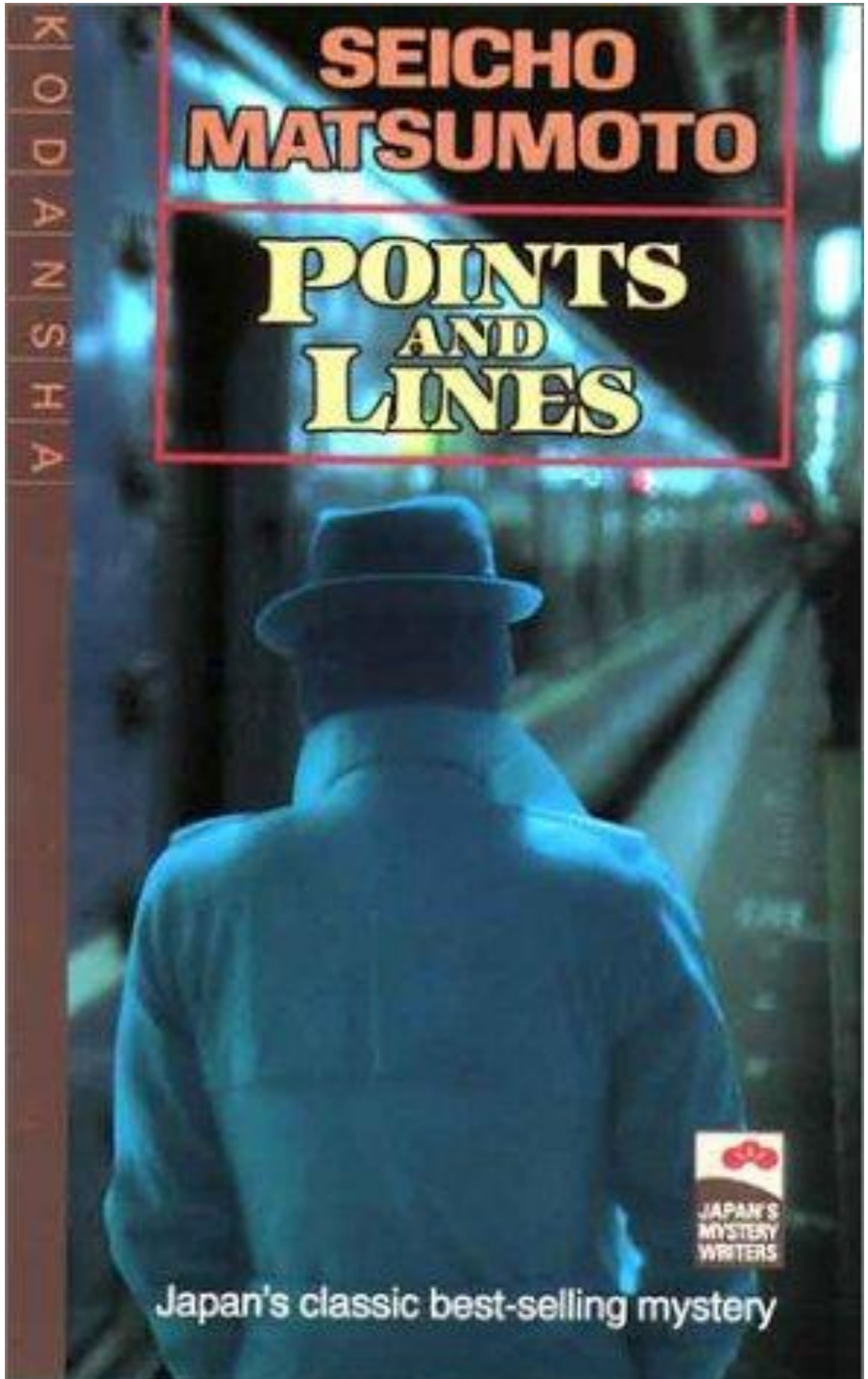
Vašek Chvátal
Department of Computer Science and Software Engineering
Concordia University
Montreal, Quebec, Canada


SEICHO
MATSUMOTO
POINTS
AND
LINES
KODANSHA
JAPAN'S
MYSTERY
WRITERS
Japan's classic best-selling mystery

Educational Times, March 1893

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James Joseph Sylvester

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A four-line solution … containing two distinct flaws

Sylvester-Gallai theorem:

If n points in the plane ($n > 1$) do not lie on a single line, then some line passes through precisely two of them

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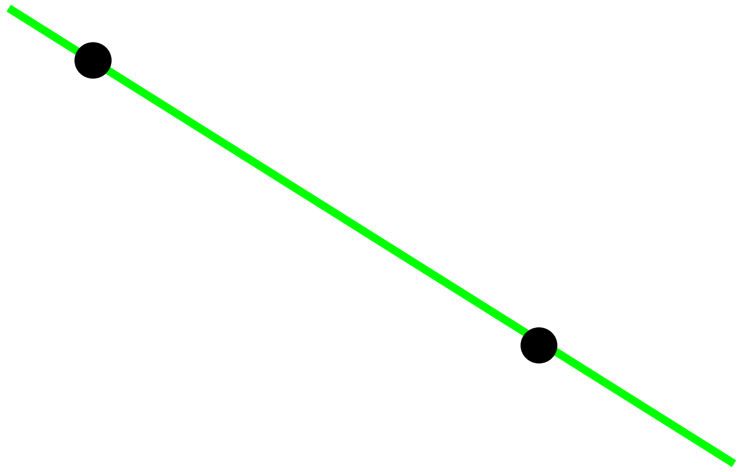

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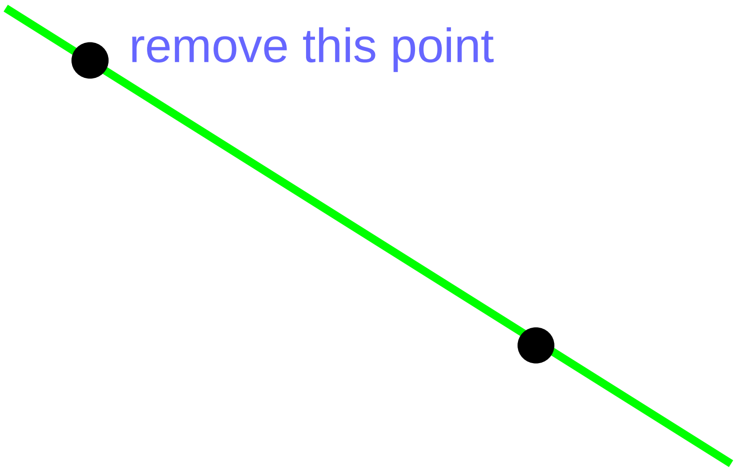

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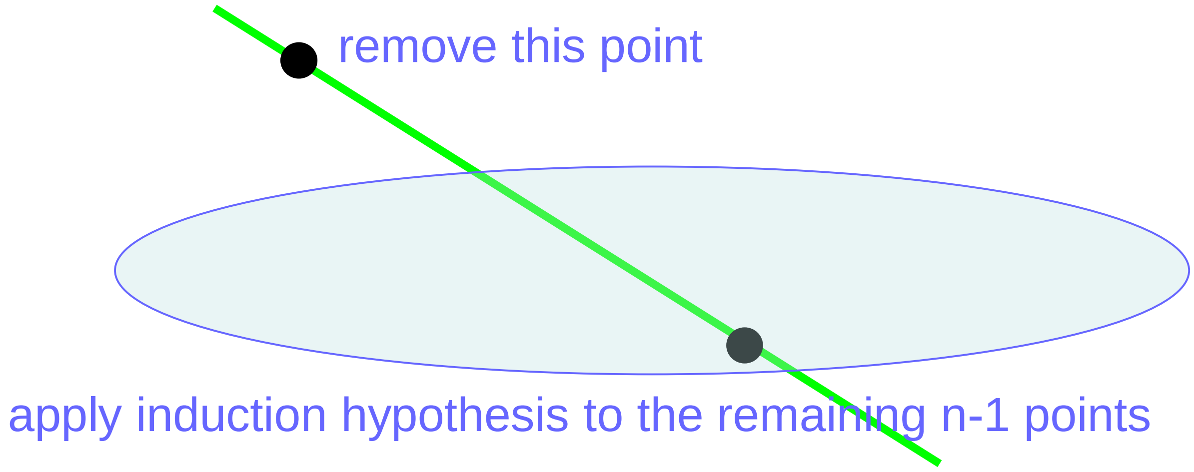

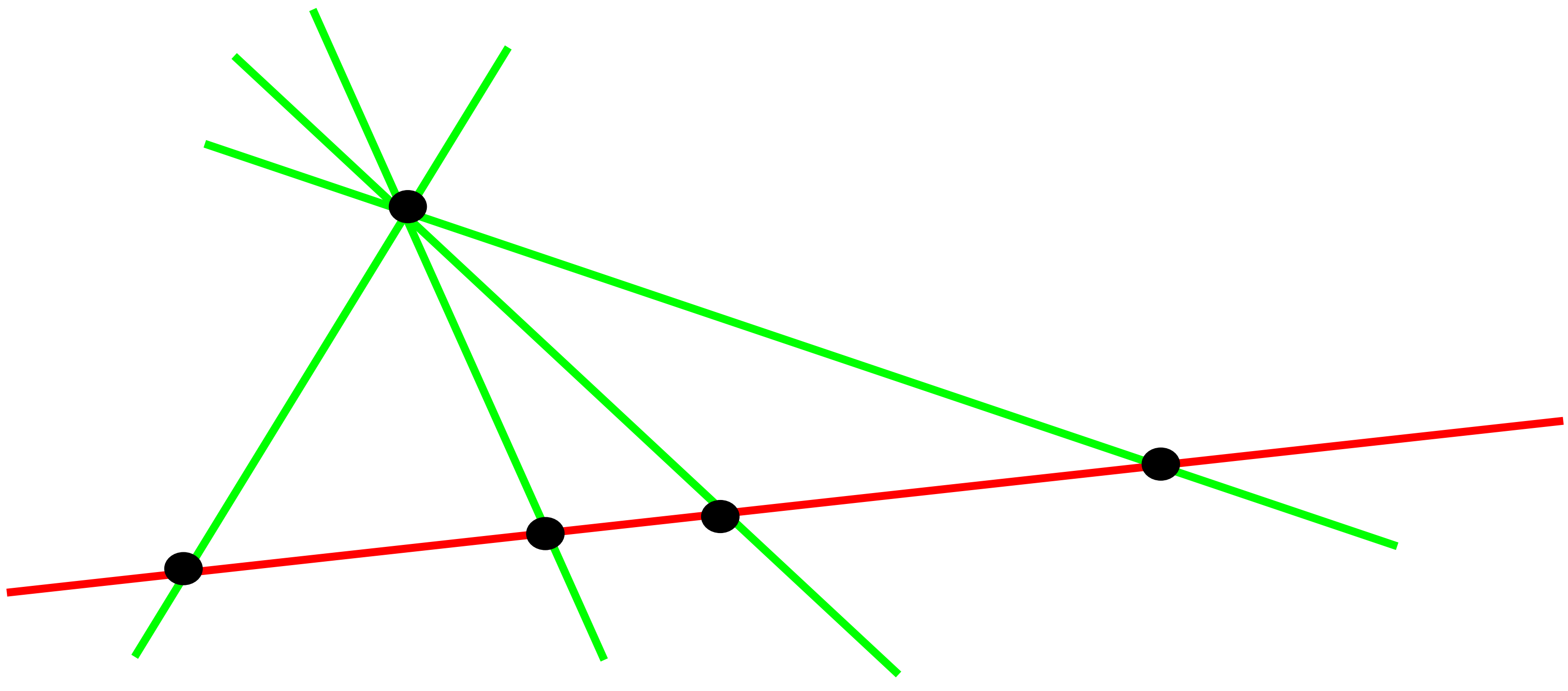

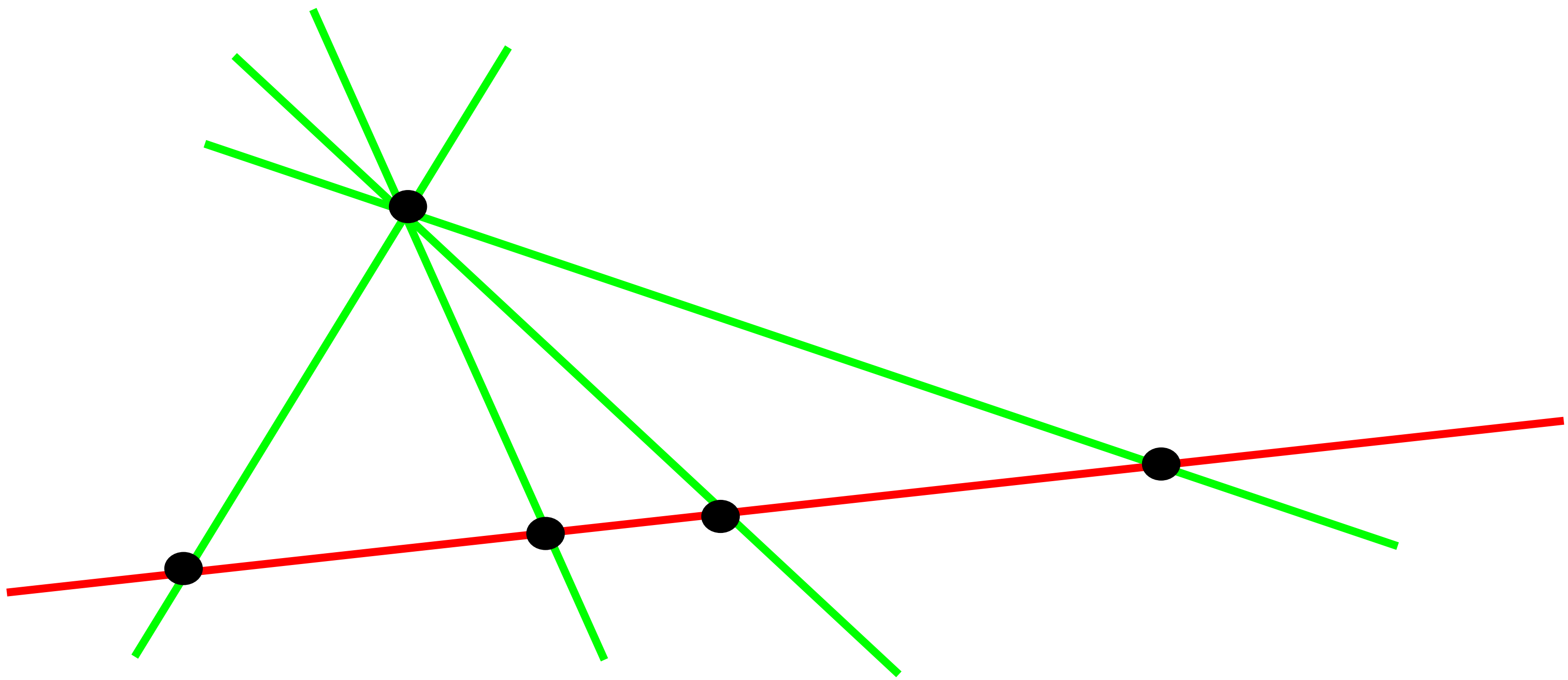

near-pencil

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Combinatorial generalization

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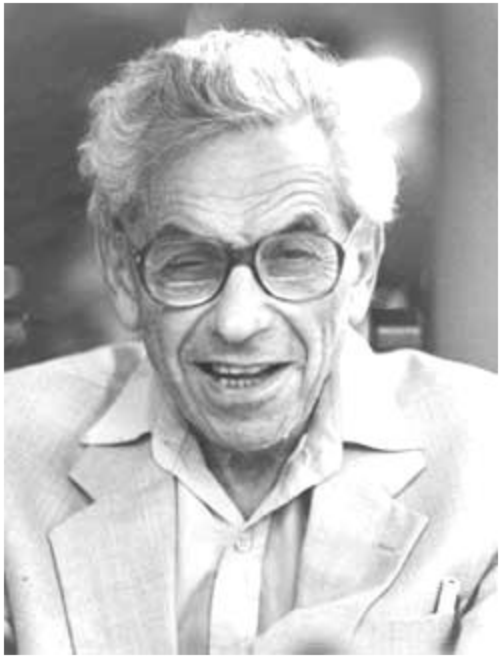
Combinatorial generalization



Nicolaas de Bruijn

Paul Erdős

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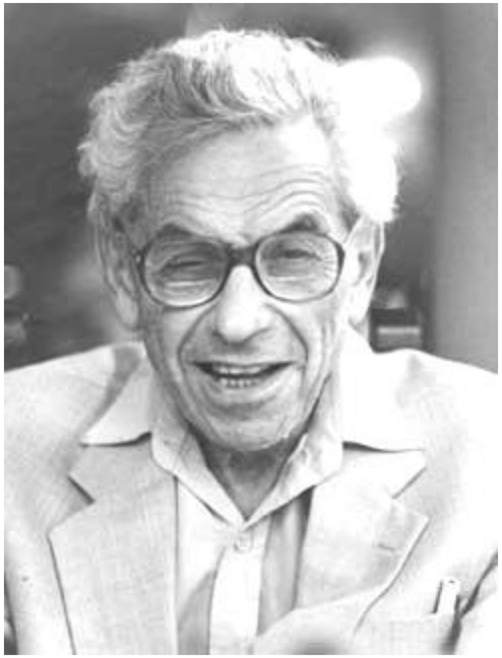
Combinatorial generalization



Nicolaas de Bruijn **Paul Erdős**

Let V be a finite set and let E be a family of of proper subsets of V such that every two distinct points of V belong to precisely one member of E. Then the size of E is at least the size of V. Furthermore, the size of E equals the size of V if and only if E is either a near-pencil or else the family of lines in a projective plane.

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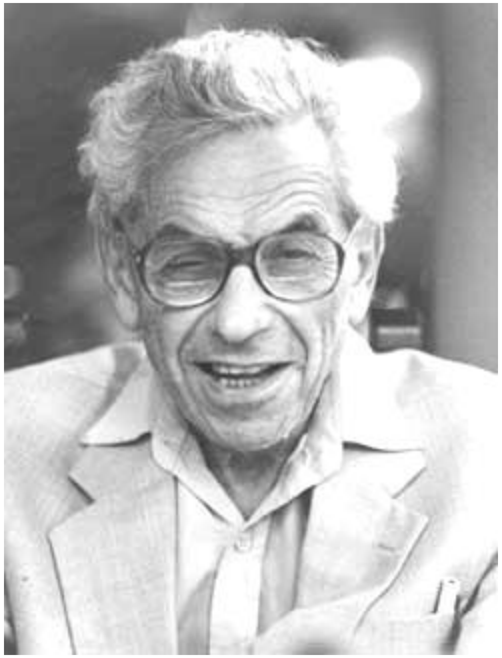

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On a combinatorial problem, *Indag. Math.* **10** (1948), 421--423

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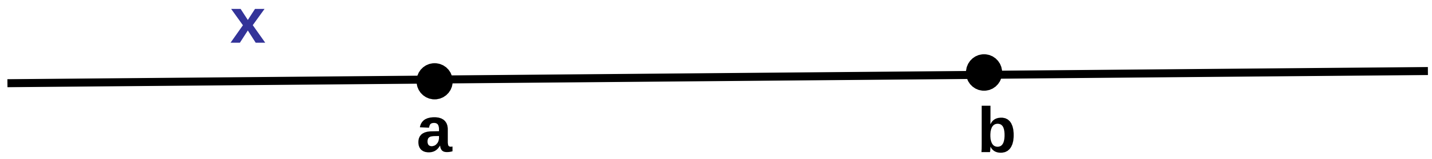
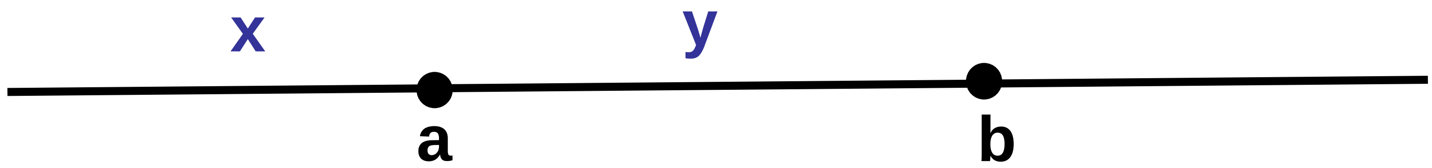
Question (Xiaomin Chen and V.C. 2006):

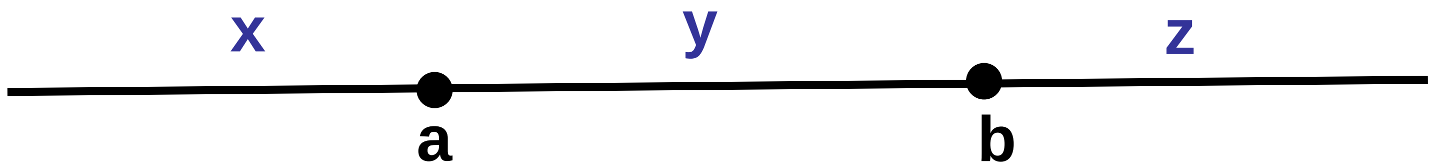
True or false? In every metric space on n points, there are at least n distinct lines or else some line consists of all these n points.


a
b


a
b


x
a
b


x
y
a
b


x
y
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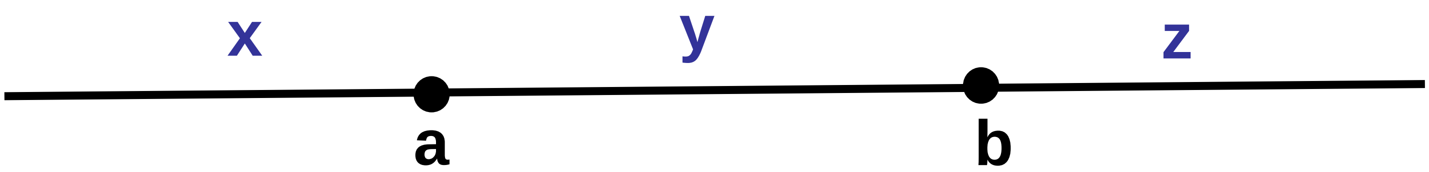
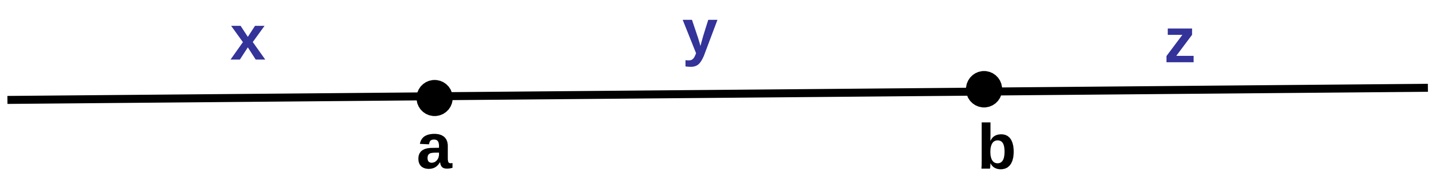
b



Observation

Line *ab* consists of
all points *x* such that *dist(x,a)+dist(a,b)=dist(x,b),*
all points *y* such that *dist(a,y)+dist(y,b)=dist(a,b),*
all points *z* such that *dist(a,b)+dist(b,z)=dist(a,z).*



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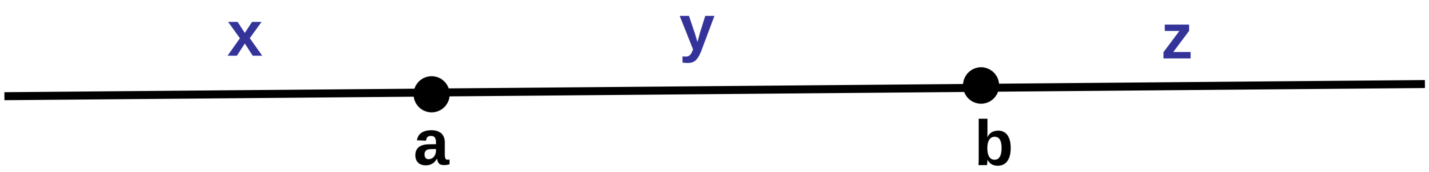
This can be taken for a definition of a line L(*ab*)

in an arbitrary metric space



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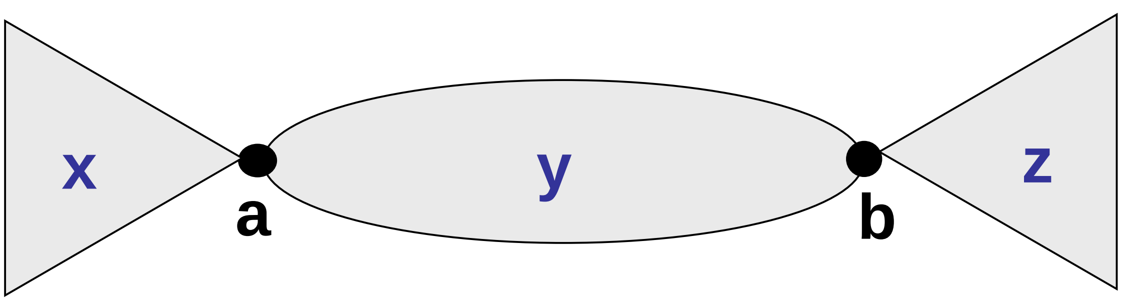

Lines in metric spaces can be exotic

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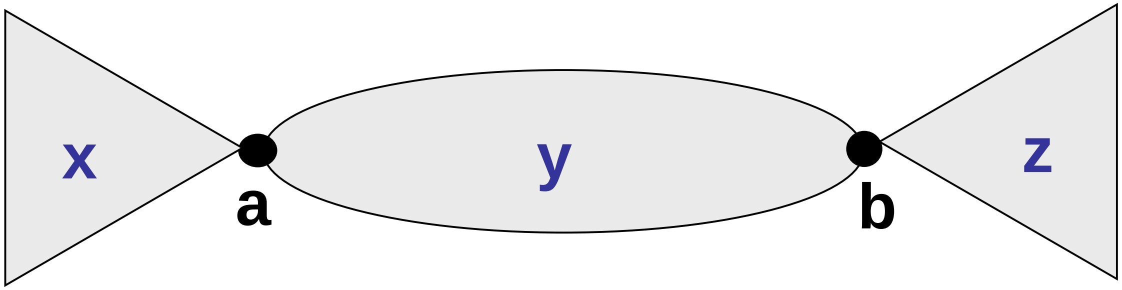
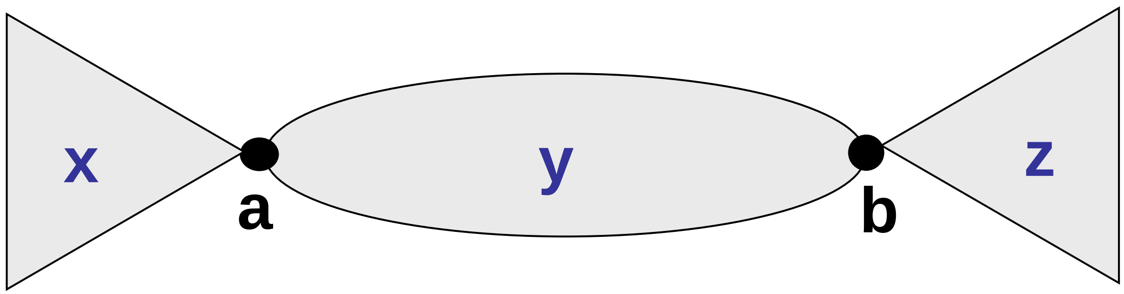
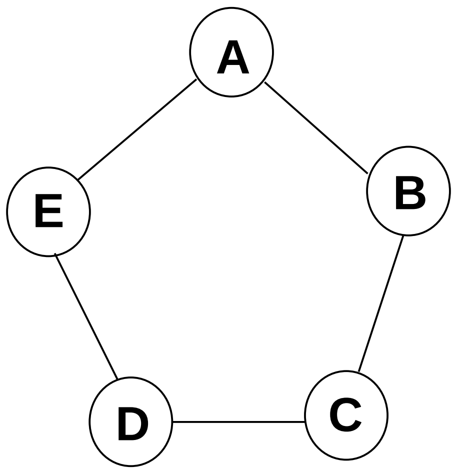
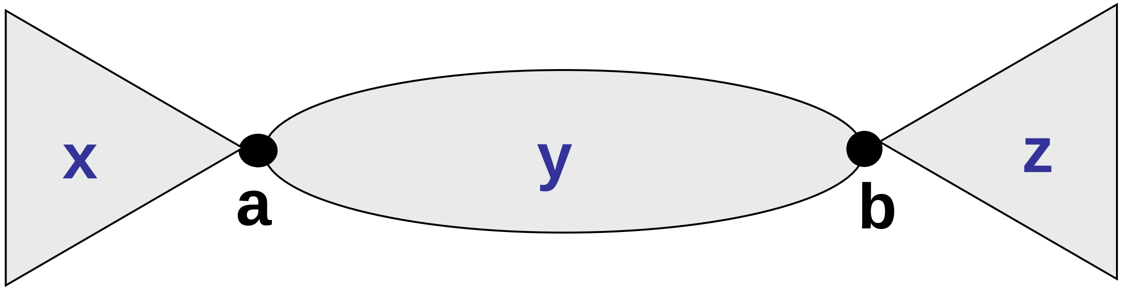
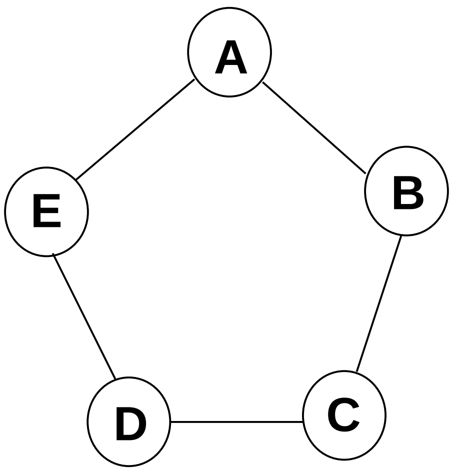
One line can hide another!


x
y
z
a
b

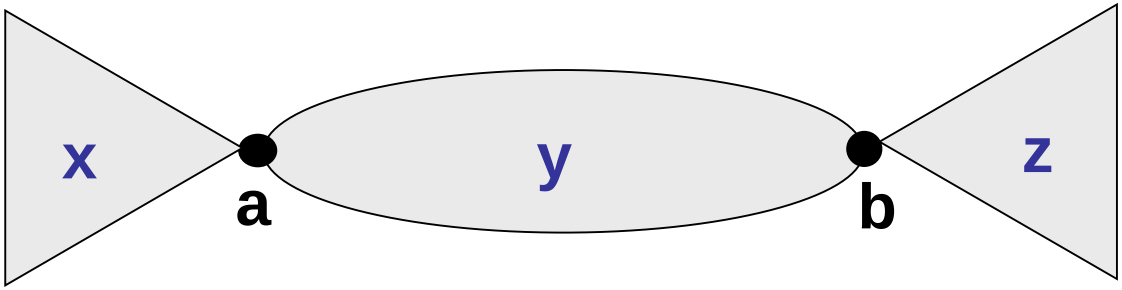

x
y
z
a
b

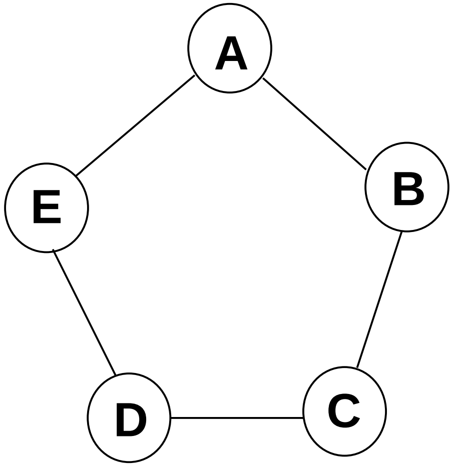

A
B
C
D
E





L(AB) = {E,A,B,C}





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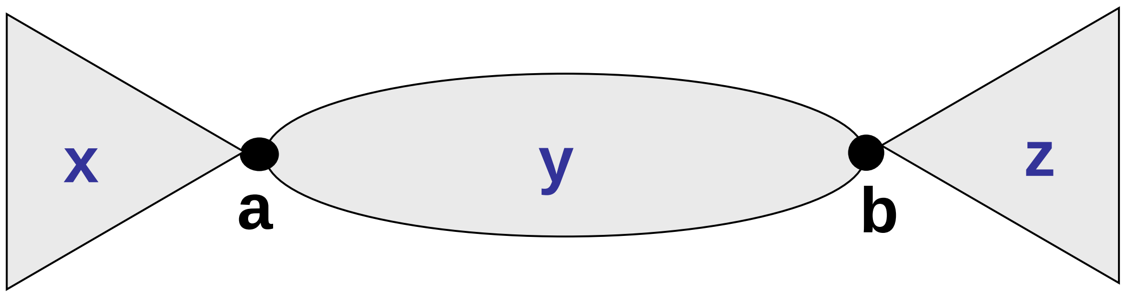
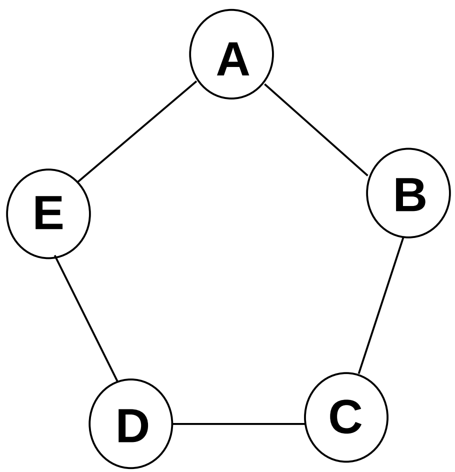
$L(AC) = \{A,B,C\}$





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One line can hide another!

True or false? In every metric space on n points, there are at least n distinct lines or else some line consists of all these n points.

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Partial answer (Ida Kantor and Balász Patkós 2012):

Every *nondegenerate* set of n points in the plane determines at least n distinct *Manhattan lines* or else one of its Manhattan lines consists of all these n points.

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Manhattan lines **=** lines in the L_1 metric

Manhattan distance

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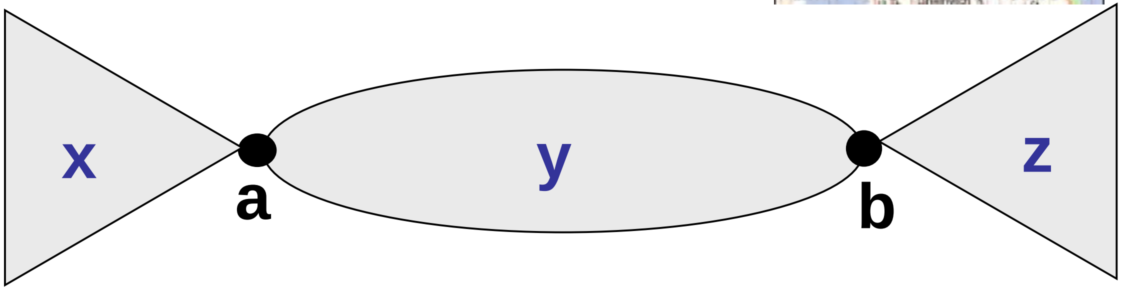
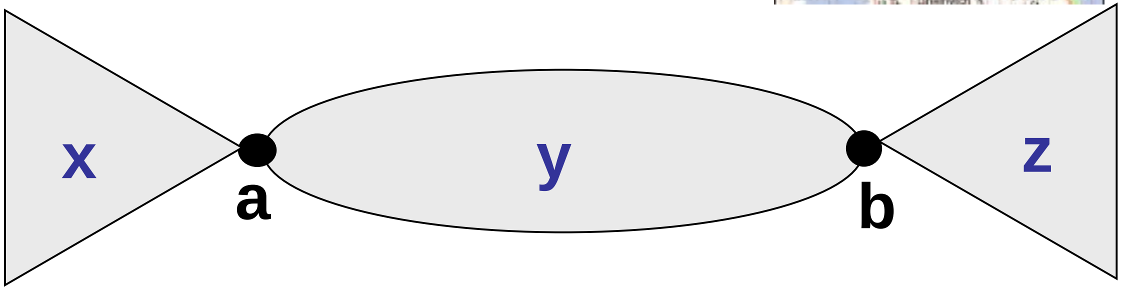


becomes

Manhattan distance



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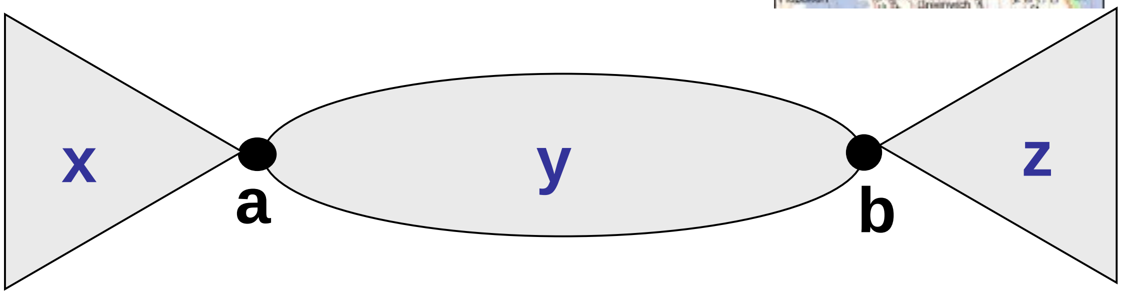

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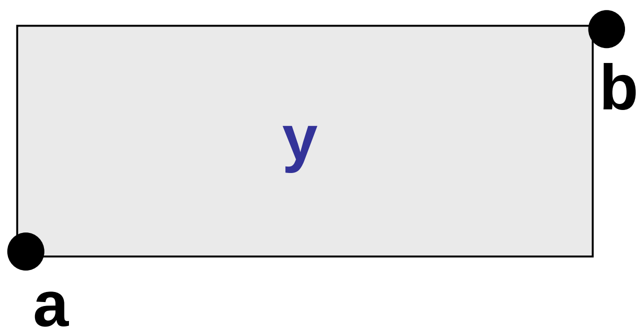
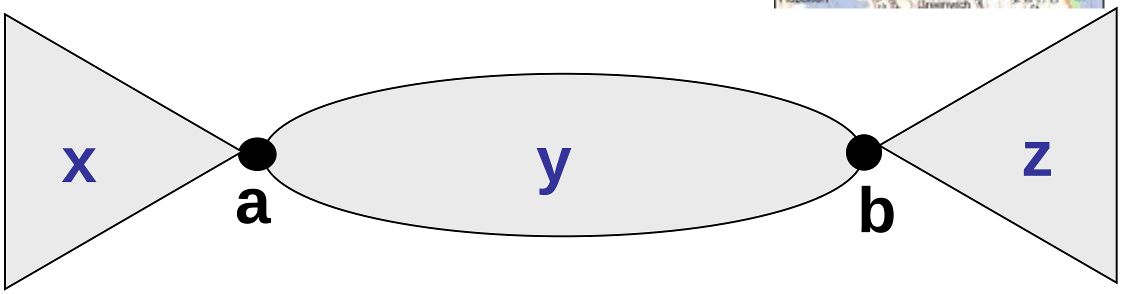


Manhattan distance



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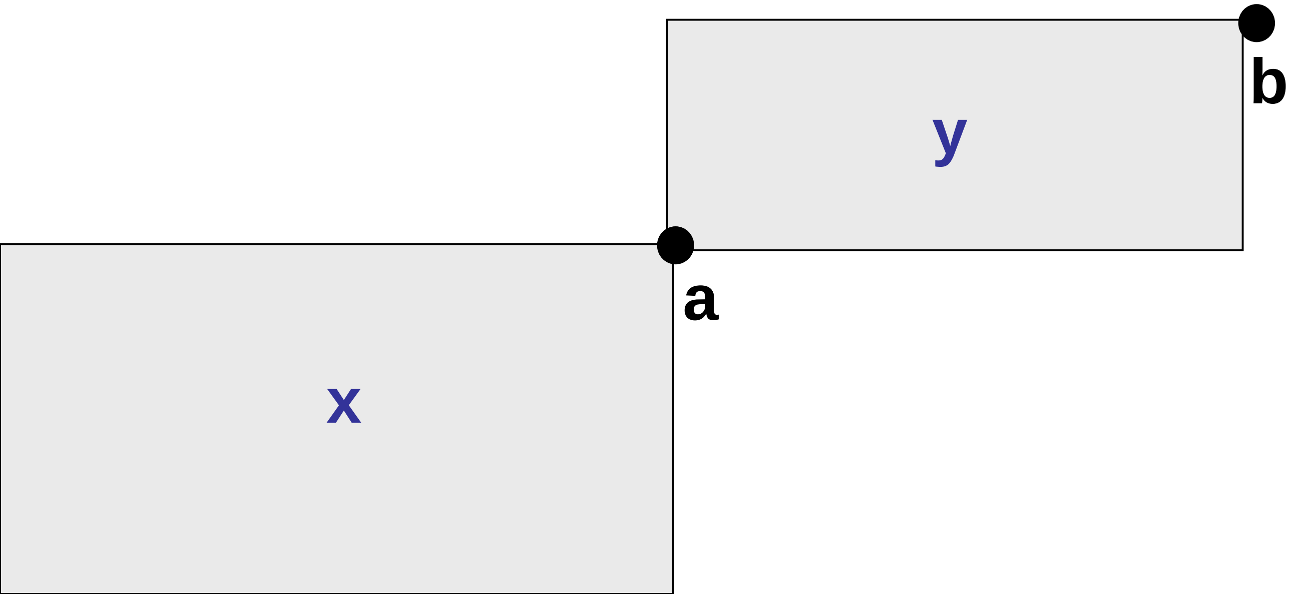
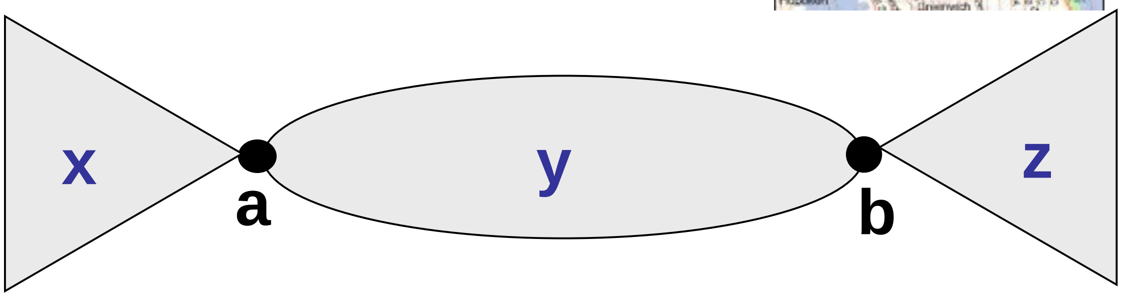
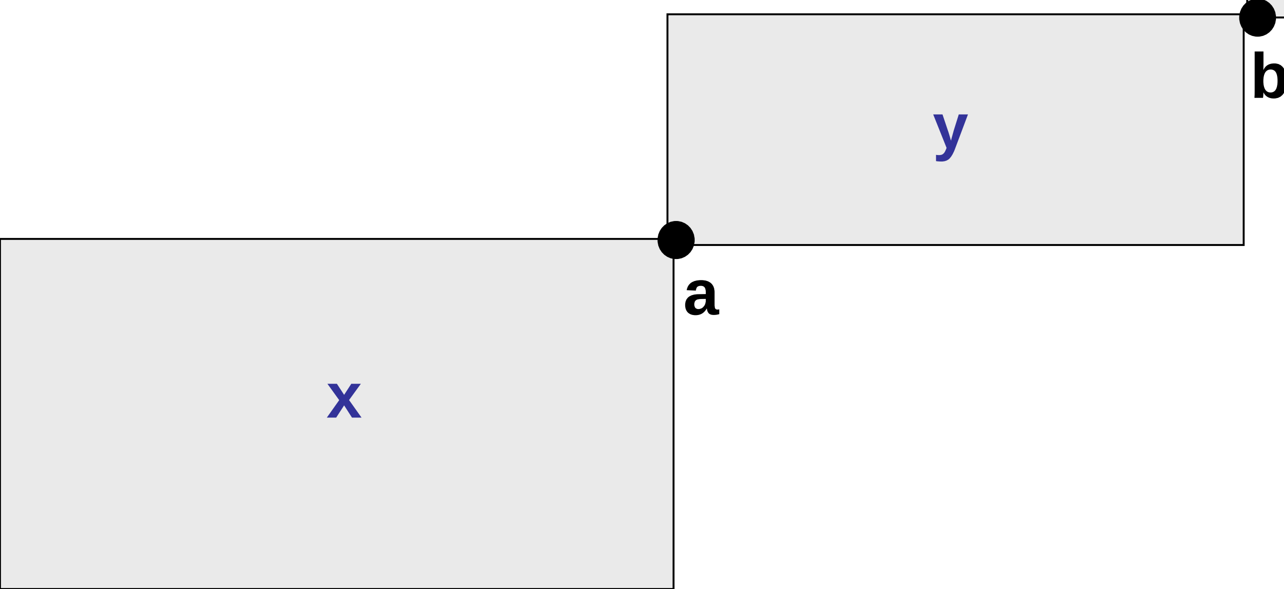
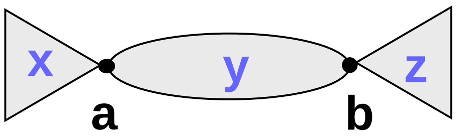


Manhattan distance



becomes




x
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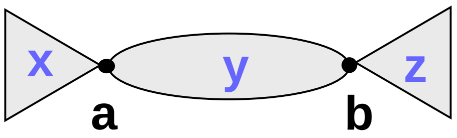
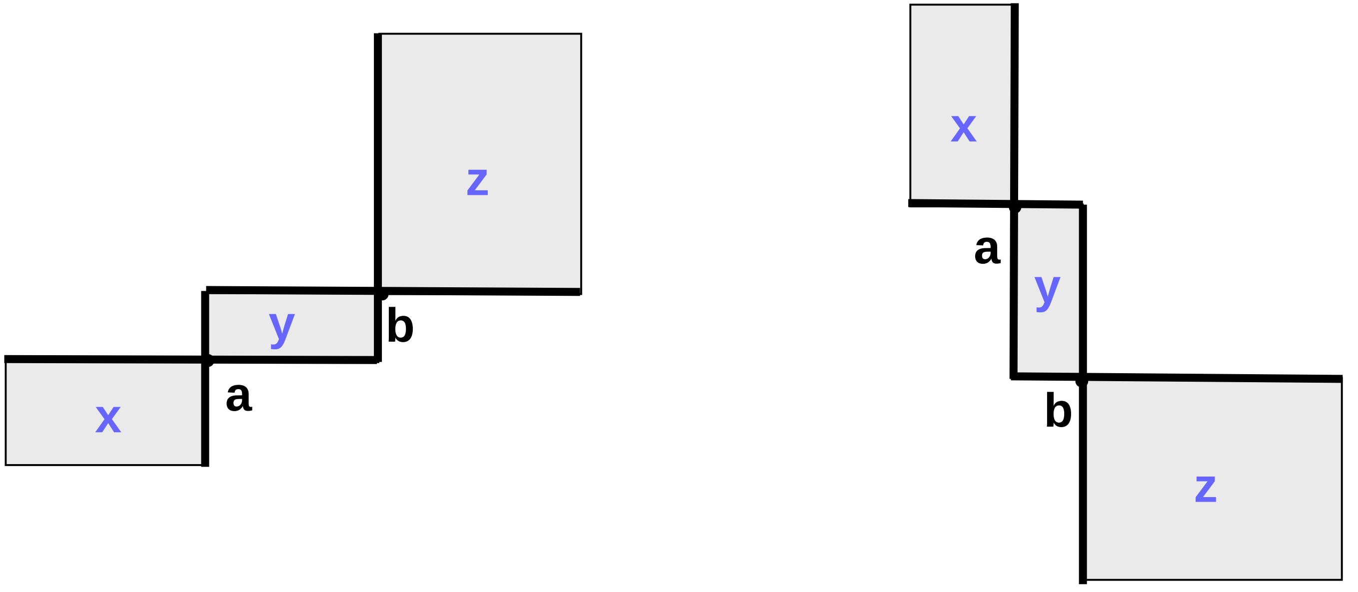
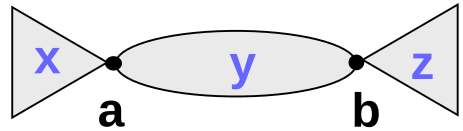


typical Manhattan lines:





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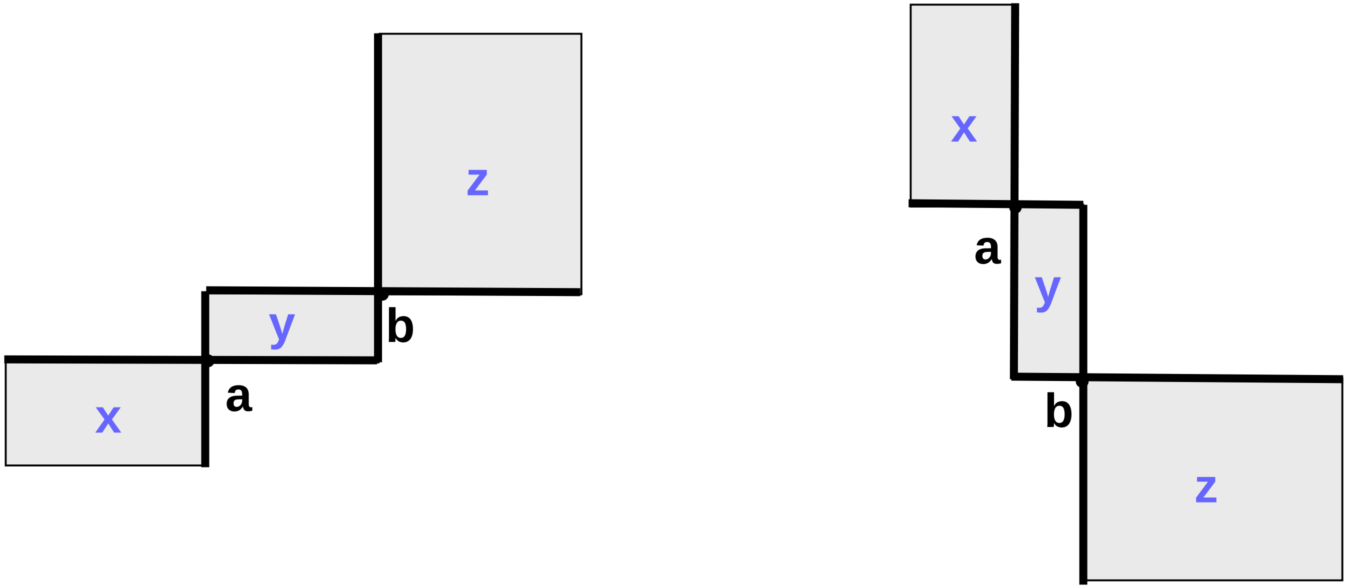
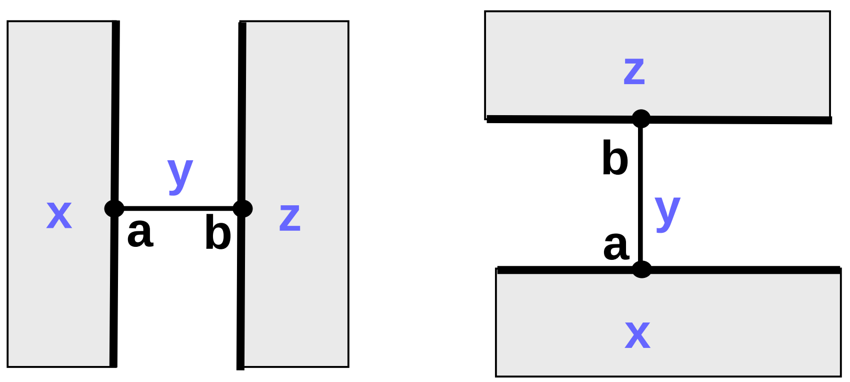


degenerate Manhattan lines:



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Theorem (Ida Kantor September 2018):

Every set of *n* points in the plane
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True or false? In every metric space on n points, there are at least n distinct lines or else some line consists of all these n points.

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Theorem (Pierre Aboulker, Xiaomin Chen, Guangda Huzhang, Rohan Kapadia, Cathryn Supko 2014):

In every metric space on n points
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there are at least $n/15$ and $\Omega(n^{4/3})$ distinct lines or else
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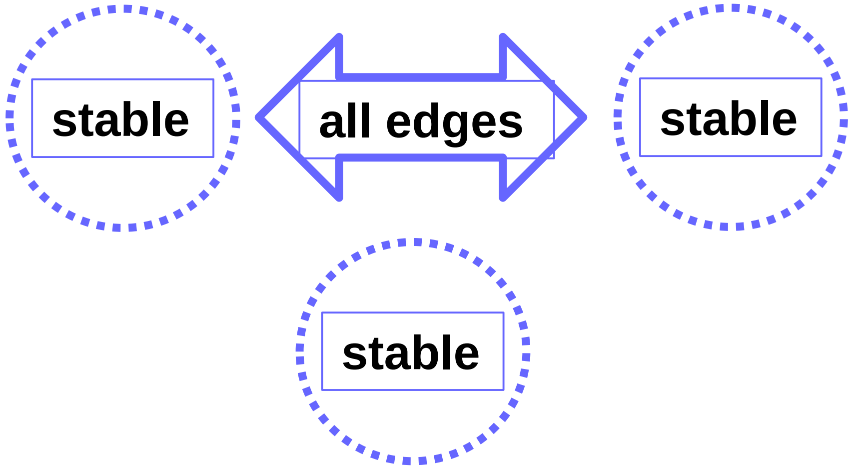
Bisplit graphs



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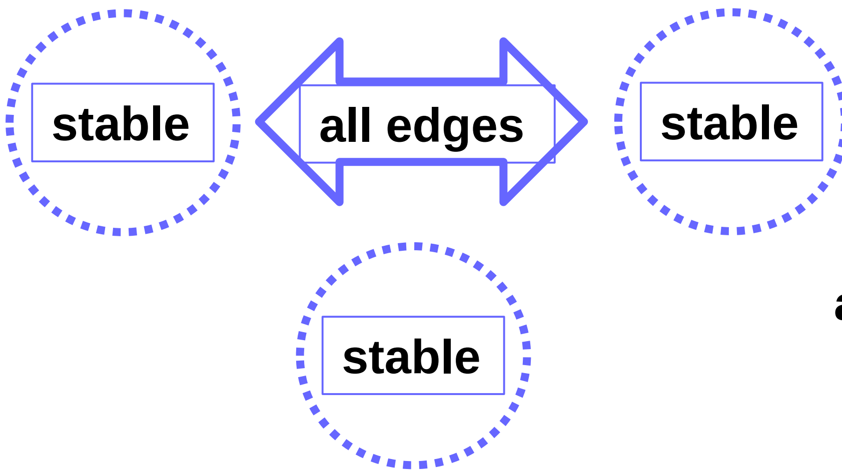

are comparability graphs

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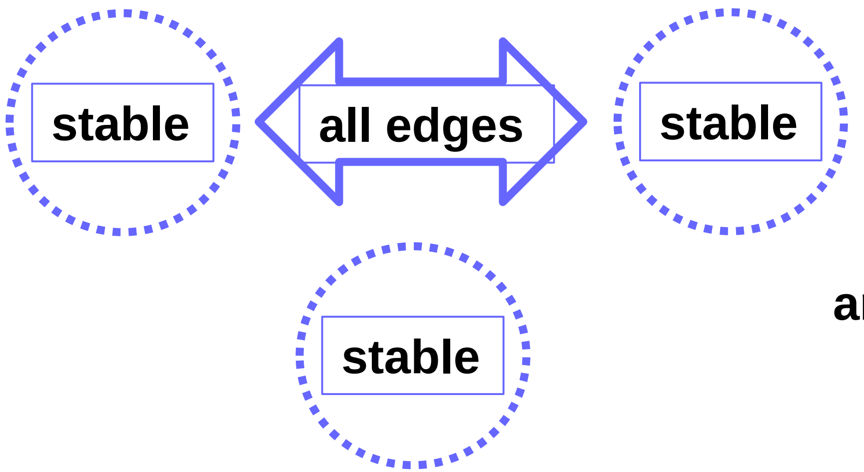

are comparability graphs

Do all comparability graphs have the De Bruijn-Erdős property?

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True in every metric space defined by a graph arising from chordal graphs by repeated substitutions and gluing vertices.

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True in every metric space defined by a graph arising from chordal graphs by repeated substitutions and gluing vertices.

All these graphs are (house, hole, domino)-free

Do all (house, hole, domino)-free graphs have the De Bruijn-Erdős property?

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Theorem (P.A, X.C., G.H., R.K., C.S. 2014):

> In every *n*-vertex graph,
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Exercise: In every n- vertex complete multipartite graph, there are $\Omega(n^{3/2})$ distinct lines and at least n distinct lines or else some line consists of all these n points.

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Each of $K(3,3,4)$, $K(1,3,3,3)$ and the complement of the Petersen graph has 15 lines.

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Theorem (X.C., G.H., P.M., K.Y. 2015):

Every two lines in a graph meet in at most one vertex
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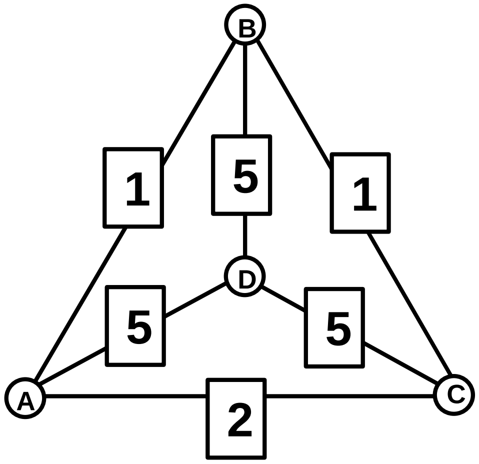
{{A,B,C},{A,D},{B,D},{C,D}}
is the family of lines in the metric space



but it is NOT the family of lines in any (connected) graph.

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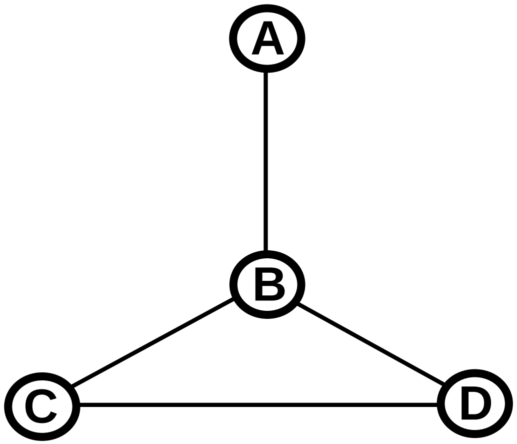

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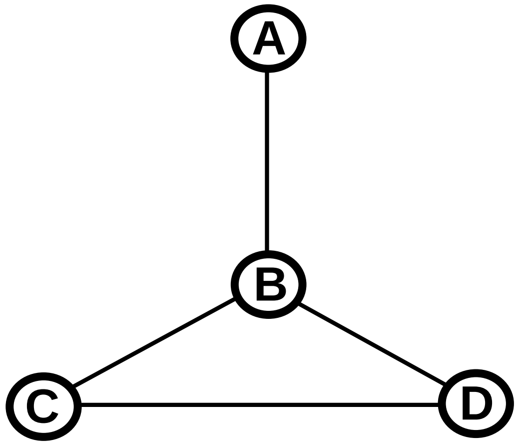

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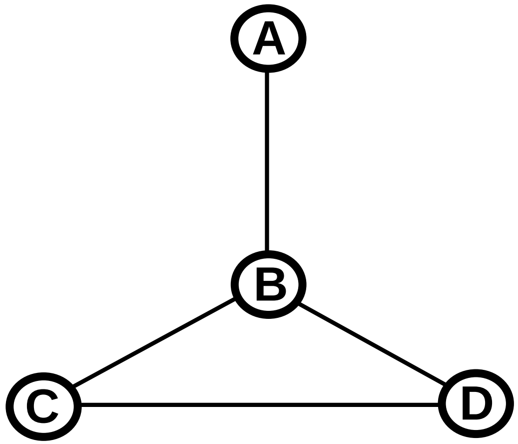

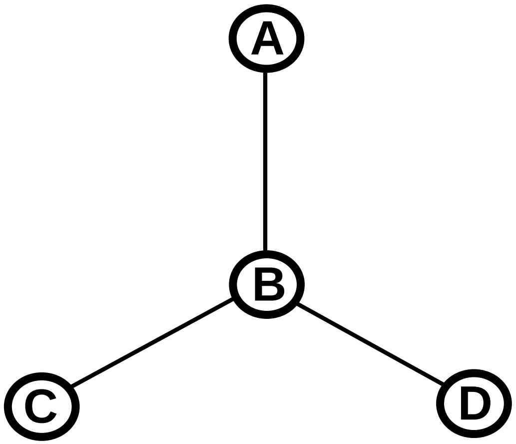

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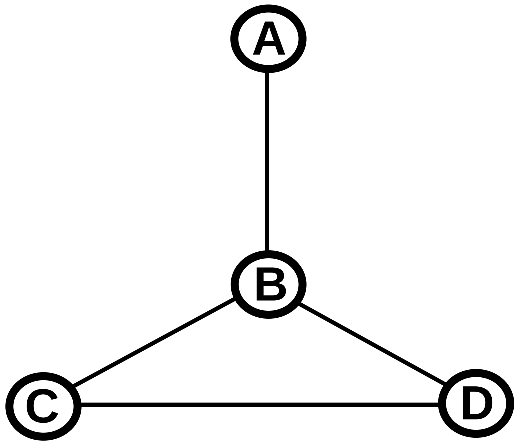

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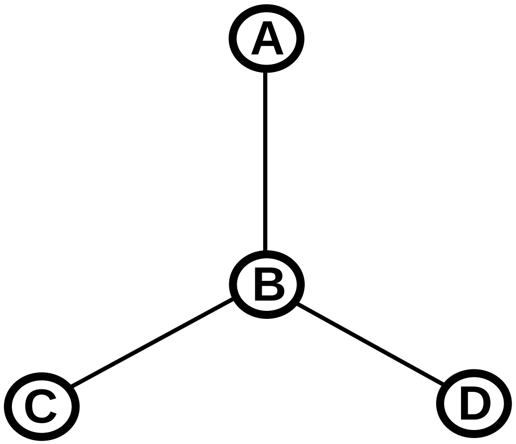

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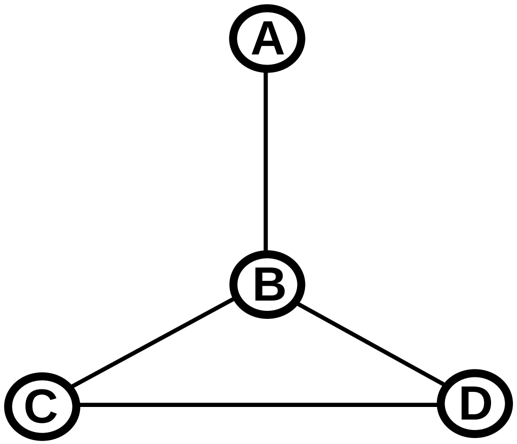

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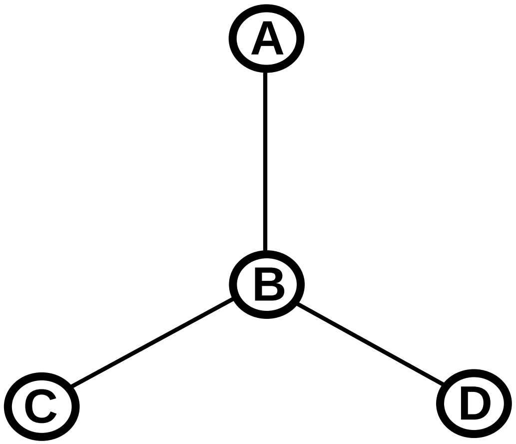

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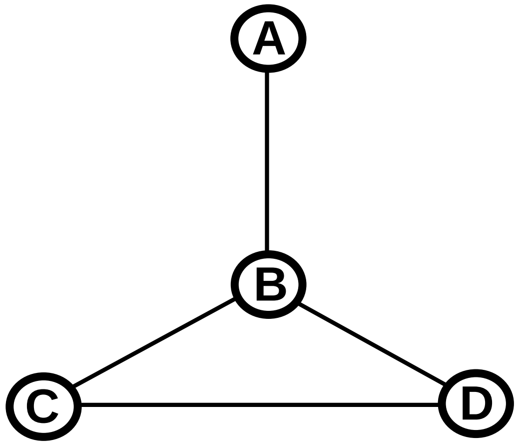

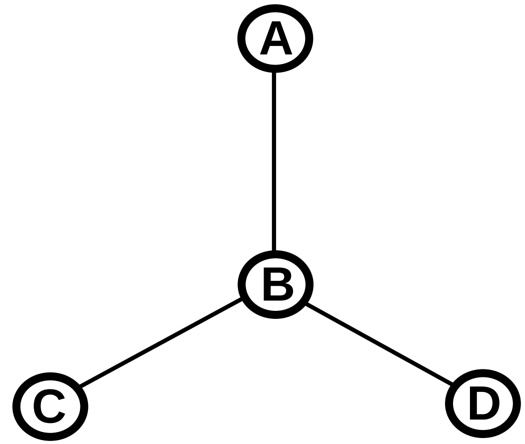

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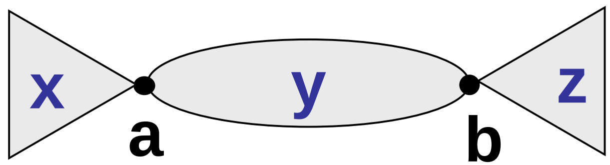
How difficult is it to recognize partitions that are induced by some metric space?

Partition into blocks {AB,AC}, {AD}, {BC}, {BD}, {CD}
is not induced by any metric space

Definition: line *ab* in a metric space consists of
all points *x* such that $dist(x,a)+dist(a,b)=dist(x,b)$,
all points *y* such that $dist(a,y)+dist(y,b)=dist(a,b)$,
all points *z* such that $dist(a,b)+dist(b,z)=dist(a,z)$.



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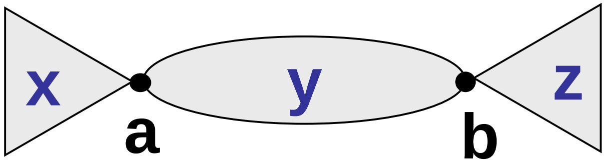
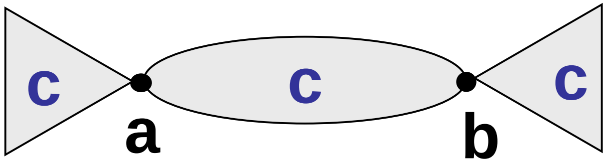


In short:
line *ab* in a metric space consists *a, b,* and all points *c* such that one of *a, b, c* lies between the other two.



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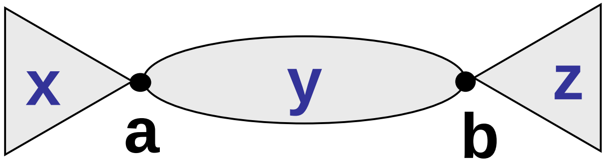

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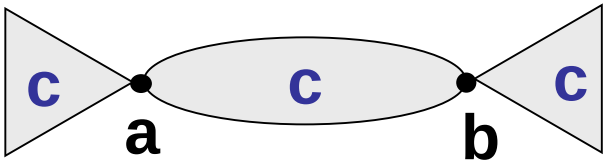

Every metric space induces a *3*-uniform hypergraph
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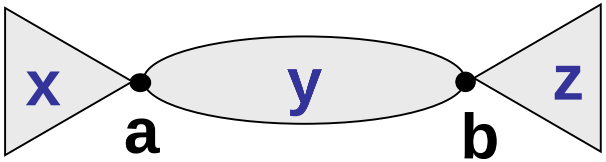

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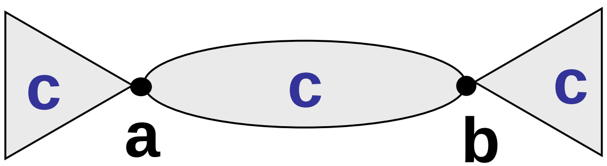

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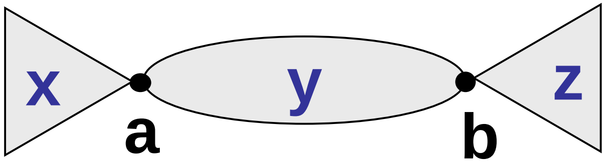

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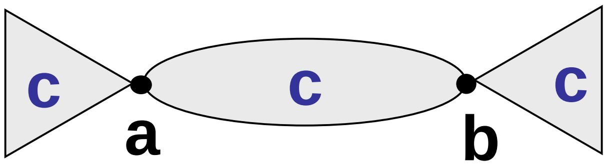

Every metric space induces a *3*-uniform hypergraph (*metric hypergraph*)
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True or false? In every metric hypergraph with *n vertices*, there are at least *n* distinct lines or else some line consists of all these *n* points.

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Xiaomin Chen and V.C. 2006

There are arbitrarily large 3-*uniform hypergraphs with n vertices, no universal line, and* $\exp(O(\log^{1/2} n))$ *distinct lines*

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Theorem (Pierre Aboulker, Adrian Bondy, Ehsan Chiniforooshan, Xiaomin Chen, V.C., and Peihan Miao 2006)

In every 3-uniform hypergraph with n vertices, there are at least $(2 - o(1)) \lg n$ *distinct lines or else some line consists of all these n points.*

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The *Fano hypergraph*
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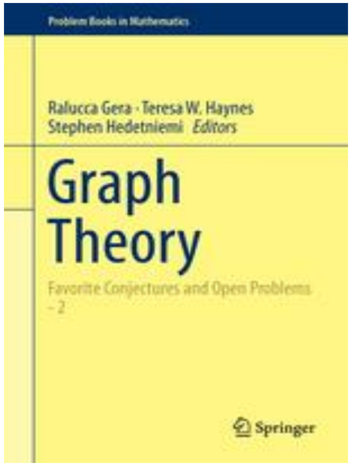
How difficult is it to recognize metric hypergraphs?



Graph Theory
Favorite Conjectures and Open Problems - 2
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Chapter 13:
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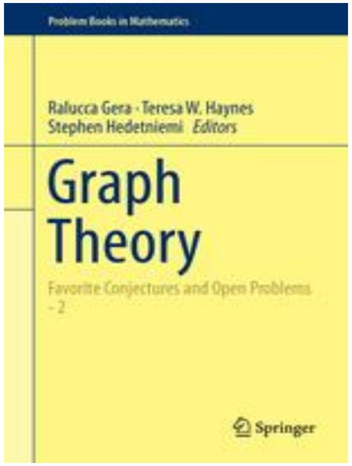

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arXiv:1812.06288 [math.CO]