Generalizing one of the De Bruijn – Erdos theorems

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joint work with

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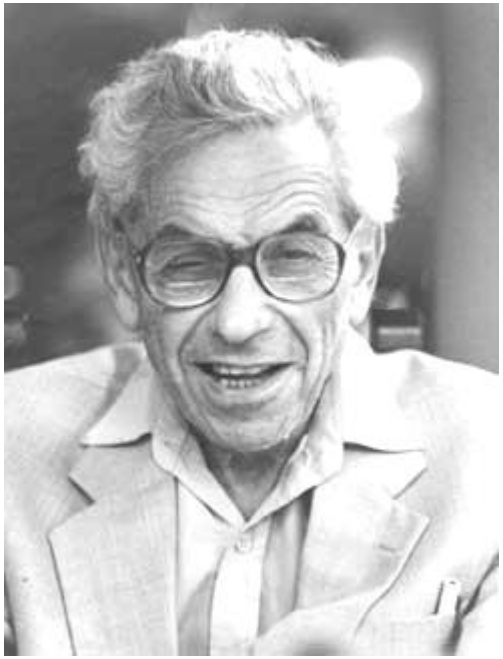
Nicolaas de Bruijn



Paul Erdős

On a combinatorial problem.
Indag. Math. **10** (1948), 421--423

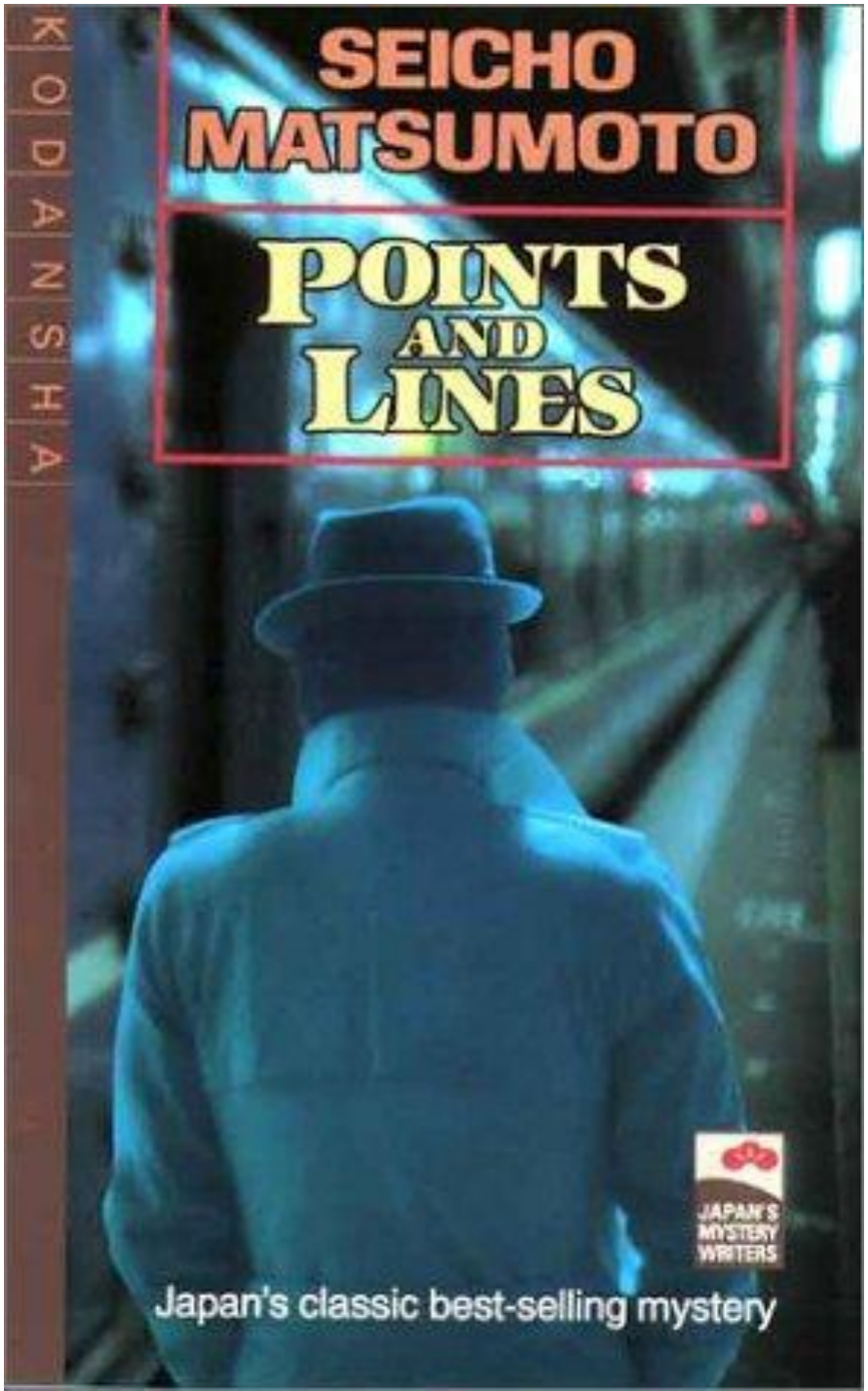
Sequences of points on a circle.
Indag. Math. **11** (1949), 46--49

A colour problem for infinite graphs and a problem in the theory of relations.
Indag. Math. **13** (1951), 369--373

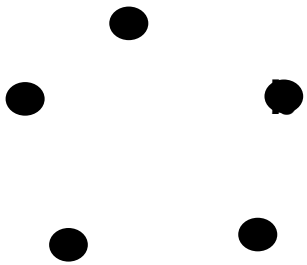
Some linear and some quadratic recursion formulas, I.
Indag. Math. **13**(1951), 374--382

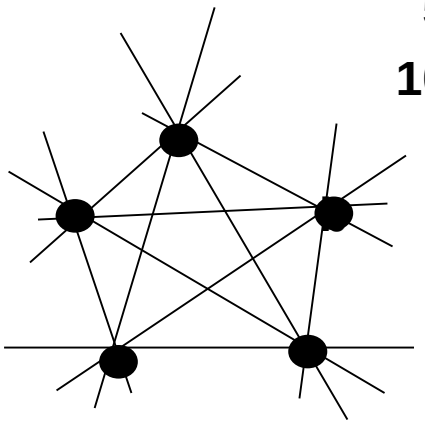
Some linear and some quadratic recursion formulas, II.
Indag. Math. **14** (1952), 152--163

On a recursion formula and on some Tauberian theorems.
J. Research Nat. Bur. Standards **50** (1953), 161--164


KODANSHA
SEICHO MATSUMOTO
POINTS AND LINES
JAPAN'S MYSTERY WRITERS
Japan's classic best-selling mystery

5 points



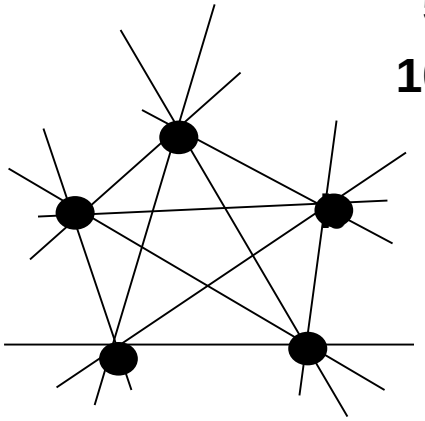


5 points

10 lines

5 points

10 lines



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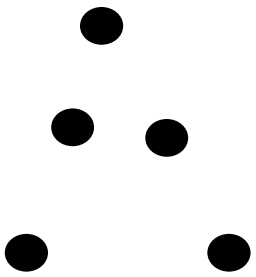

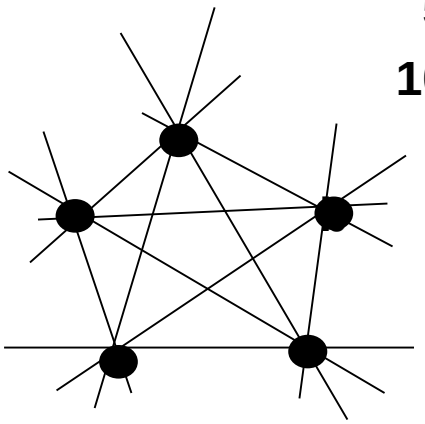

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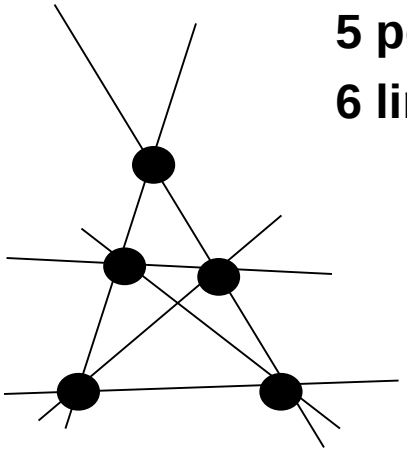
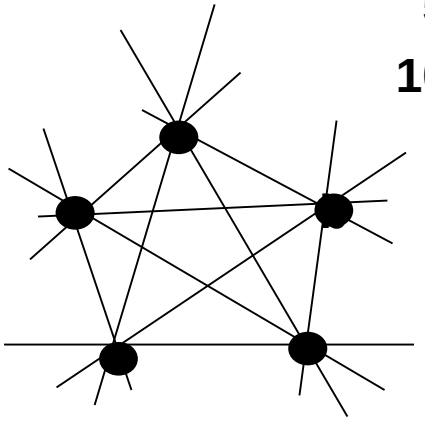


5 points

6 lines



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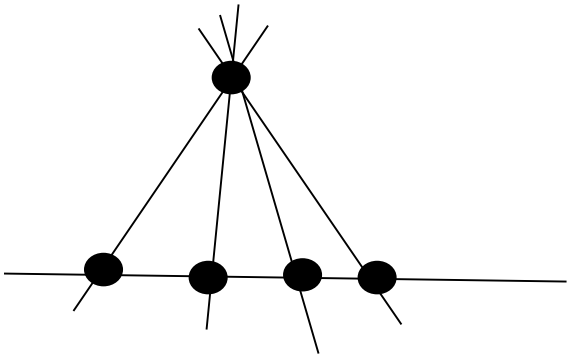
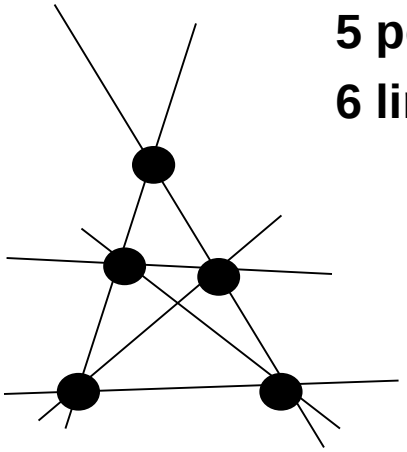


5 points, 5 lines



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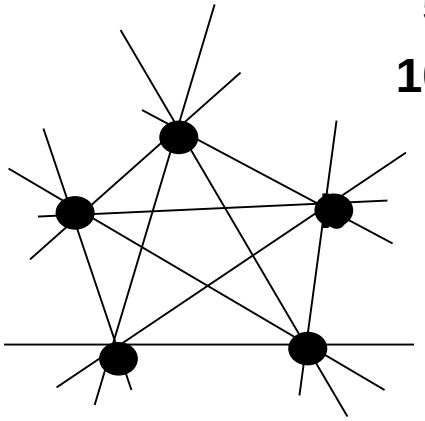

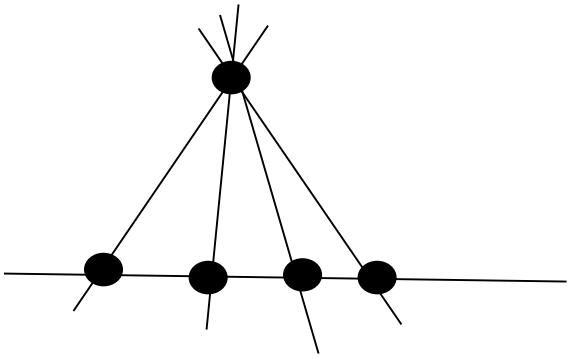

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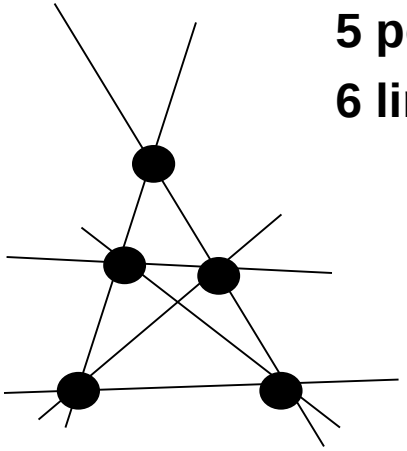
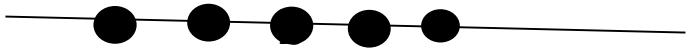
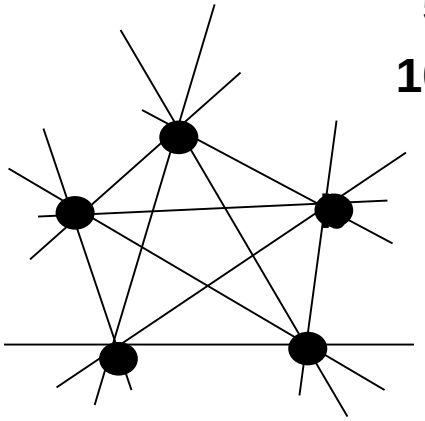


5 points, 1 line





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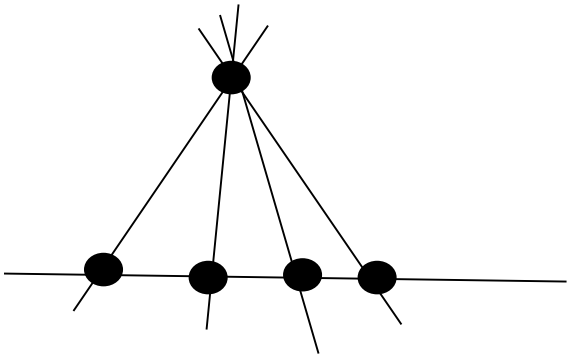
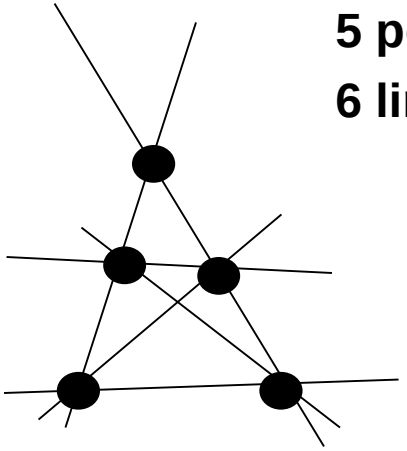


5 points, 5 lines

nothing between these two



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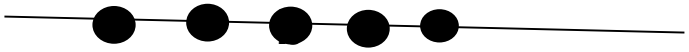

Every set of *n* points in the plane determines at least *n* distinct lines unless all these *n* points lie on a single line.

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The De Bruijn – Erdos theorem is a generalization of this theorem.

De Bruijn – Erdos theorem

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Set V of n elements called points

Family L of m subsets of V called lines

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Hypothesis

Every two points belong to precisely one line

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1. If m > 1, then m is at least n

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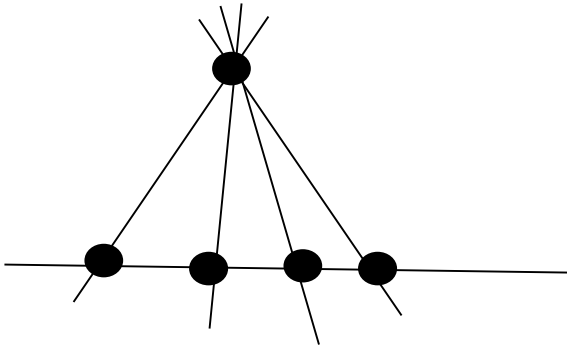
1. If m > 1, then m is at least n

2. m = n if and only if L is a near-pencil



or the family of lines of a projective plane

Lines in hypergraphs

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Definition: The *line uv in a 3-uniform hypergraph* consists of u, v, and all w such that {u,v,w} is a hyperedge.

Lines in hypergraphs can be exotic

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Vertices 1,2,3,4 and hyperedges {1,2,3}, {1,2,4}:

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line 12 = {1,2,3,4}, line 23 = {1,2,3}

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One line can hide another!

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Observation:

If 4 vertices induce 2 or 3 hyperedges,

then 2 of these vertices are in at least 2 lines.

Easy theorem

For every 3-uniform hypergraph,
these four properties are logically equivalent:

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For every family L of sets, these two properties are logically equivalent:

A. Every two points belong to precisely one member of L.

Easy theorem

**For every 3-uniform hypergraph,
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Easy corollary

For every family L of sets, these two properties are logically equivalent:

A. **Every two points belong to precisely one member of L.**

B. **L is the family of lines in a 3-uniform hypergraph, in which no 4 vertices induce 2 or 3 hyperedges.**

De Bruijn – Erdos theorem restated:

3-uniform hypergraph with n vertices

Hypothesis

No 4 vertices induce 2 or 3 hyperedges.

Conclusions

1. **If there are at least two lines,**

   then there at least n lines.

2. **There are precisely n lines if and only if the hypergraph**

   generates a near-pencil or the family of lines of a projective plane

De Bruijn – Erdos theorem restated:

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A warning example

3-uniform hypergraph with 11 vertices,
no 4 vertices induce 3 hyperedges,
no line consists of all 11 vertices,
there are only 10 distinct lines

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3-uniform hypergraphs in general

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These bounds are from Xiaomin Chen and V.C., Problems related to a De Bruijn – Erdos theorem, *Discrete Applied Mathematics* 156 (2008), 2101 - 2108.

A special case of the De Bruijn – Erdos theorem:

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Definition: A *metric hypergraph* has for its vertex set the ground set of a metric space and it has for its hyperedges all three-point sets {x,y,z} such that *dist* (x,y) + *dist* (y,z) = *dist* (x,z).

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True or false?

In every metric hypergraph with *n* vertices, there are at least *n* distinct lines or else some line consists of all *n* vertices.

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- in metric spaces arising from graphs, there are $\Omega(n^{2/7})$ distinct lines

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These bounds come from Ehsan Chiniforooshan and V.C.,
A De Bruijn - Erdos theorem and metric spaces,
Discrete Mathematics & Theoretical Computer Science
Vol 13 No 1 (2011), 67 - 74.