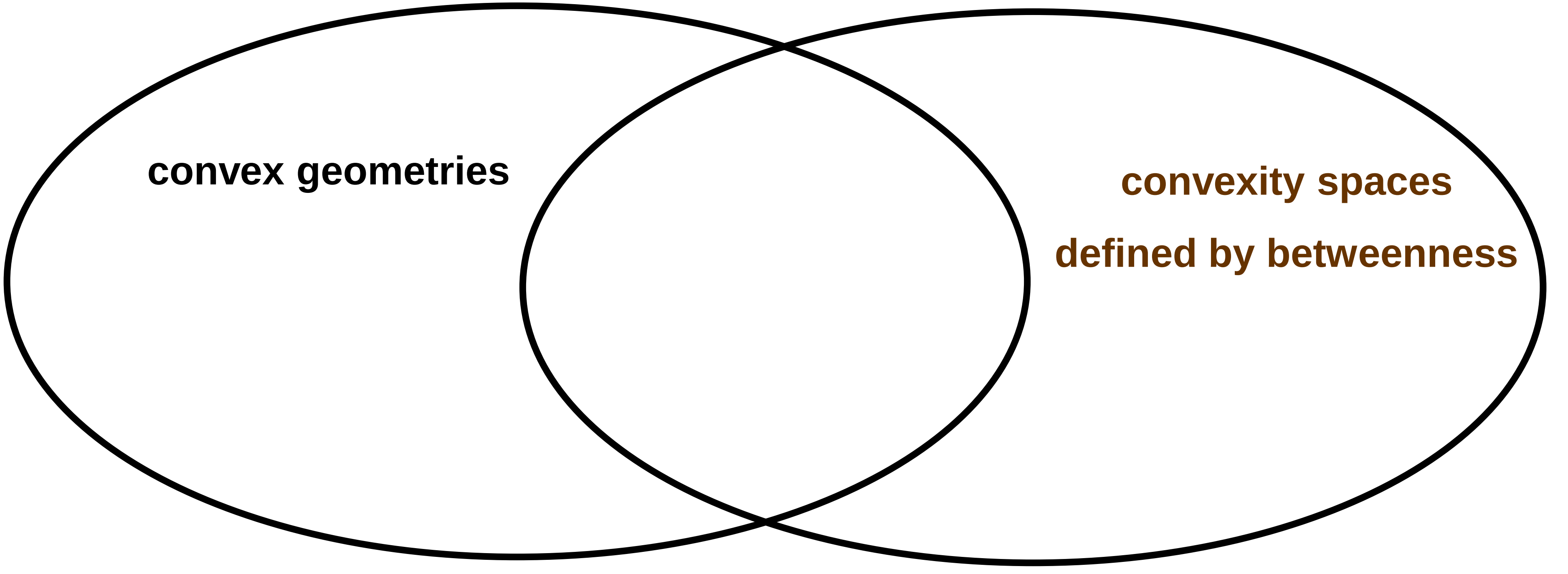
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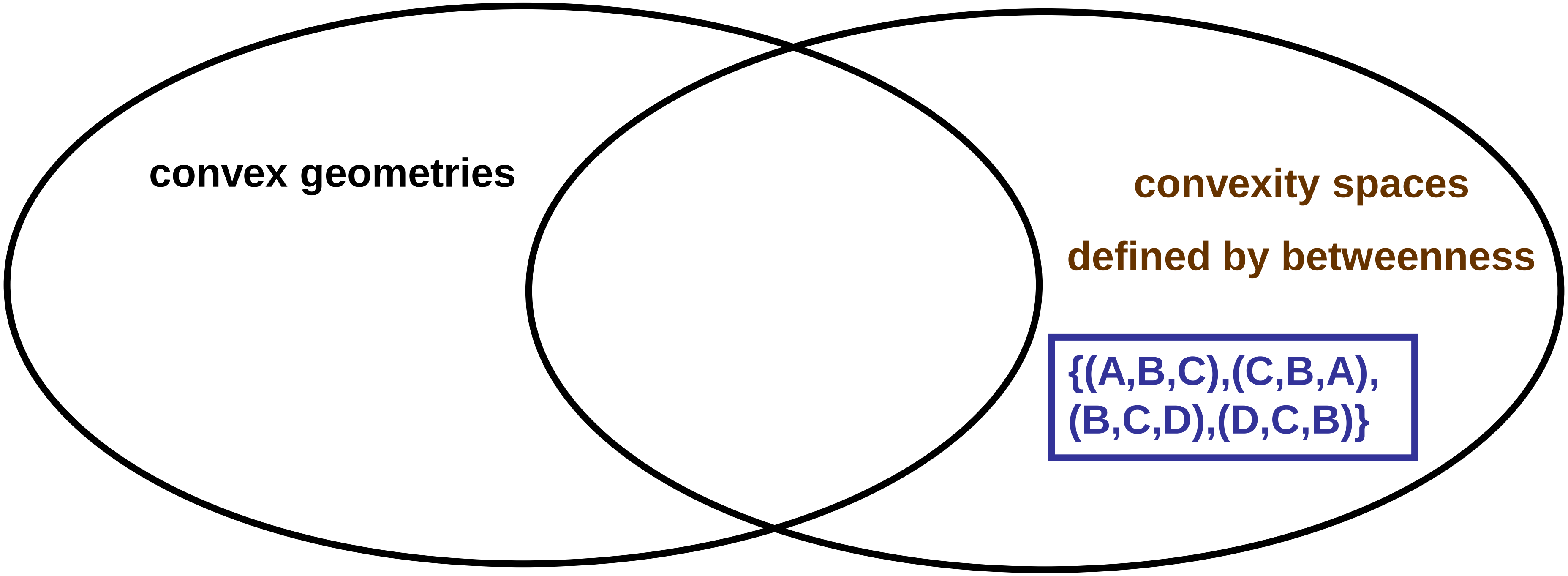
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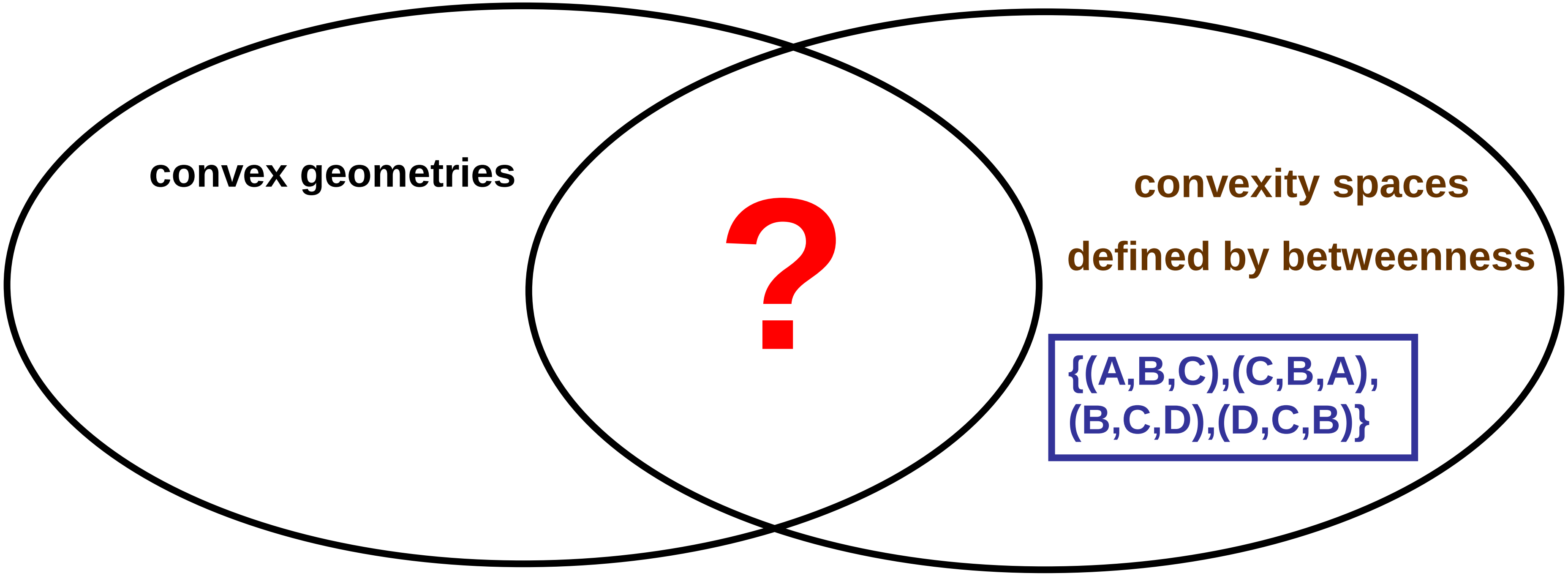
defined by betweenness
{(A,B,C),(C,B,A),
(B,C,D),(D,C,B)}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?
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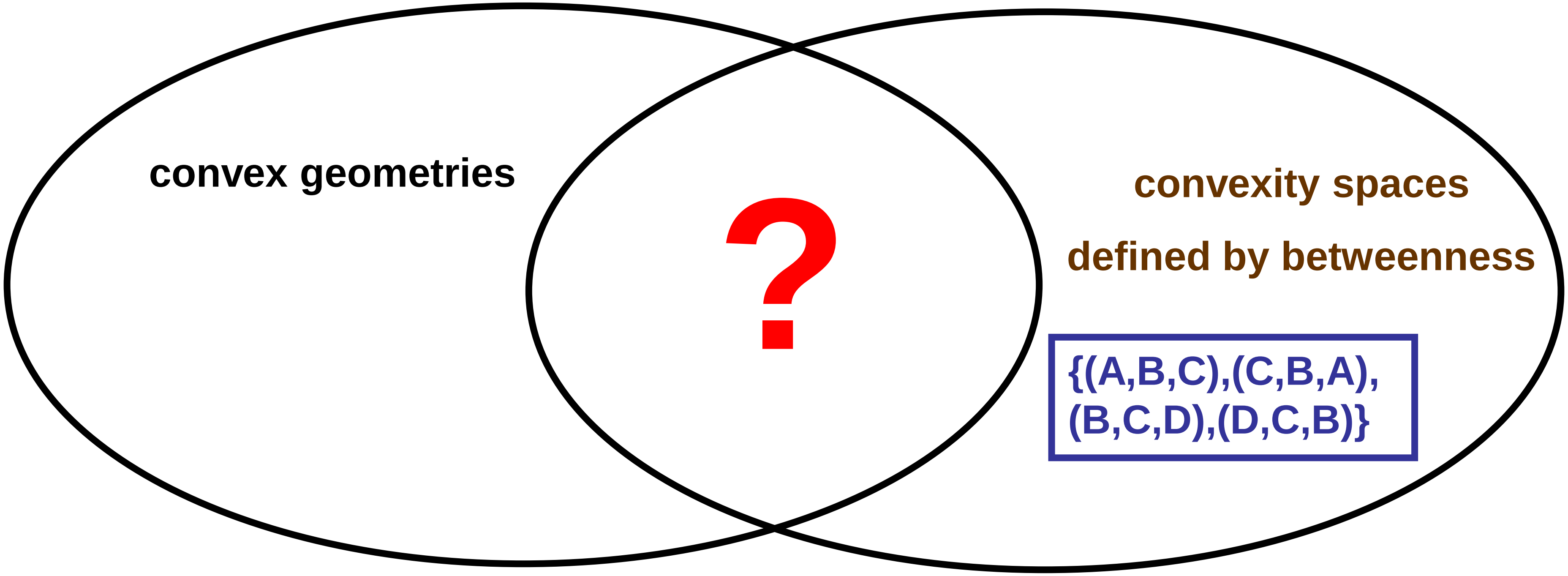
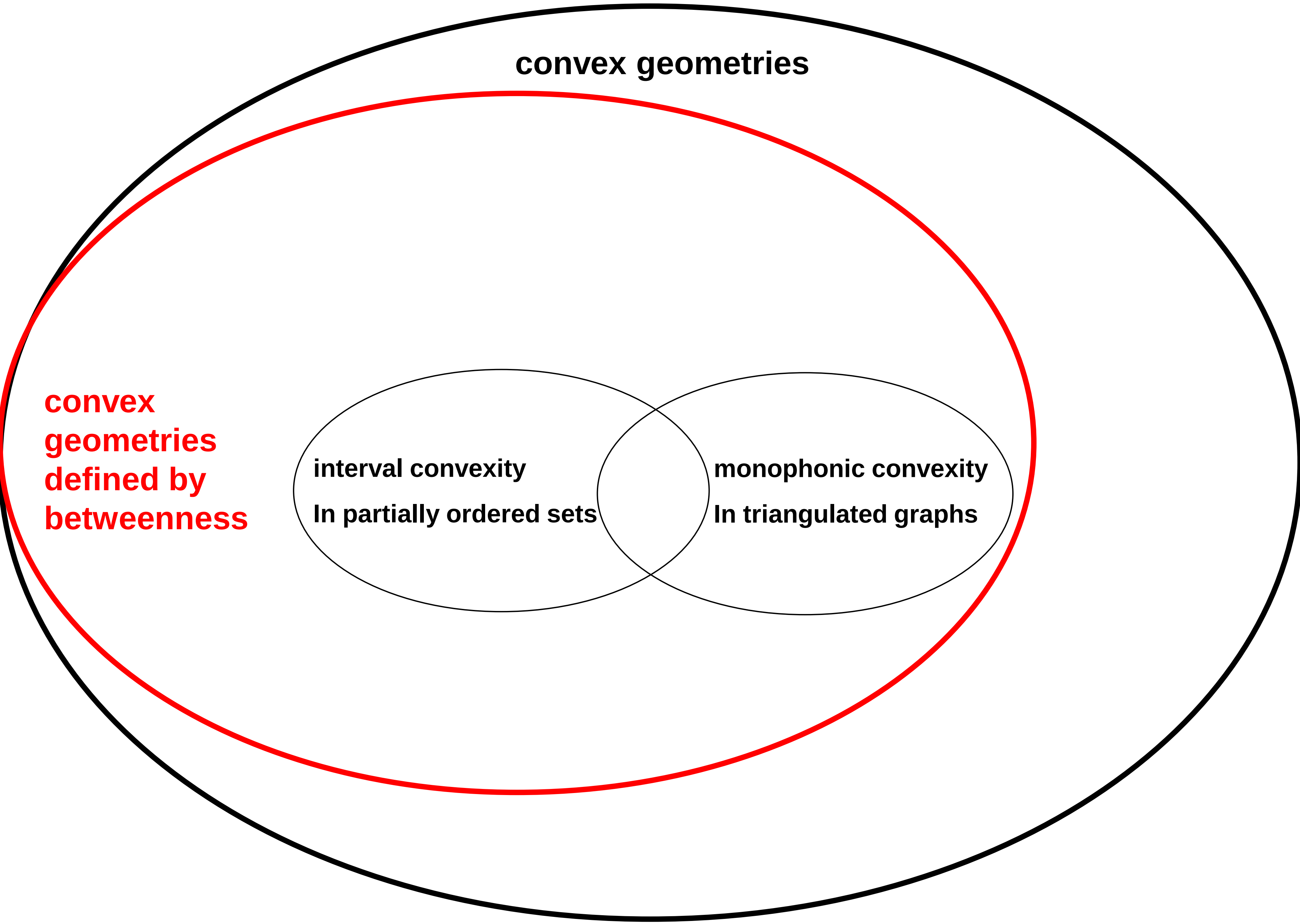


Theorem
(Laurent Beaudou, Ehsan Chiniforooshan, V.C. 2008):

The problem of

recognizing betweennesses that define convex geometries

is co$\mathcal{NP}$-complete.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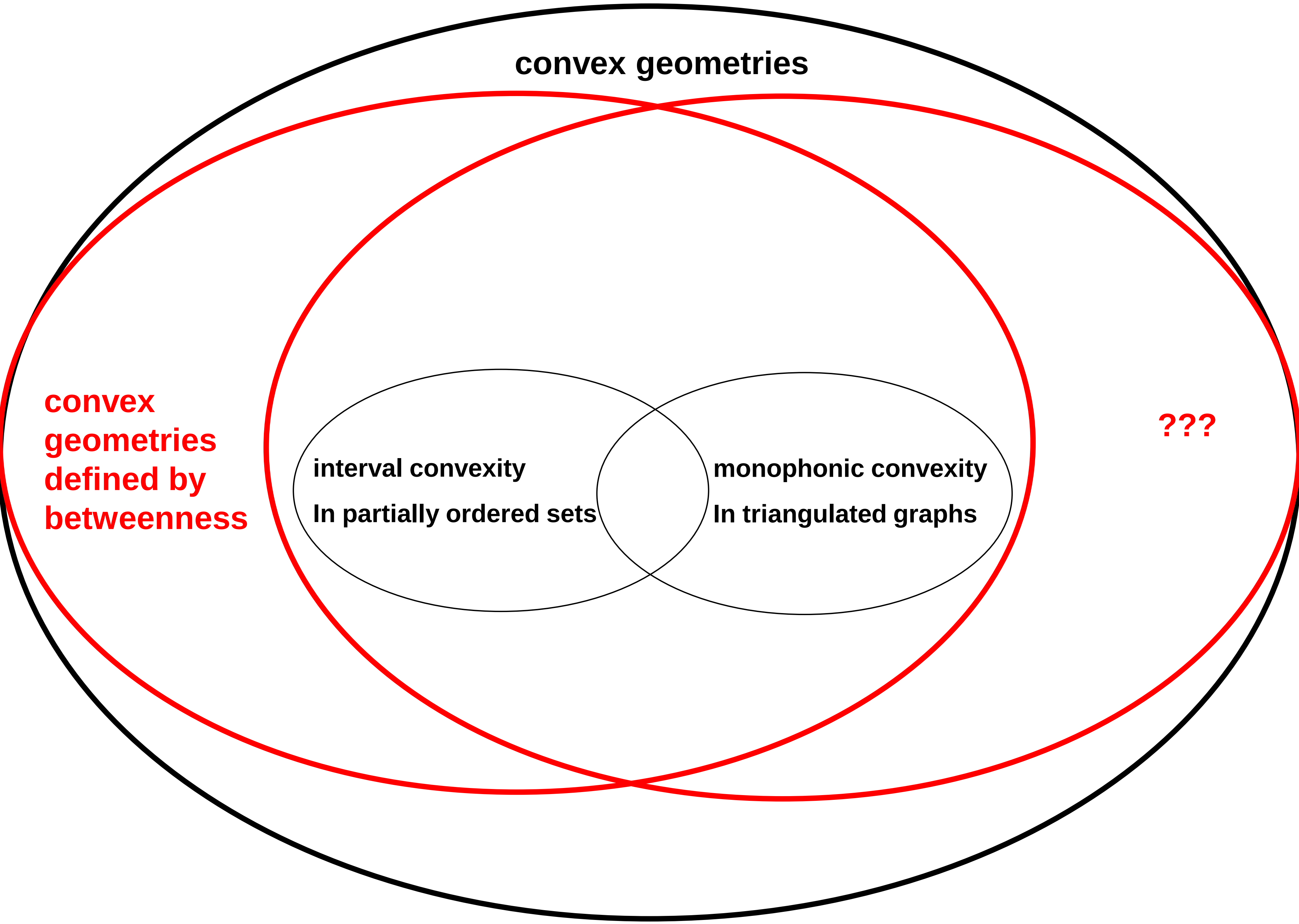

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if, and only if, every point of every convex set *C*

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Theorem (Farber & Jamison 1986):

Monophonic convexity in triangulated graphs

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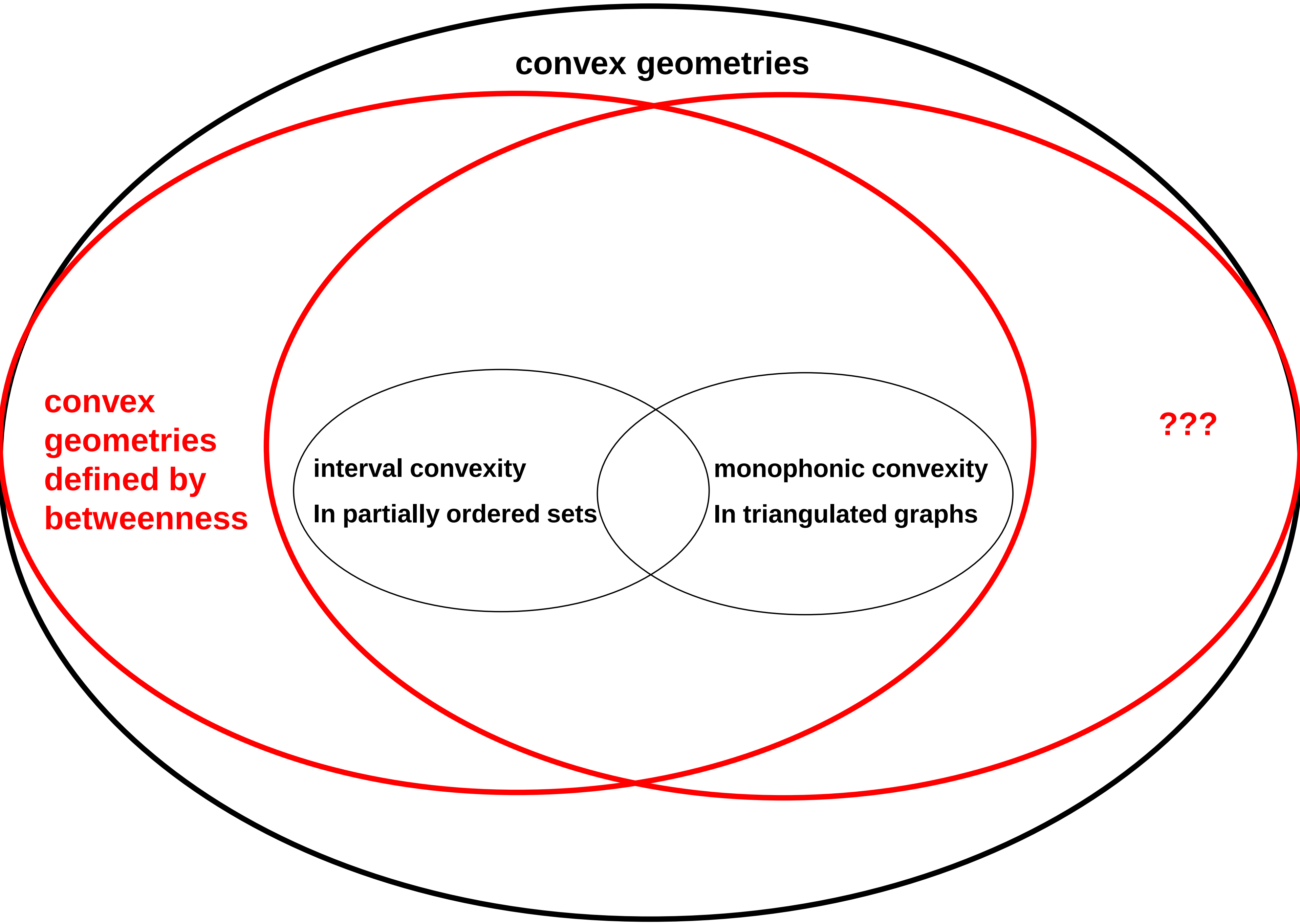
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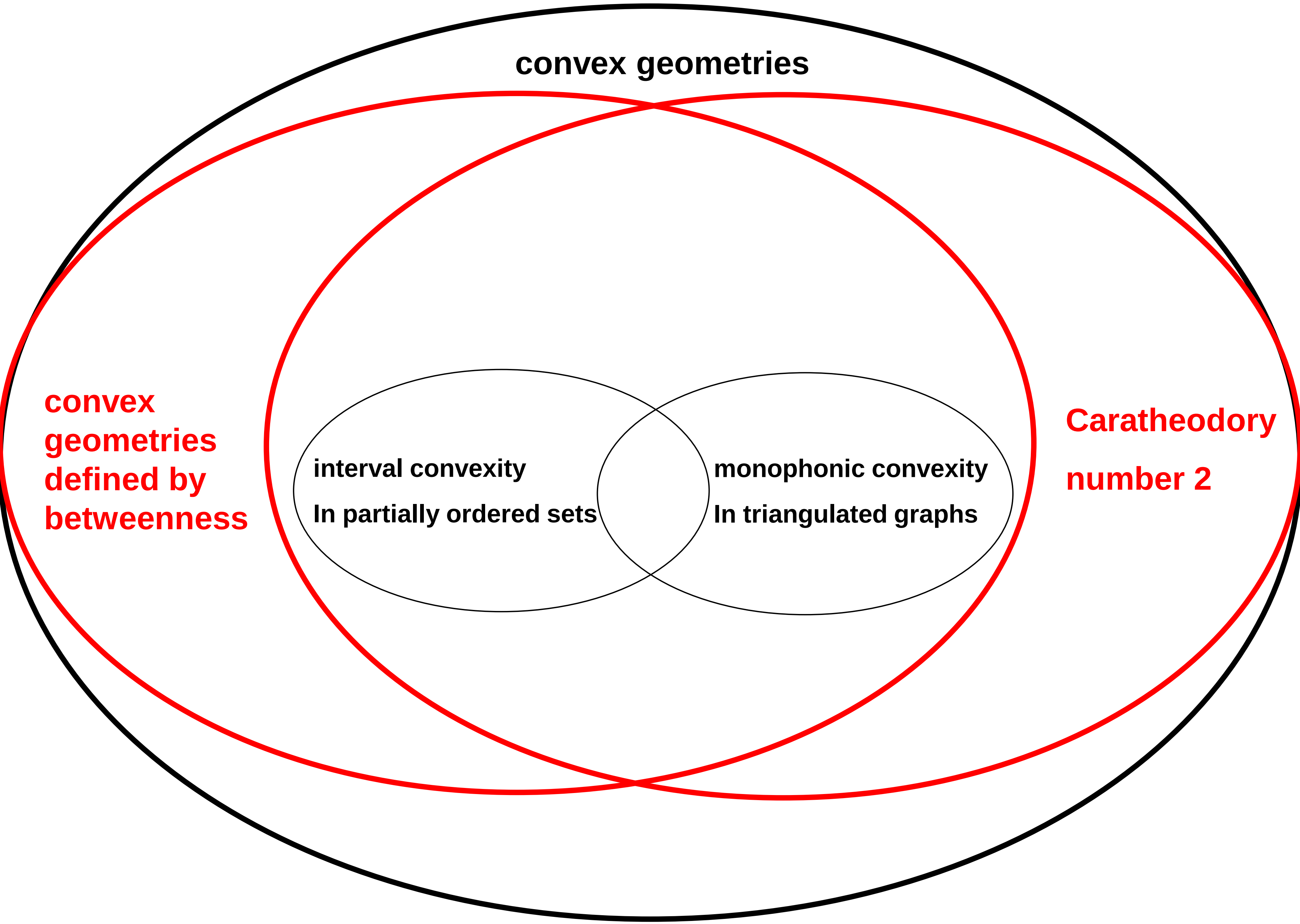
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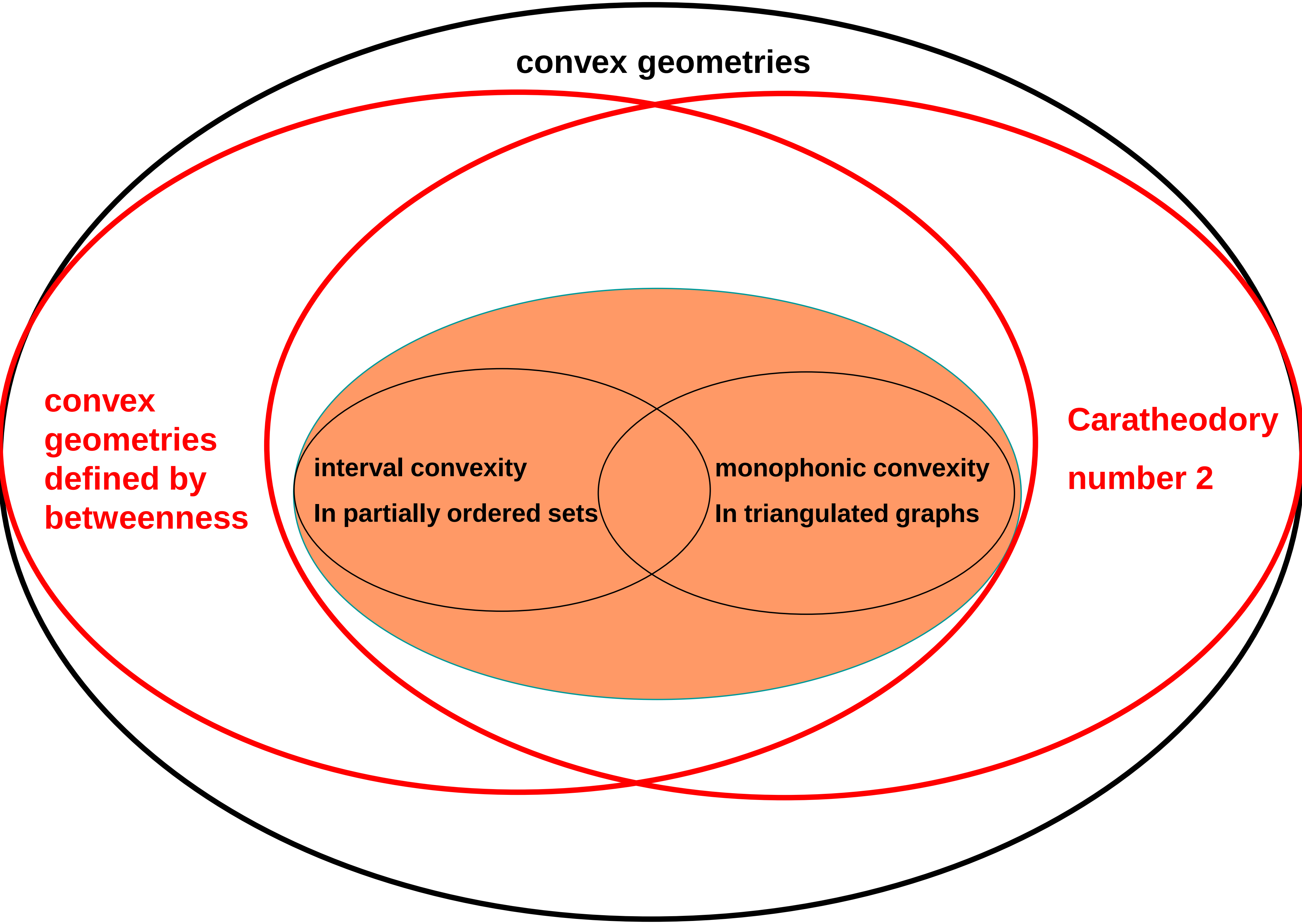
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A betweenness $\mathscr{B}$ defines

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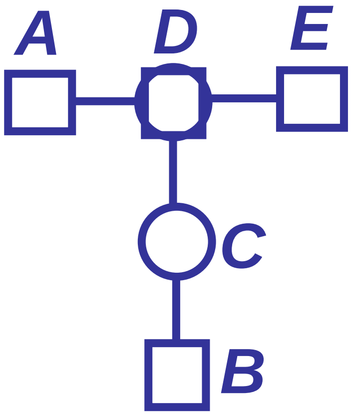

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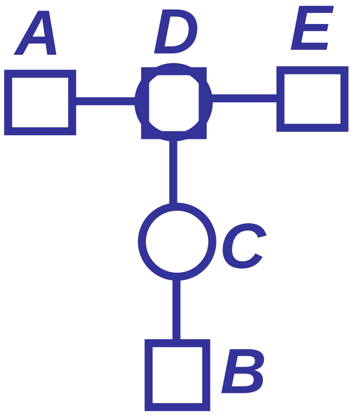

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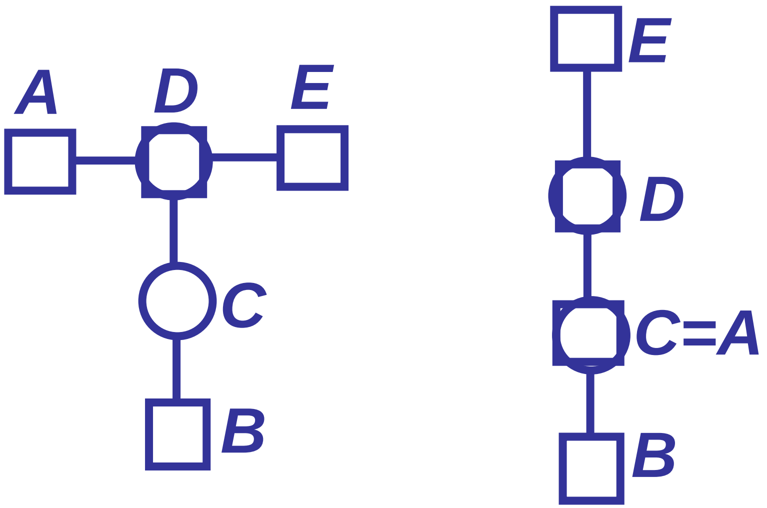

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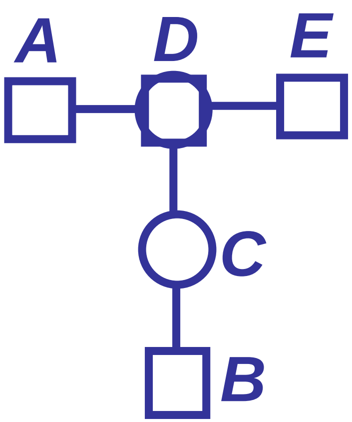
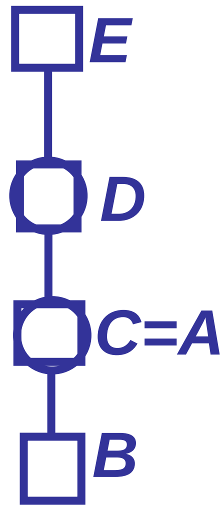
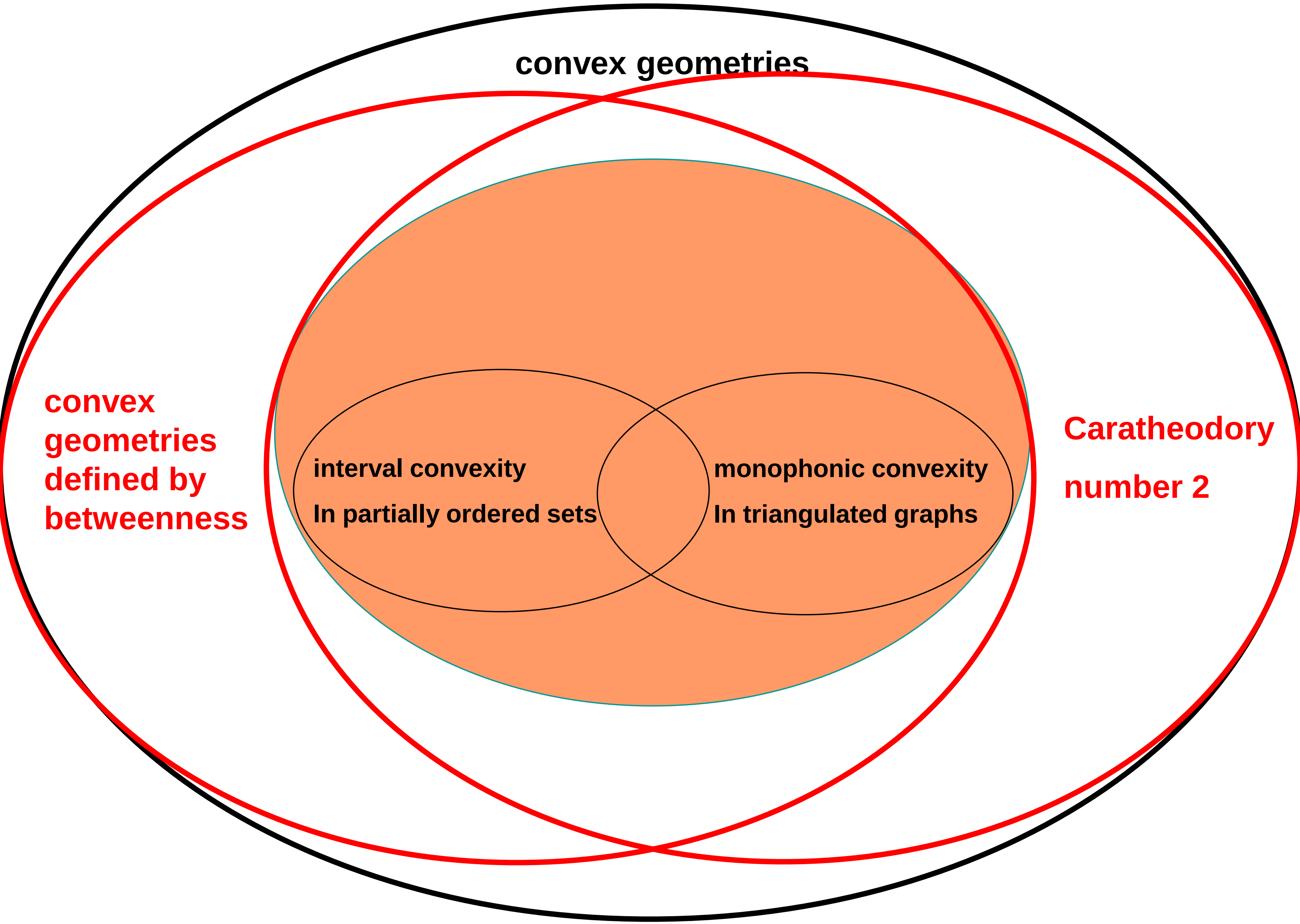




etc.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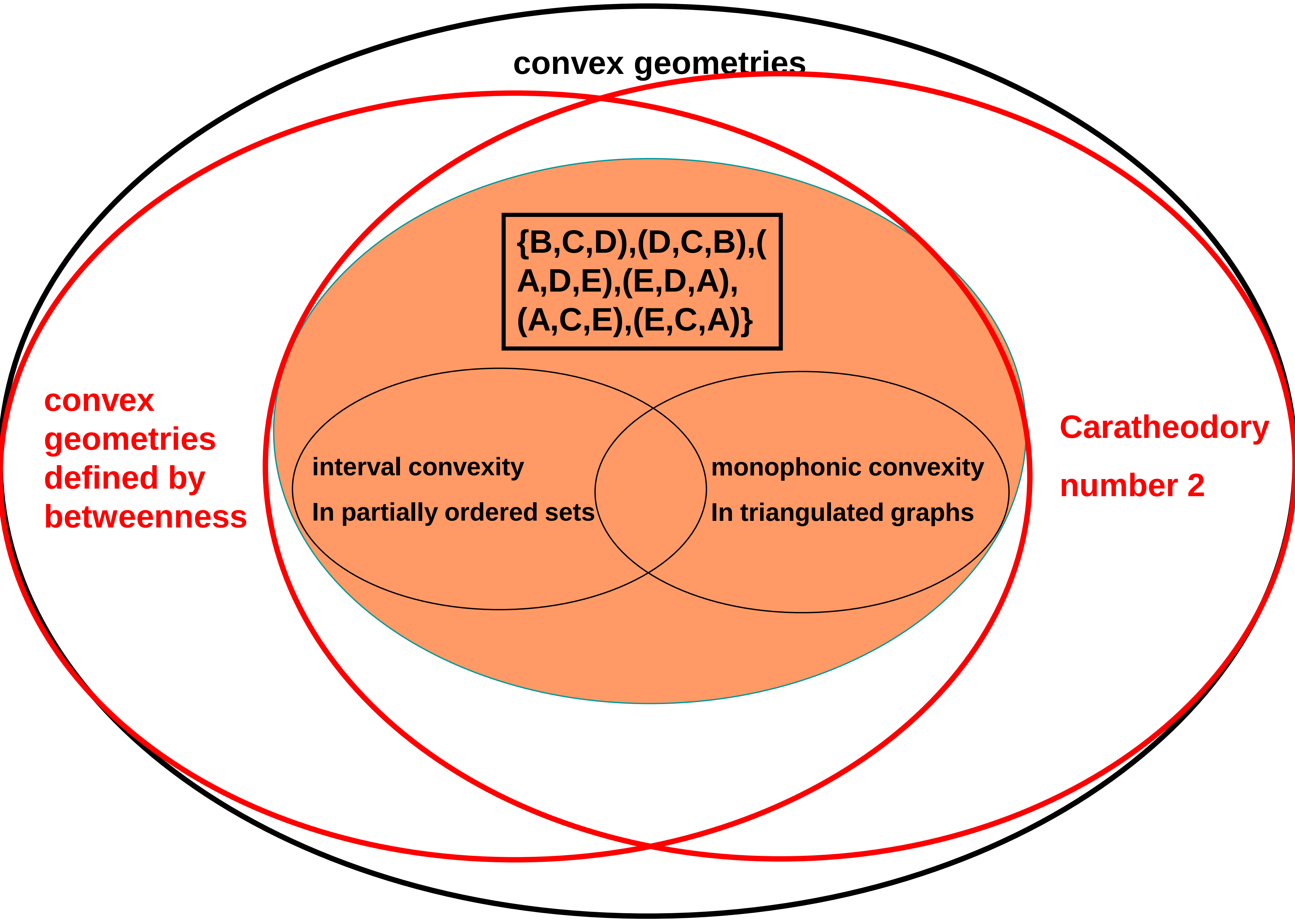
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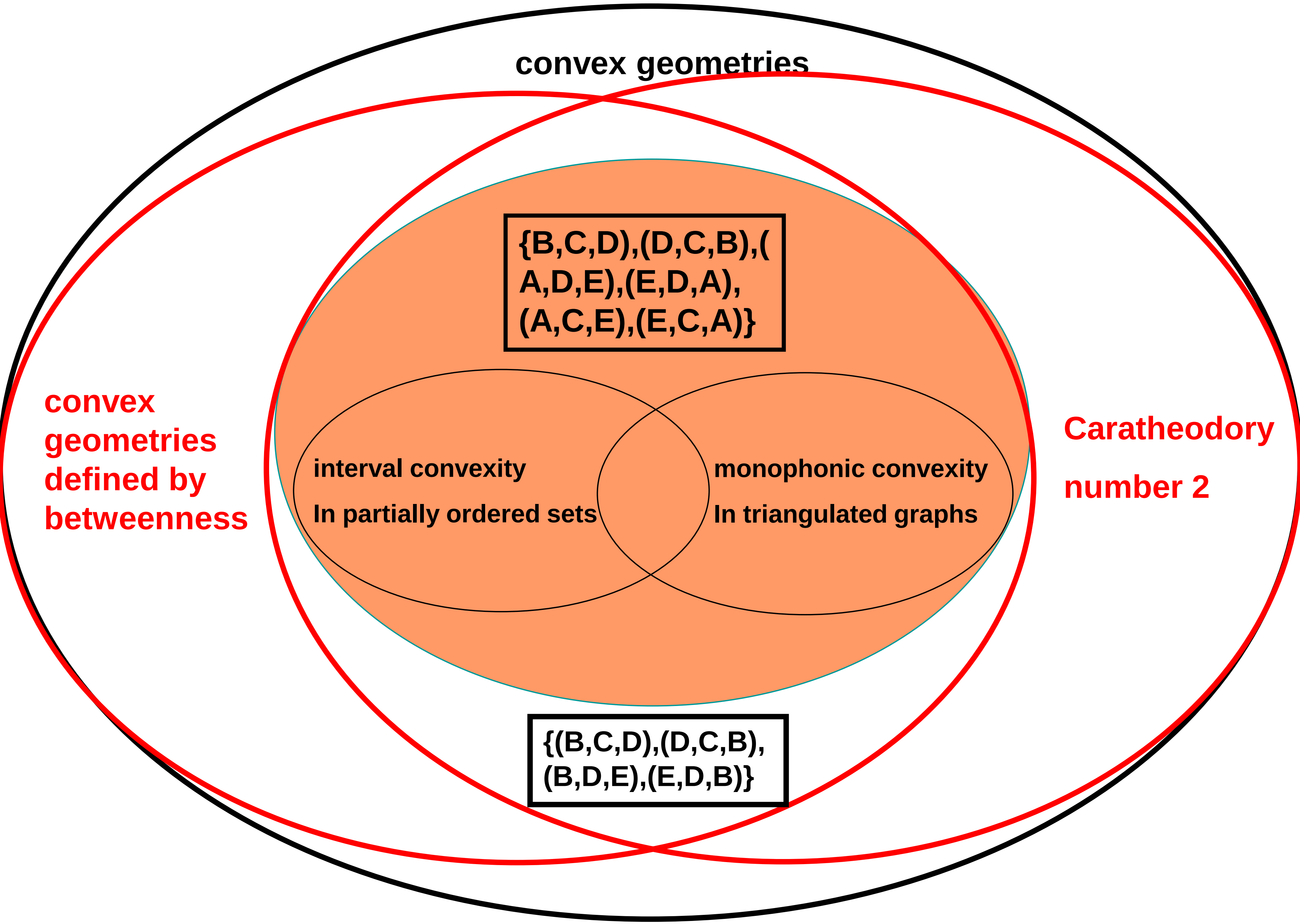
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BBC

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1891 - 1953

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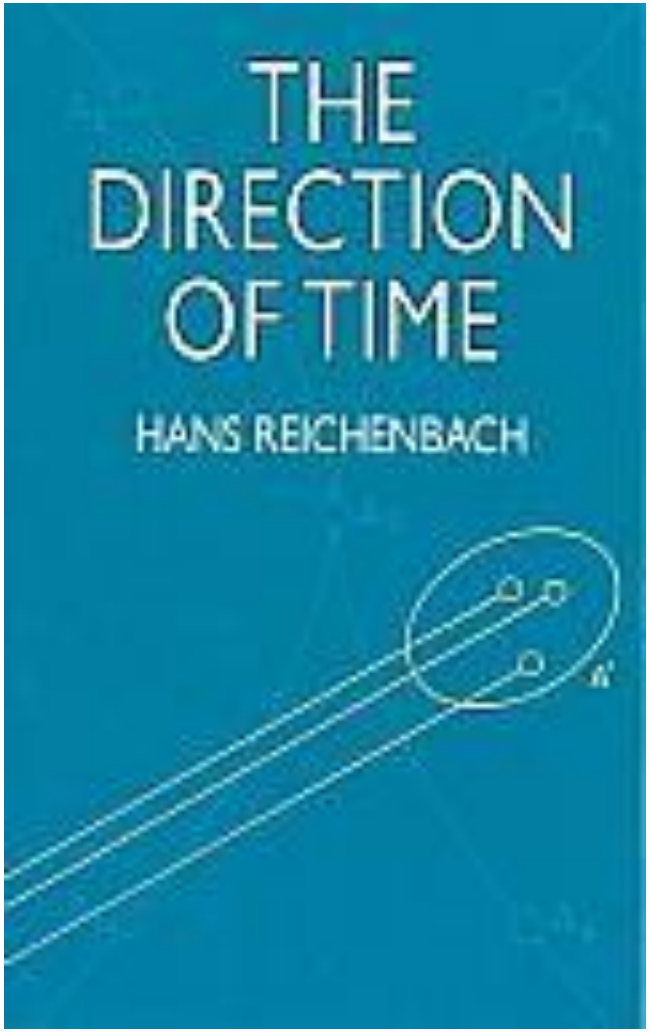
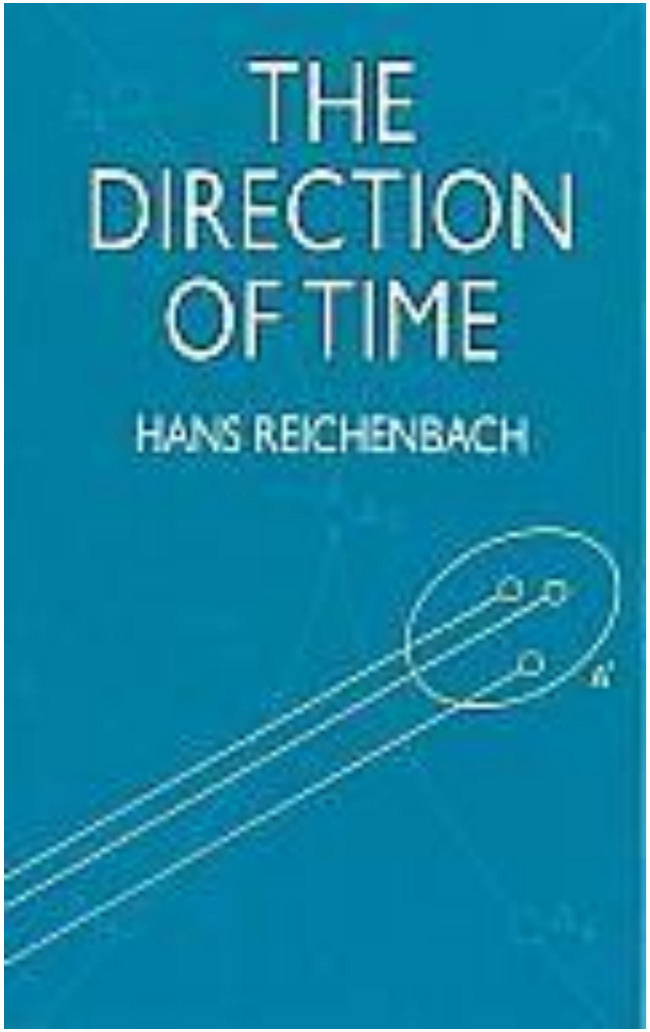


University of California Press, 1956



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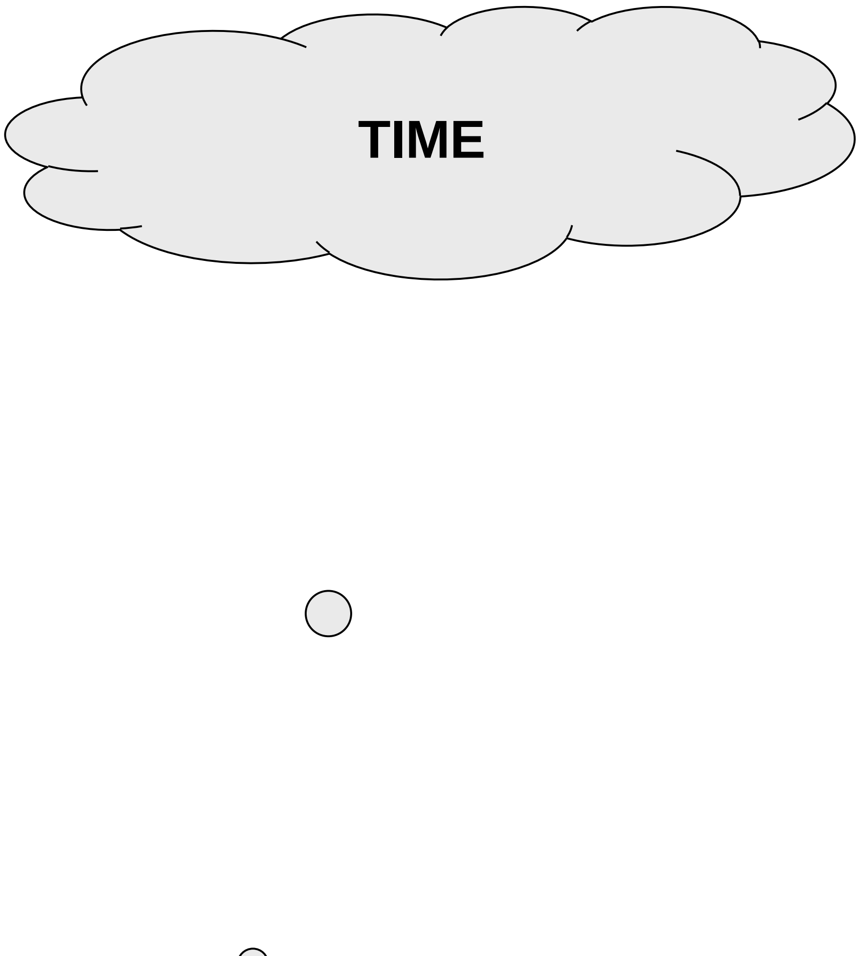
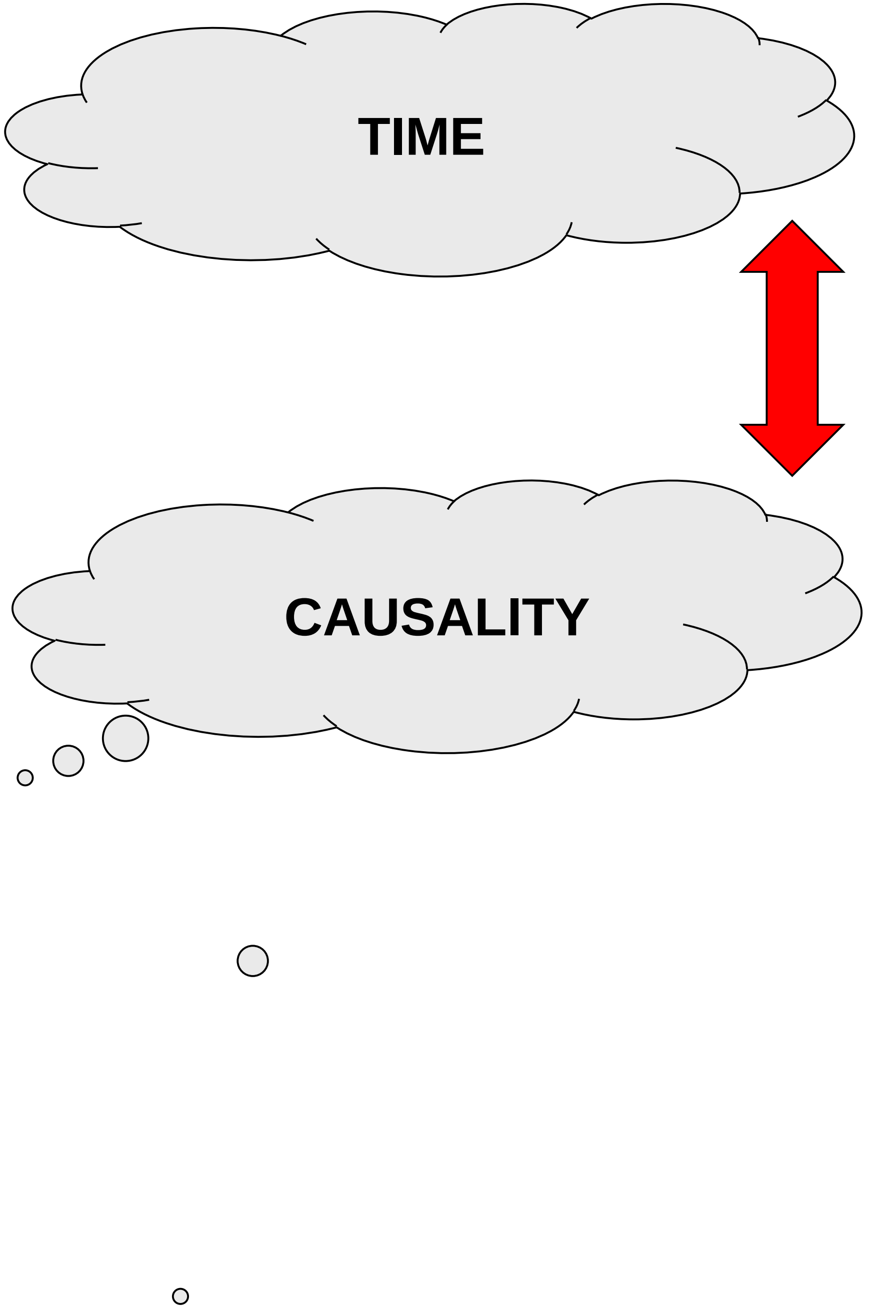
Google Scholar BETA **found 922 works citing it**


TIME

TIME

CAUSALITY


TIME
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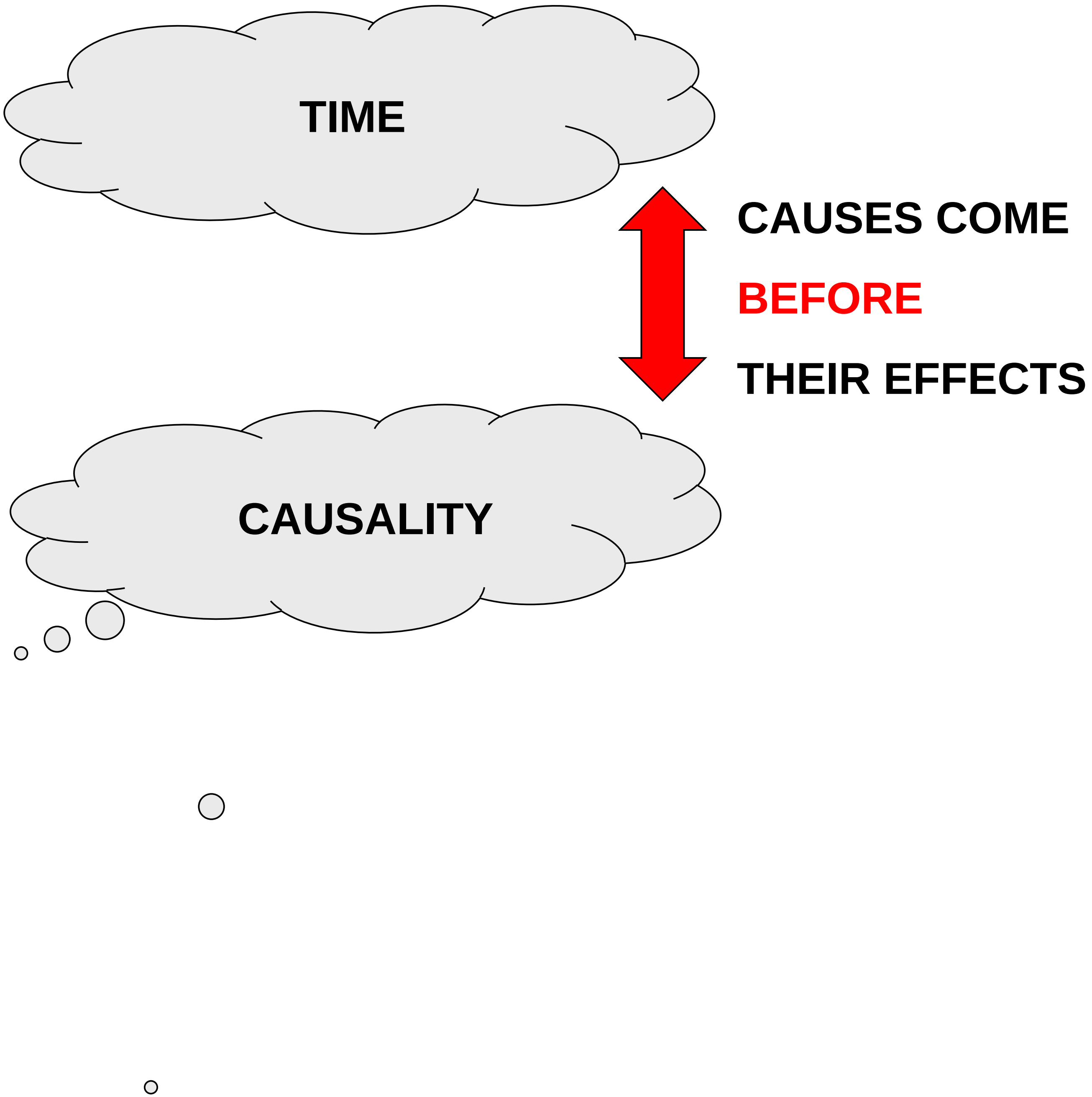
TIME
CAUSES COME
BEFORE
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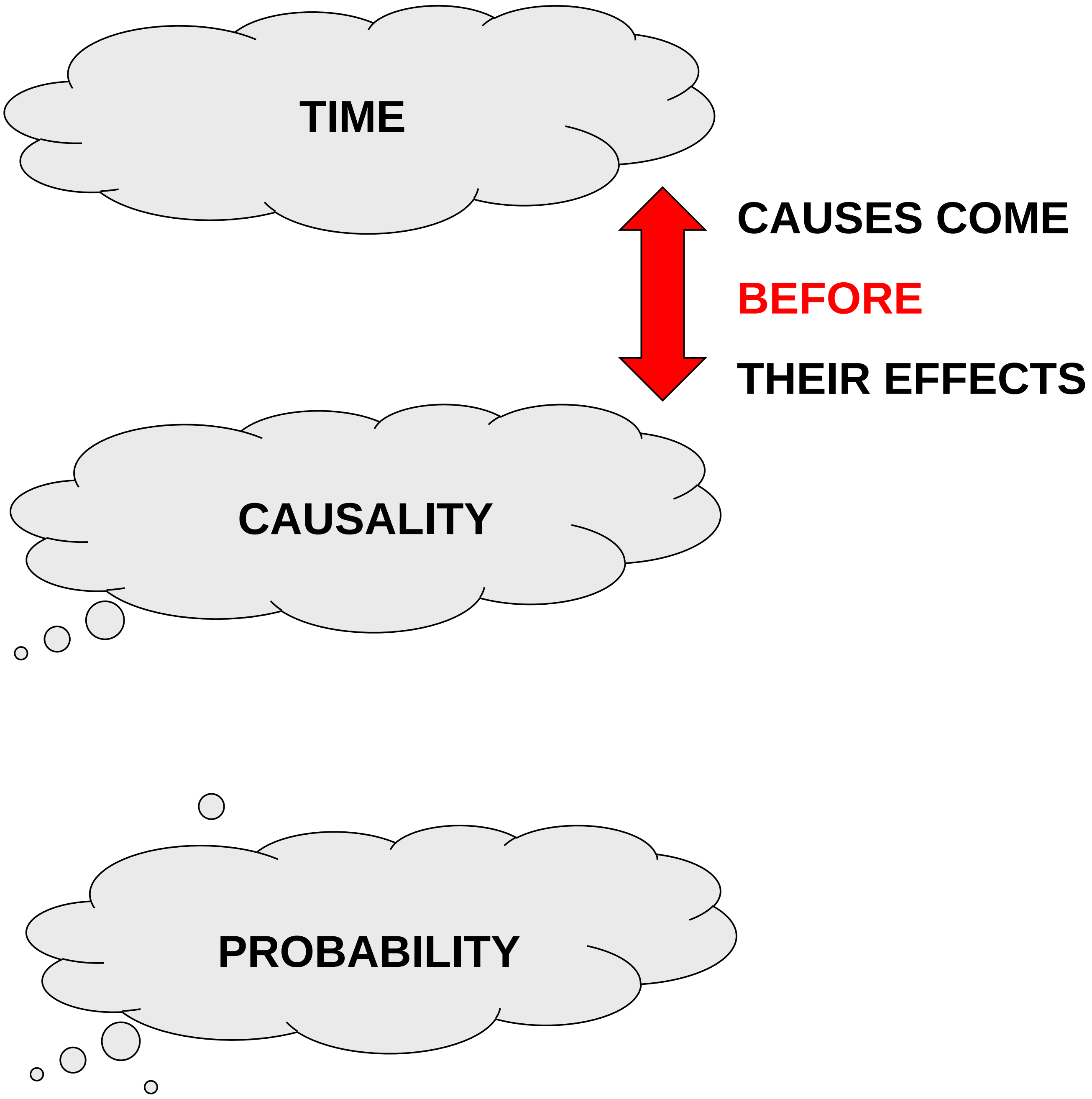
TIME
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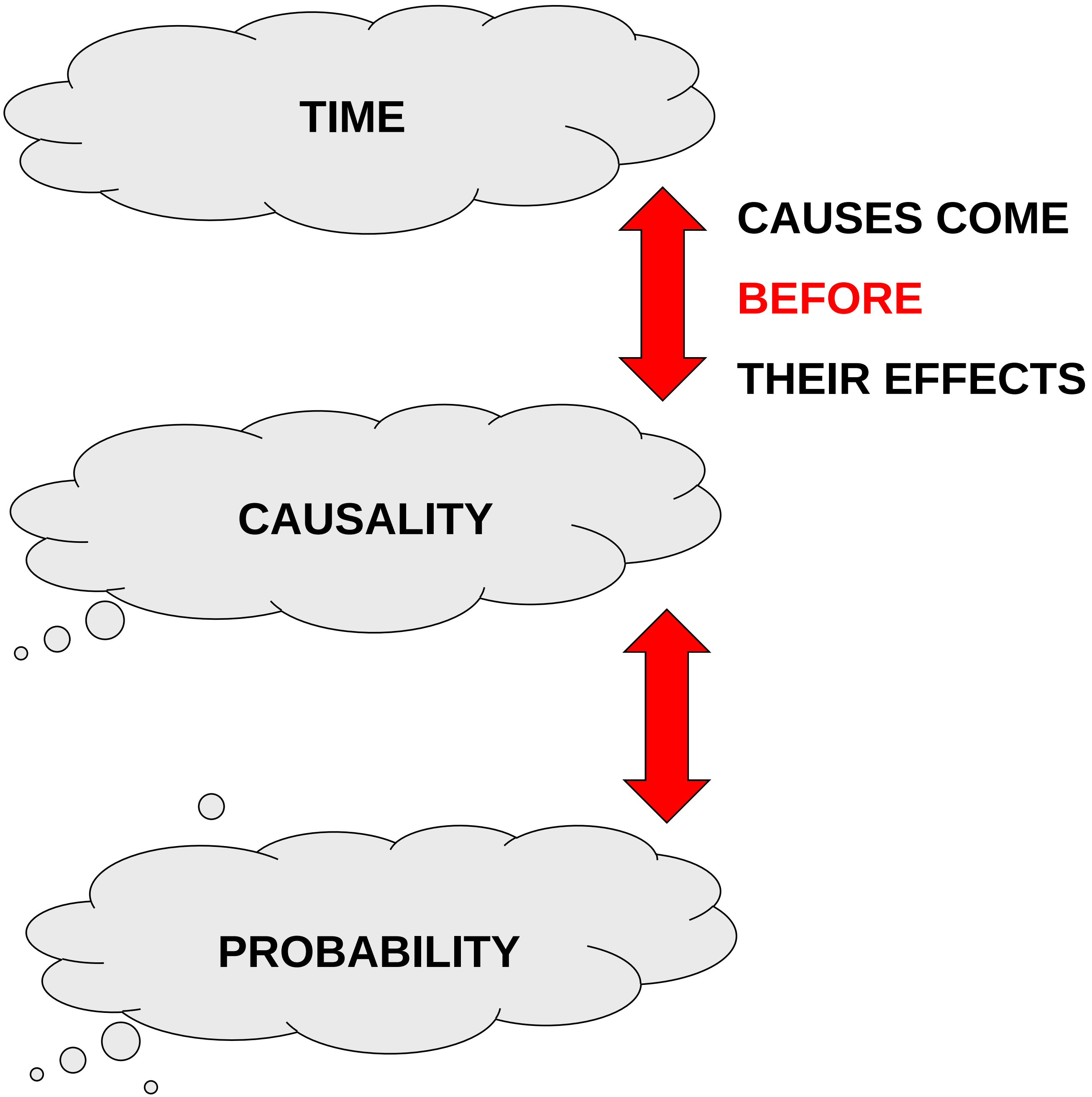
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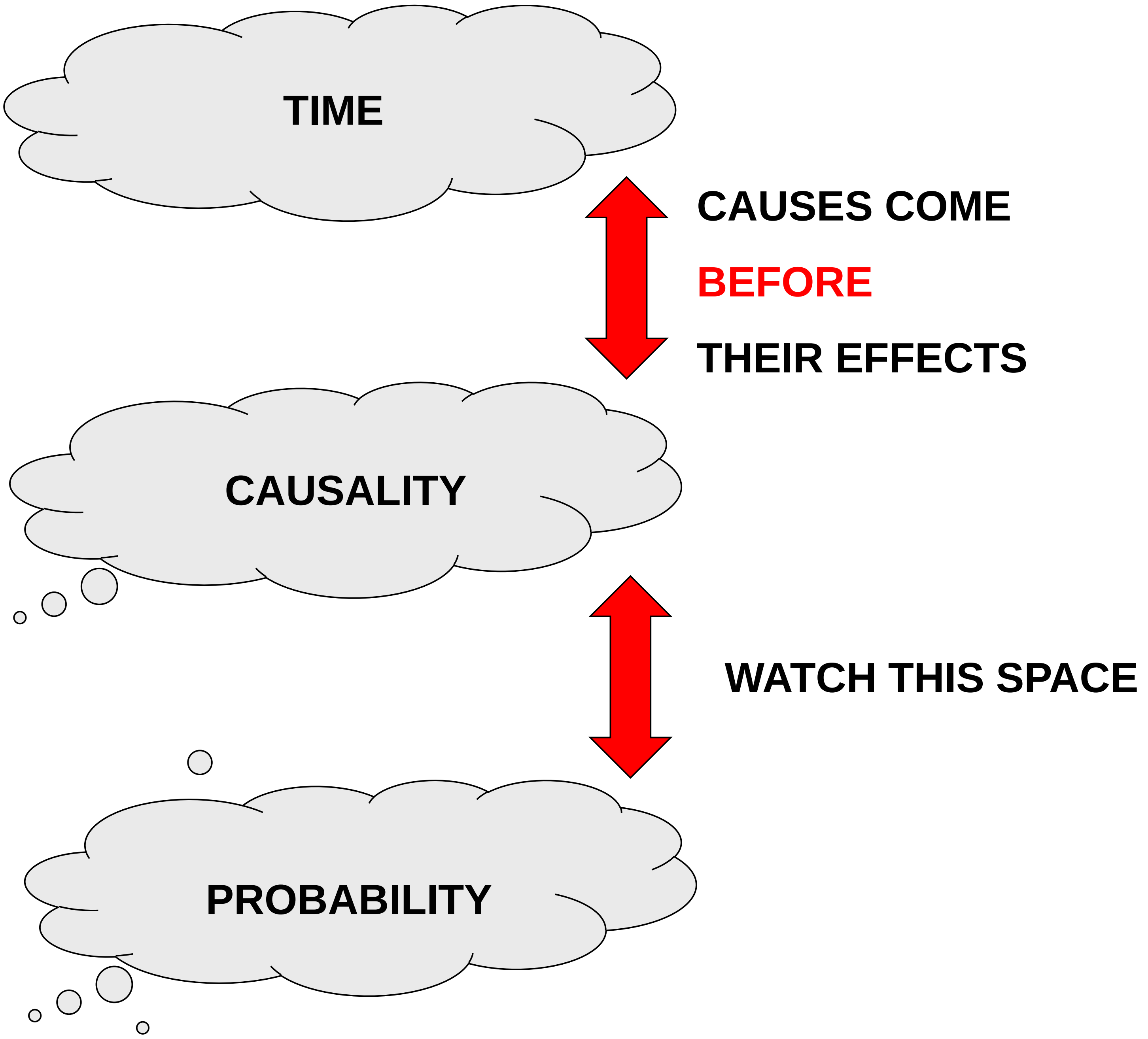
TIME
CAUSES COME
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WATCH THIS SPACE
PROBABILITY

Finite probability spaces

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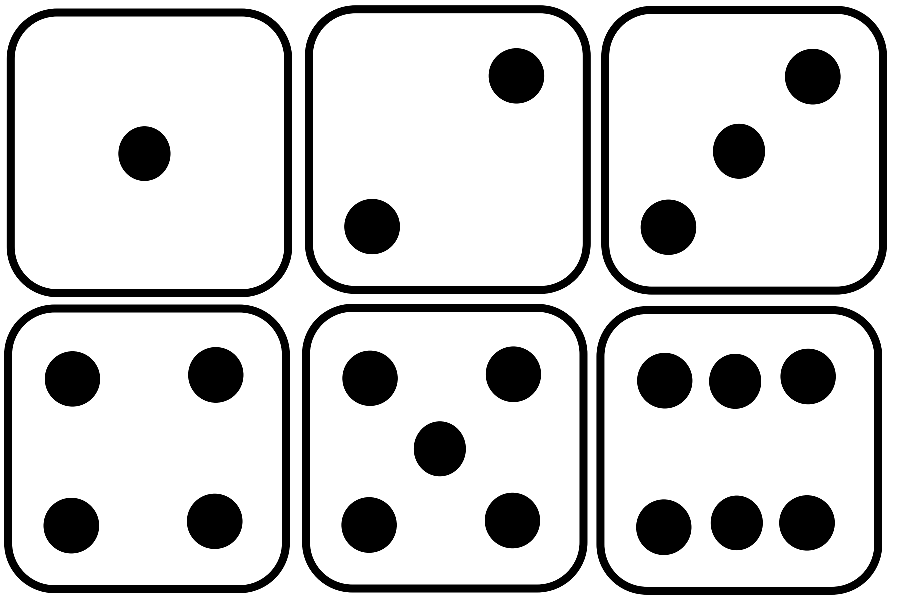

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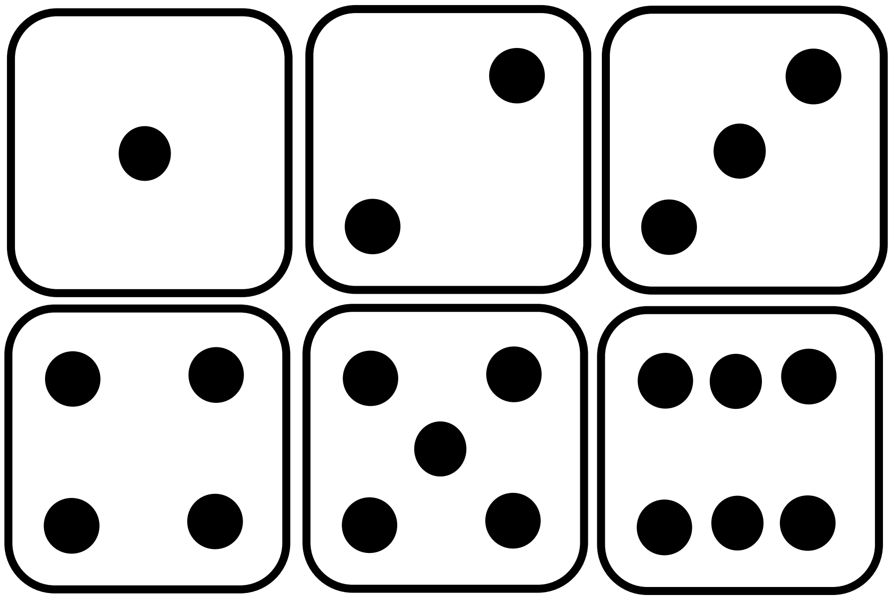
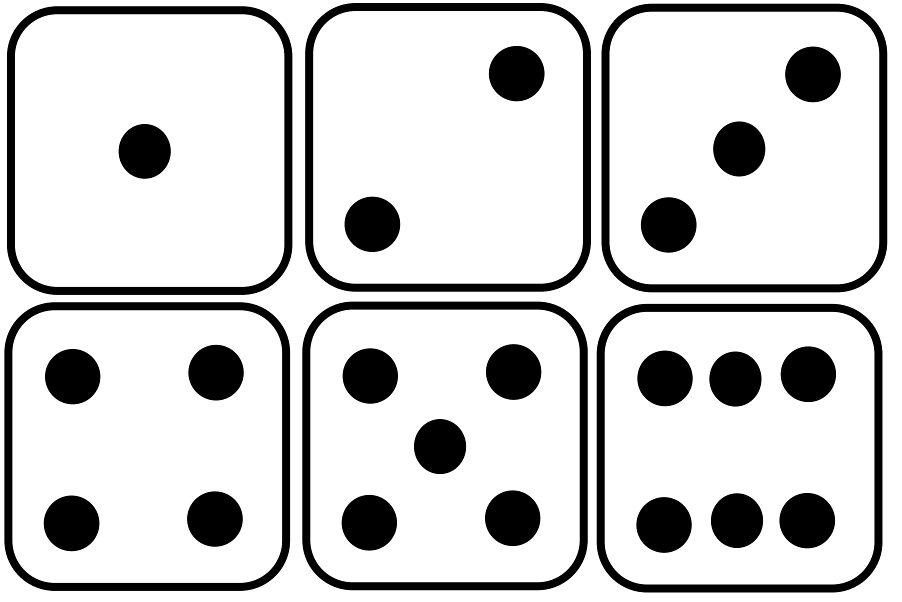


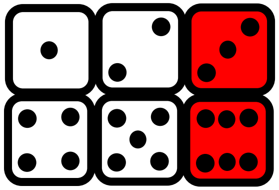
Six *outcomes*

Finite probability spaces

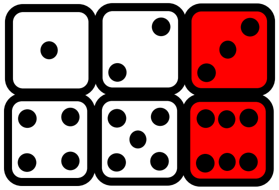


Six *outcomes*

2^6 *events*



Prob(A)=1/3



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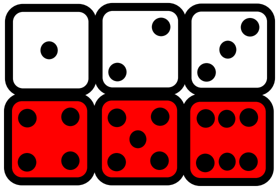
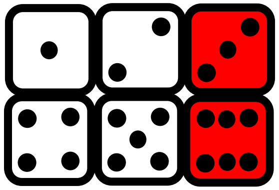


Prob(B)=1/2



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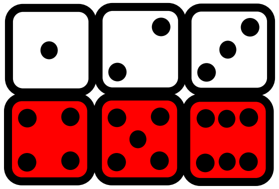

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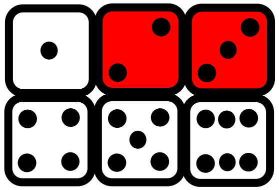
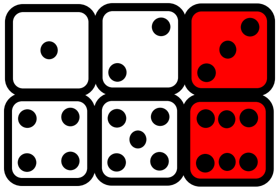


Prob(C)=1/3



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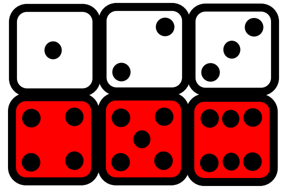
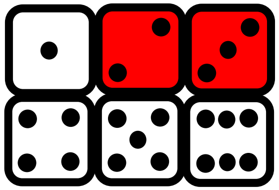


Prob(B)=1/2



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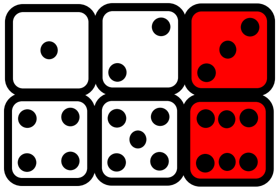
Conditional probability of event E given event F:

Prob(E|F) = Prob(E&F) / Prob(F)



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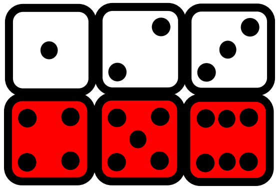

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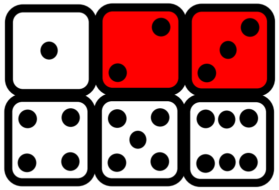

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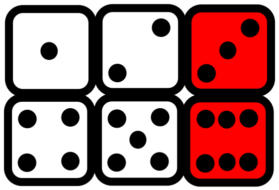

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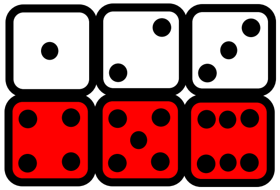

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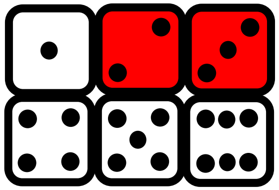

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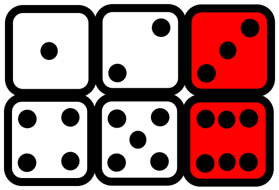

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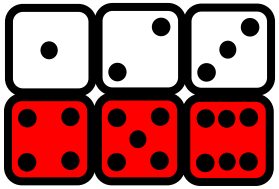

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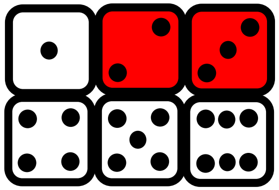

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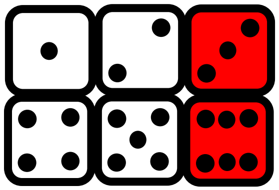
A and C are *positively correlated*



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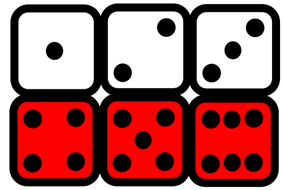

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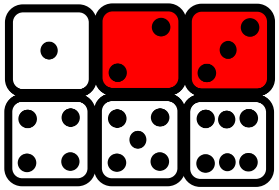

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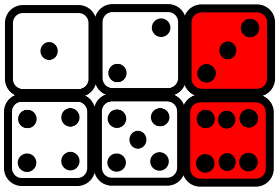

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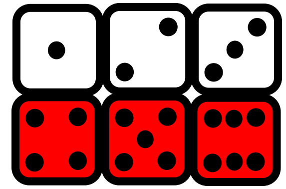

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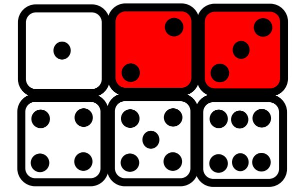

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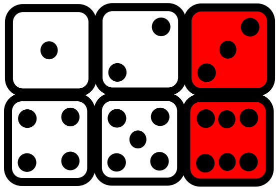
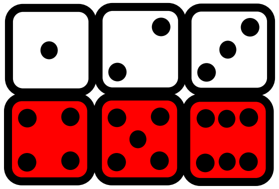
Prob(A|B)= 1/3 , Prob(B|A)= 1/2

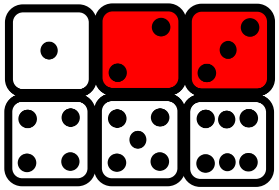
A and B are *independent*



Prob(A)=1/3



Prob(B)=1/2



Prob(C)=1/3

Conditional probability **of event E given event F:**

Prob(E|F) = Prob(E&F) / Prob(F)

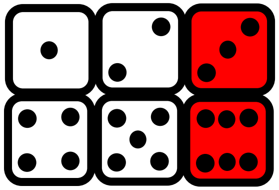
Prob(A|C)= 1/2 , Prob(C|A)= 1/2

A and C are *positively correlated*

Prob(A|B)= 1/3 , Prob(B|A)= 1/2

A and B are *independent*

Prob(B|C)= 0 , Prob(C|B)= 0



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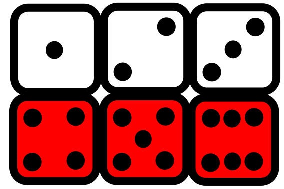

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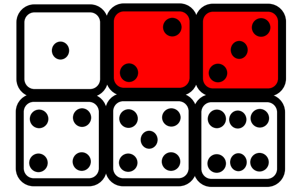

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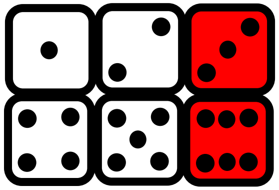

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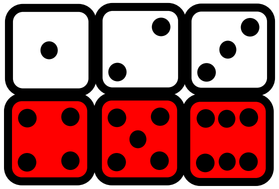

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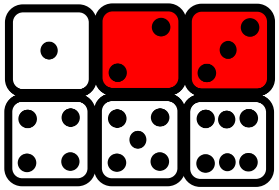

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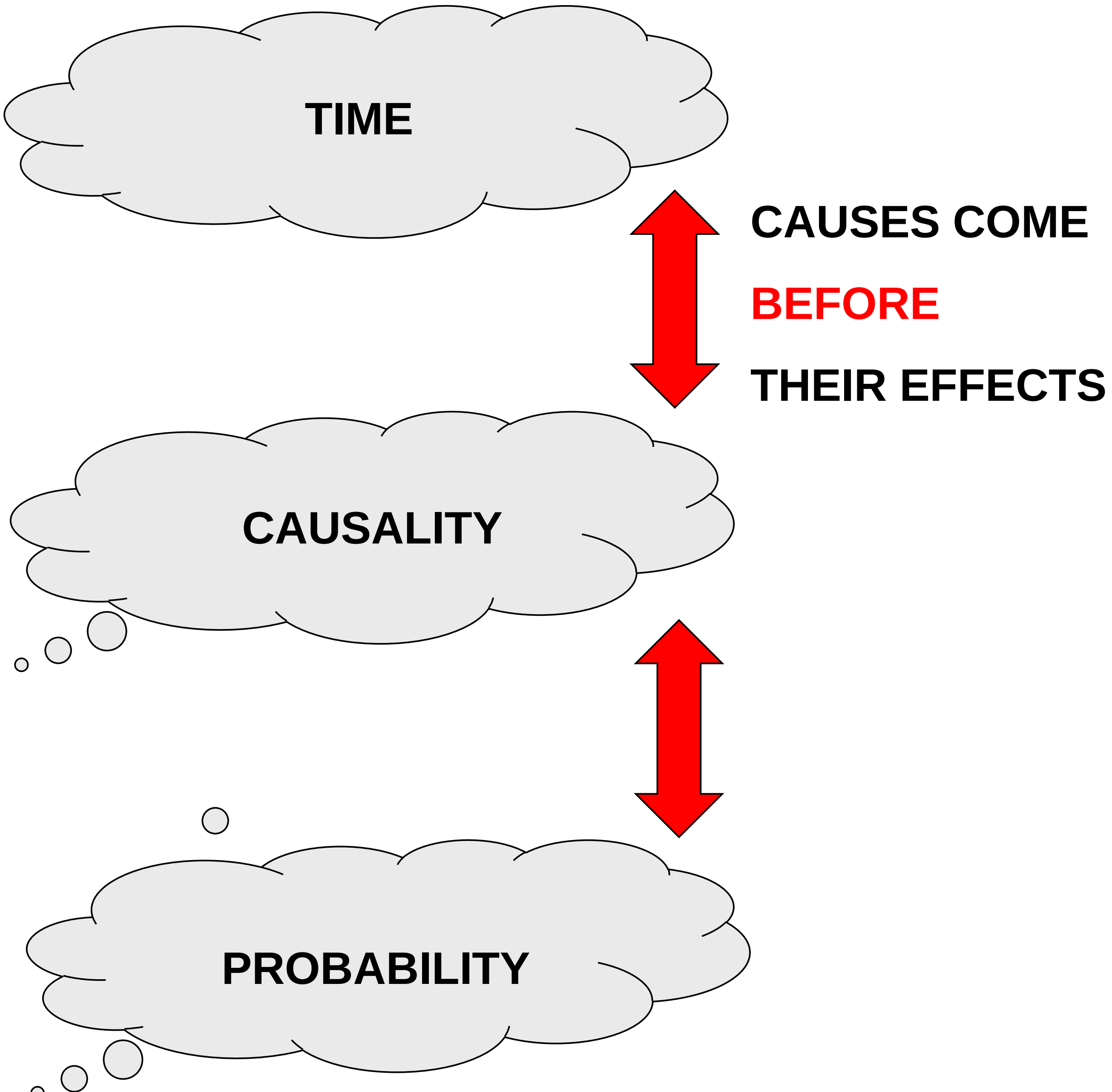
TIME
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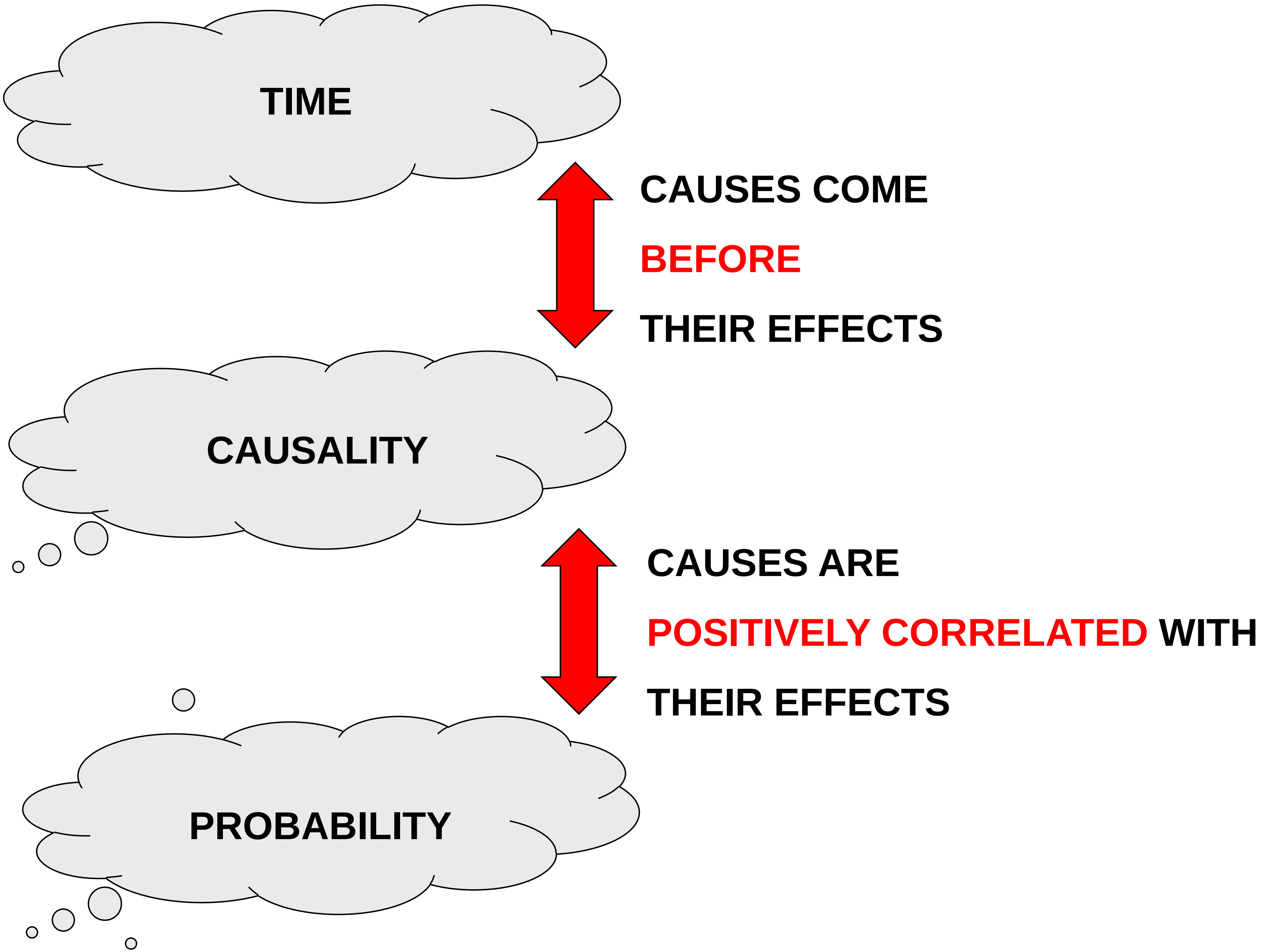
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Definition (Reichenbach, p. 190):

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B *screens off* A from C

Question: When is a ternary relation $\mathscr{B}$ isomorphic to a causal betweenness?

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and a certain directed graph G($\mathscr{B}$) contains no directed cycles.

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Example: $\mathscr{B}$ consists of

(C,A,B), (B,A,C),

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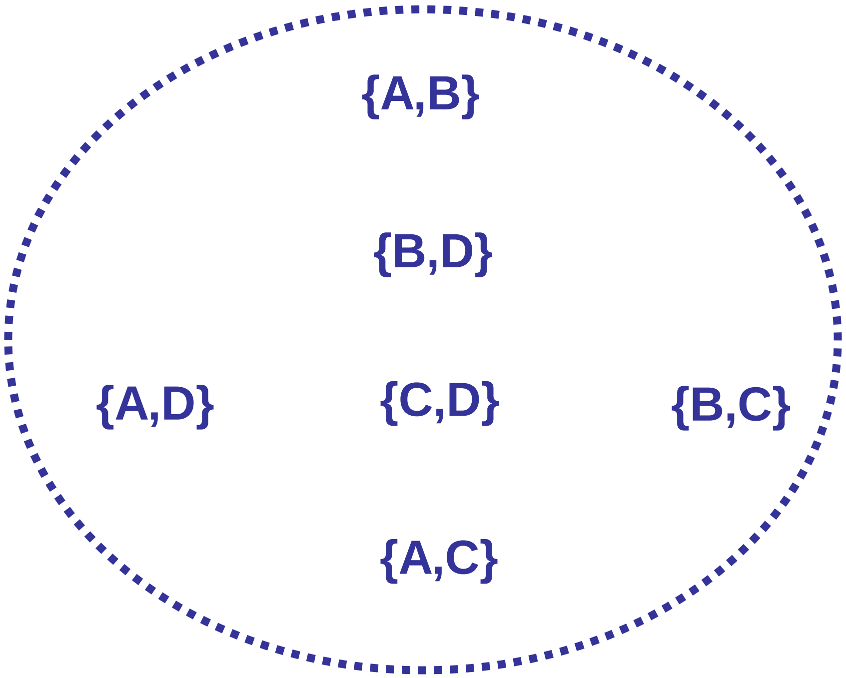

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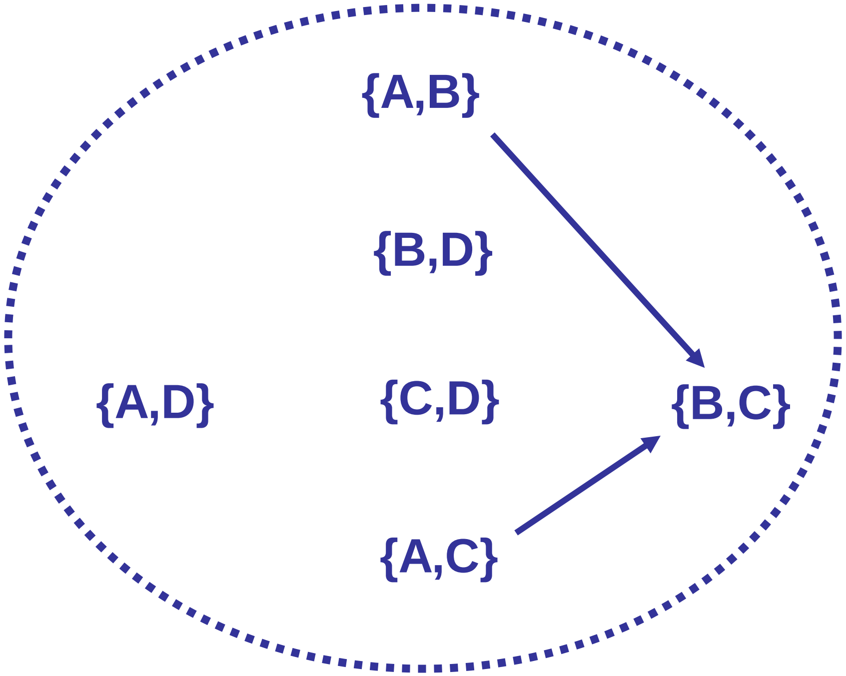

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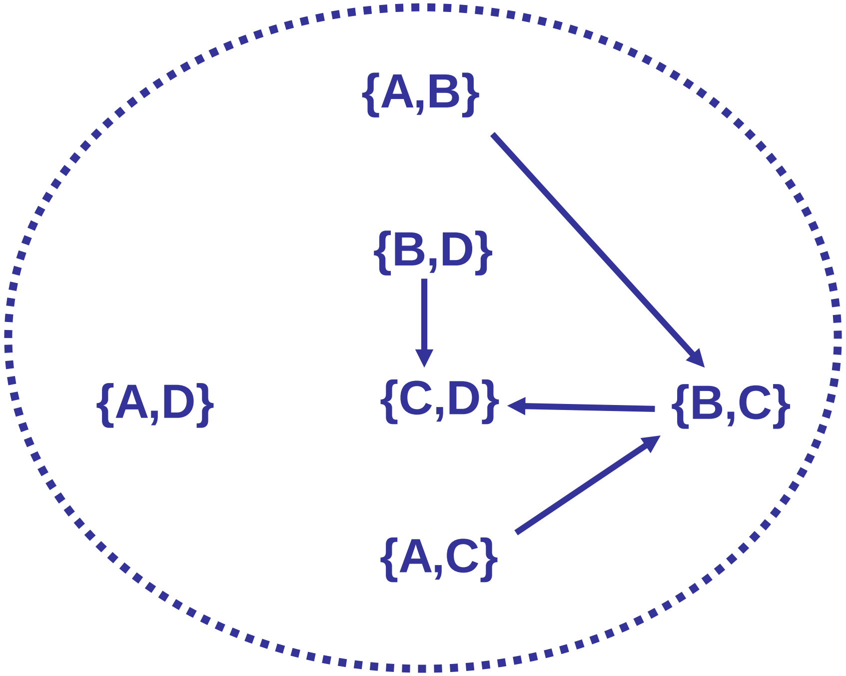

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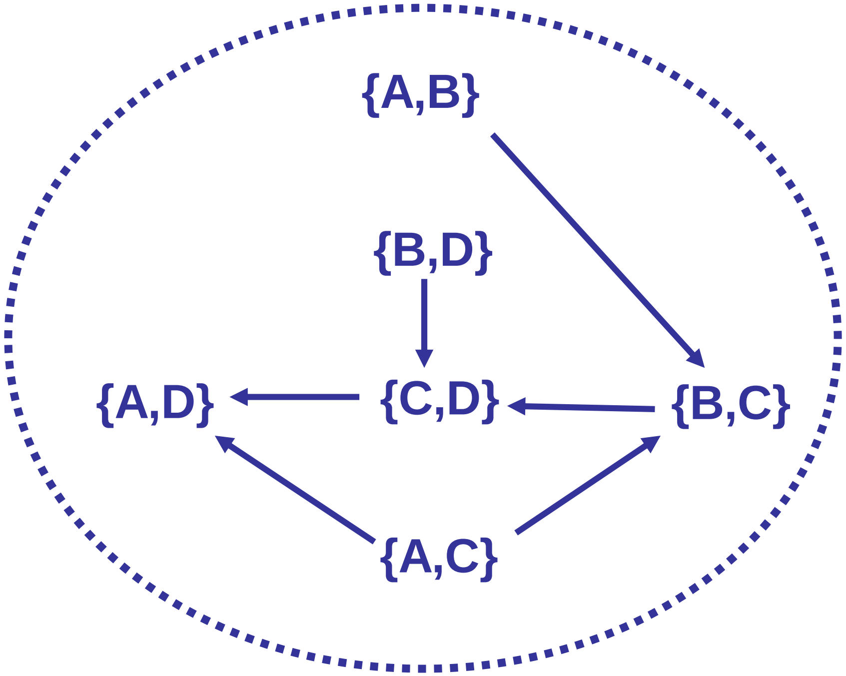

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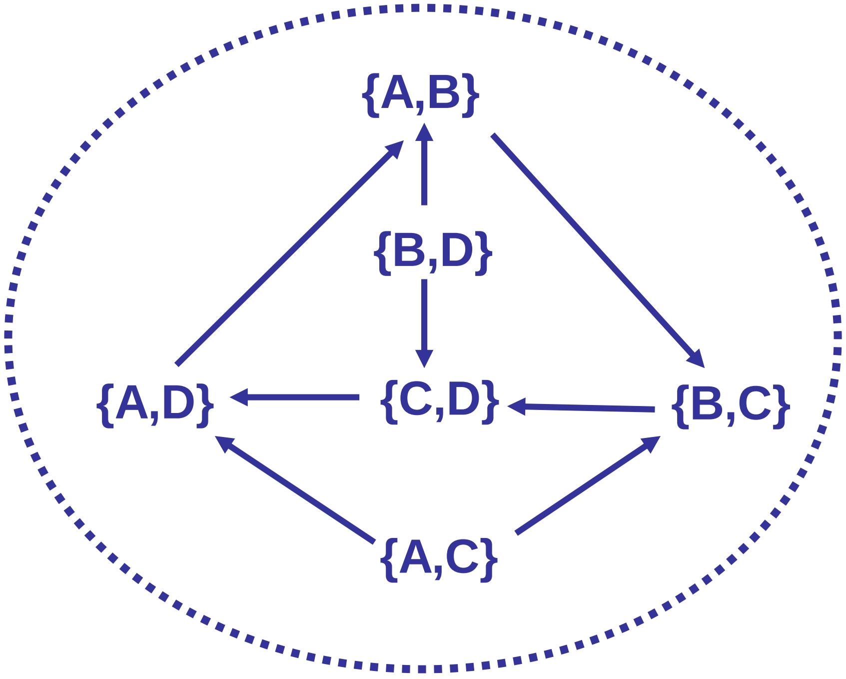

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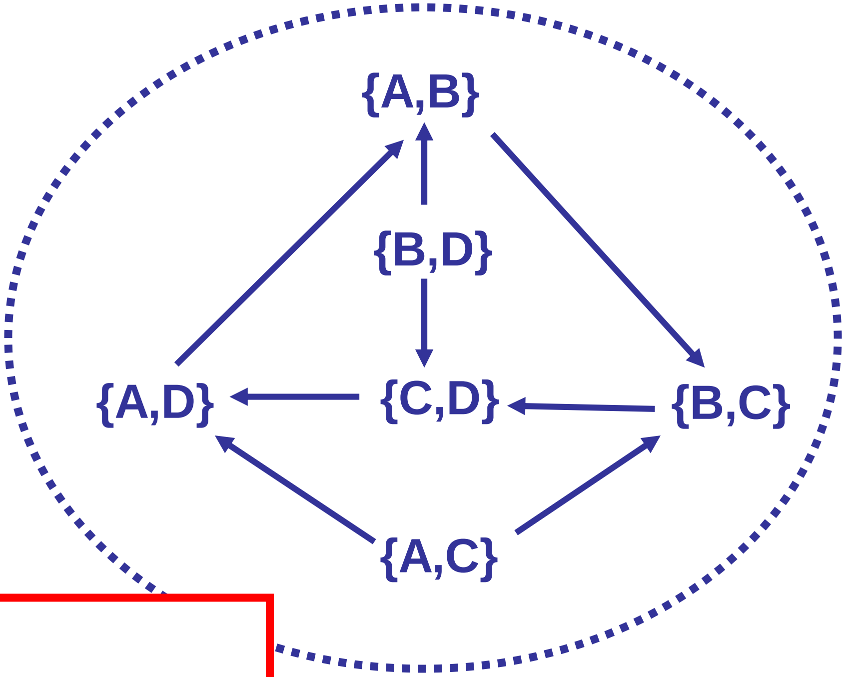

Abstract causal betweennesses can be recognized in polynomial time

Definition (Reichenbach, p. 159):

Events A,B,C constitute a *conjunctive fork* if, and only if

B and A are positively correlated,

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Question: When is a ternary relation $\mathscr{B}$ isomorphic to the conjunctive fork betweenness?

The End

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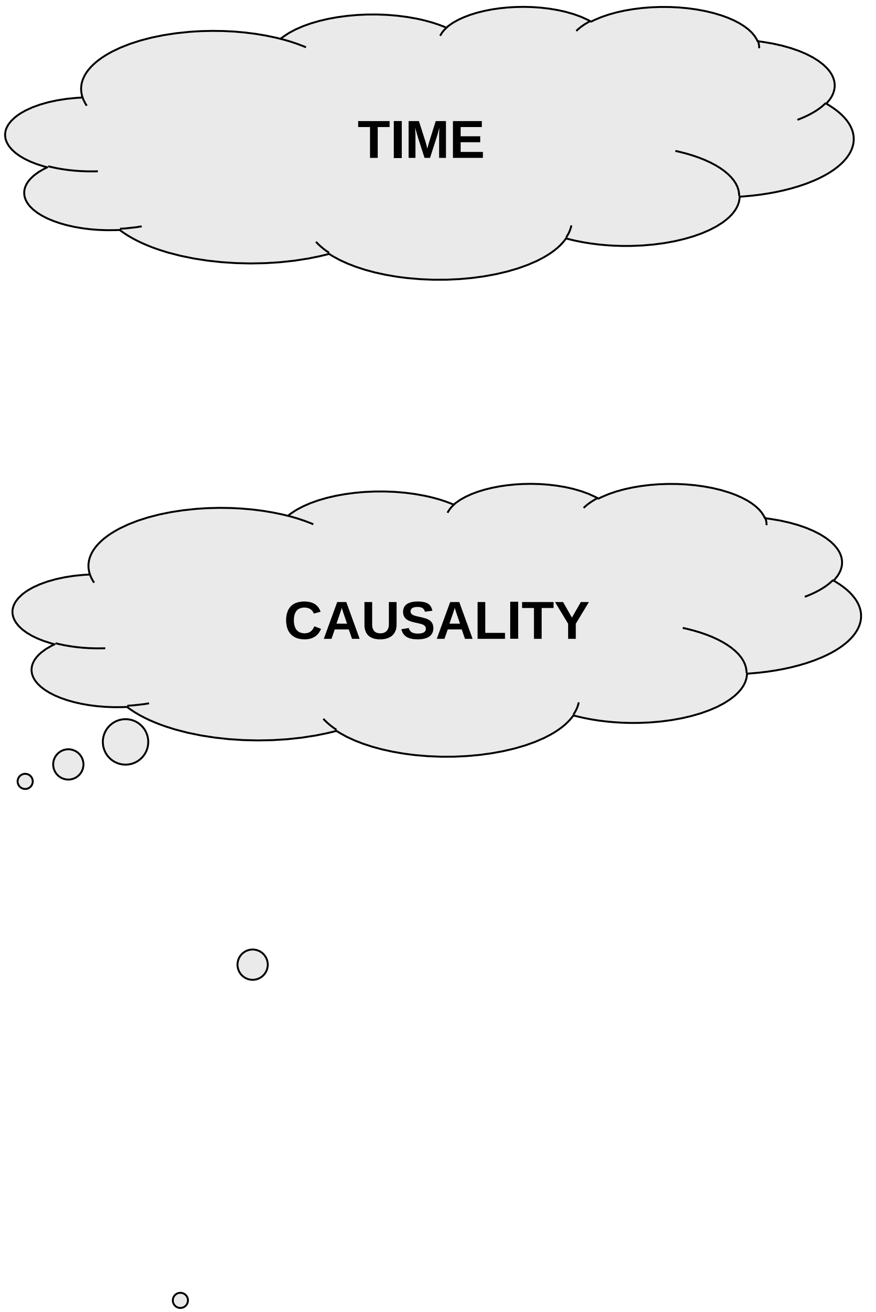
TIME
CAUSALITY

TIME

CAUSALITY

B is causally between A and C

TIME

B occurs between A and C

CAUSALITY

B is causally between A and C

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B occurs between A and C

CAUSALITY

B is causally between A and C

--- WARNING ---

If an event B is causally between events A and C,

then it does not necessarily occur between A and C:

it can occur before both A and C

and it can occur after both A and C.

How about betweennesses $\mathscr{B}$ such that

(A,B,C) is in $\mathscr{B}$ if and only if B is between A and C

both causally and in time?

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Definition

A ternary relation $\mathscr{B}$ on a set X is called *totally orderable* if, and only if,

there is a mapping *t* from X into a totally ordered set

such that, for each (A,B,C) in $\mathscr{B}$, either $t(A) < t(B) < t(C)$ or $t(C) < t(B) < t(A)$.

Fact: Every totally orderable betweenness is causal.

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***Justification*: If $\mathscr{B}$ is totally orderable, then G($\mathscr{B}$) contains no directed cycle.**

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Opatrny (1979):

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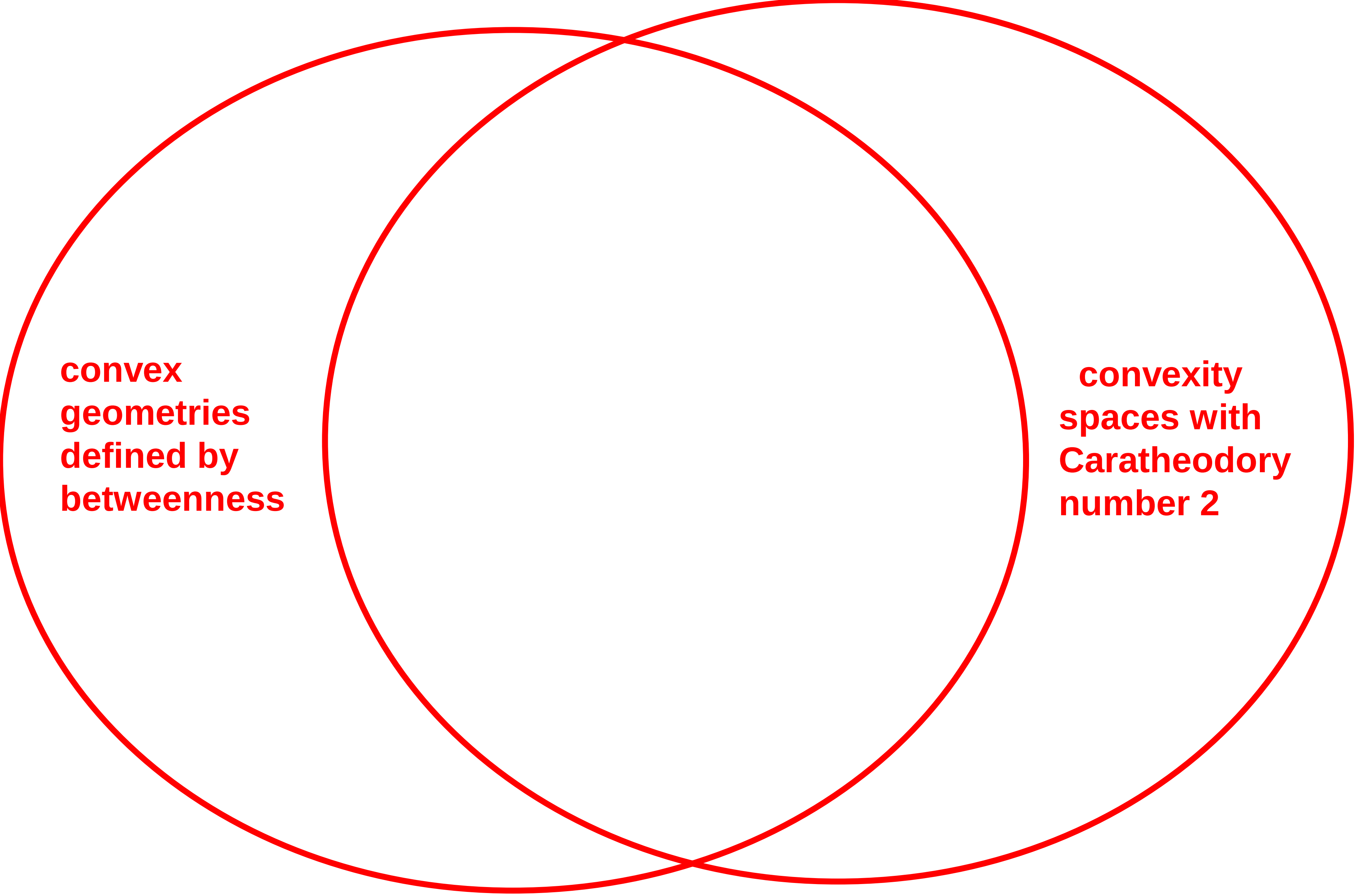
⇒ The problem of recognizing totally orderable abstract causal betwennesses is $\mathcal{NP}$-complete.


convex
geometries
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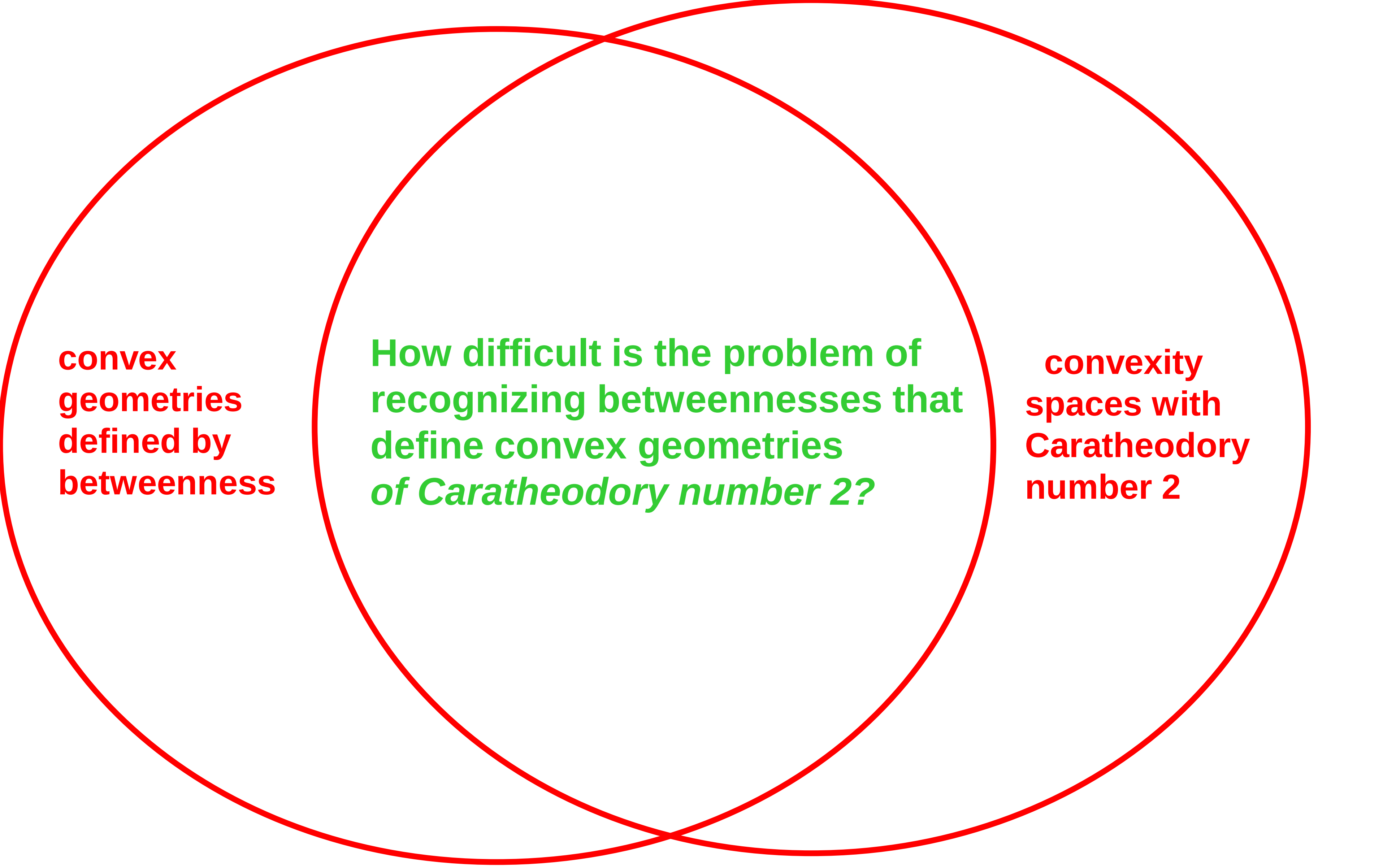
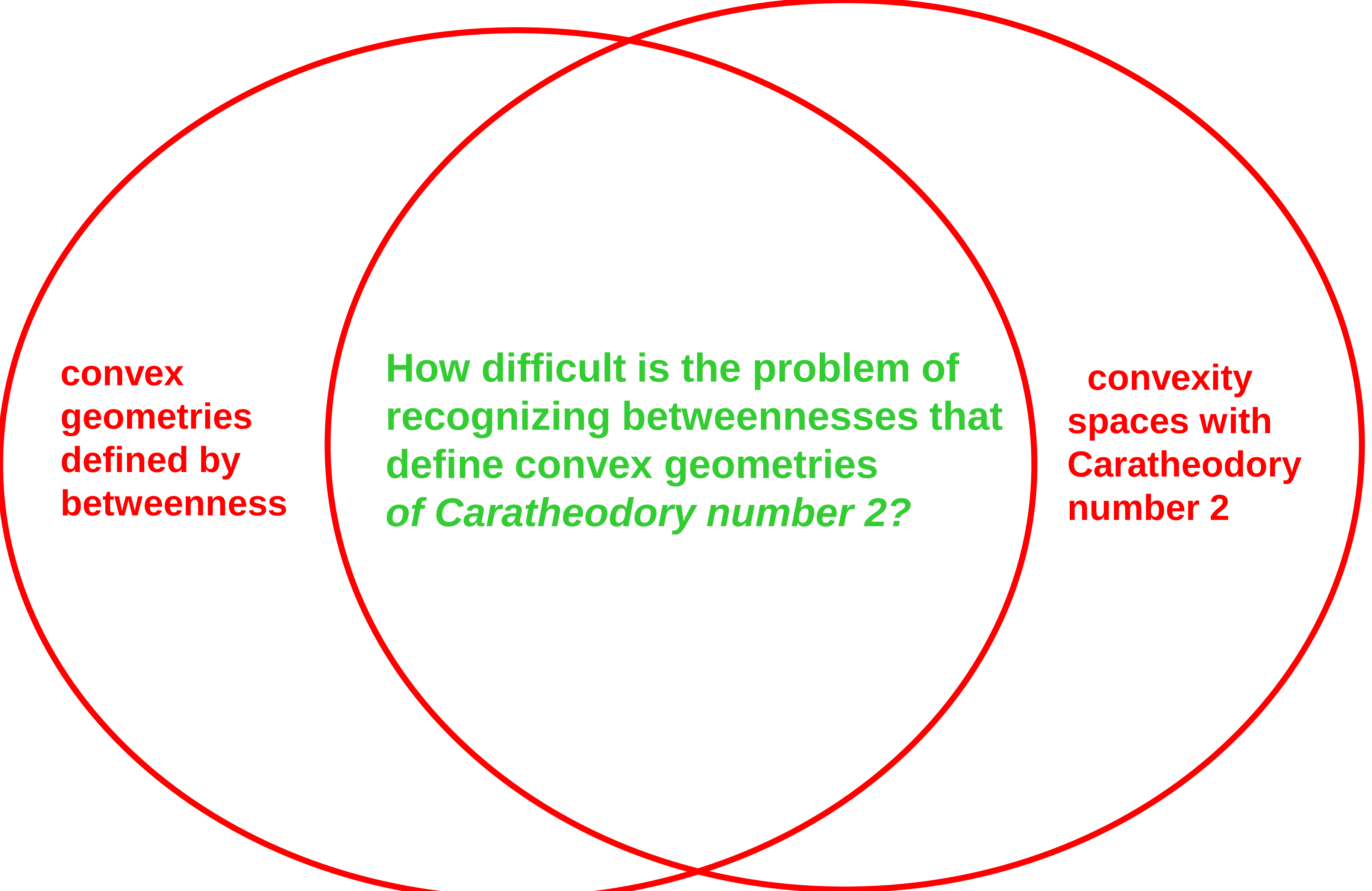

convex geometries defined by betweenness
How difficult is the problem of recognizing betweennesses that define convex geometries *of Caratheodory number 2?*
convexity spaces with Caratheodory number 2



Work in progress: Laurent Beaudou, Ehsan Chiniforooshan, V.C.