SOME UNKNOWN VAN DER WAERDEN NUMBERS

V. Chvátal

University of New Brunswick,
Fredericton, NB, Canada

Van der Waerden's theorem [1] may be formulated as follows:
Given any positive integer k and positive integers $t_1, t_2, \ldots, t_k$ there is an integer m such that given any partition

$$\{1,2,\ldots, m\} = V_1 \cup V_2 \cup \ldots \cup V_k \tag{1}$$

there is always a class V_j containing an arithmetic progression of $t_j + 1$ terms. Let us denote the least m with this property by $W(k;t_1,t_2,\ldots t_k)$. Using the computing facilities of the University of New Brunswick, I found $W(3;2,2,2) = 27$ and the following numbers $W(2;t_1,t_2)$:

$t_1 \backslash t_2$	2	3	4	5	6
2	9	18	22	32	46
3	18	35	55		
4	22	55			
5	32				
6	46				

By a *good partition*, we shall mean a partition of the form (1) such that no V_j contains an arithmetic progression at $t_j + 1$ terms. Obviously, if (1) is a good partition, then a partition

$$\{1,2,\ldots,m\} = X_1 \cup X_2 \cup \ldots \cup X_k \tag{2}$$

obtained by "reflection" (i.e. defined by $X_j = \{p; m + 1 - p \in V_j\}$) is also good.

If (1) is a good partition and $t_1 = t_2 = \ldots = t_k$, then a partition (2) obtained by "relabelling" (i.e. defined by $X_j = V_{\pi(j)}$ where π is a fixed permutation of the subscripts) is also good.

All the good partitions corresponding to the numbers $W(k;t_1,t_2,\ldots,t_k)-1$ are as follows:

W(2;2,2):

11221 122 (i.e. $V_1 = \{1,2,5,6\}$, $V_2 = \{3,4,7,8\}$)

12122 121

12211 221

and the other three obtained by relabelling.

W(2;3,2):

11121 22111 21221 11

and another one obtained by reflection.

W(2;4,2):

11212 11112 21211 11212 1

121A1 12211 11212 2111B C

where BC ≠ 11,

and seven others obtained by reflection.

W(2;5,2):

21111 12211 11121 12211 11212 11112 2

11111 221A1 11221 11211 11122 1BC11 D

where ABCD is 1112, 1211, 1212, 1221 or 2112,

and six others obtained by reflection.

W(2;6,2):

21111 12112 11111 21112 21211 11211 11121 12111 12121

11111 12121 11121 22111 11121 11221 11112 A1211 11B12

where AB ≠ 22,

and four others obtained by reflection.

W(2;3,3):

A2122 21112 1B212 22111 21C21 22211 121D

where $|\{A,B,C,D\}| = 2$,

and 14 others obtained by relabelling.

W(2;4,3):

11121 12111 12221 22122 21111 21121 11122 21221 22211 11211 2111

A2B12 212C2 11112 11211 11222 12212 22111 12112 11112 D2122 1E2F

where ABCDEF is arbitrary.

W(3;2,2,2):

11221 12323 31311 21223 13323 2

C2112 12332 23321 21123 31131 3

112AB 11313 22332 23131 1BA21 1

where AB = 23 or 32, C = 1 or 3,

and 43 others obtained by reflection and relabelling.

Incidentally, a similar algorithm was developed and some of the above results obtained also by A. Rosa and Š. Znám.

References

1. van der Waerden, B.L., Beweis einer Baudetschen Veermutung, *Nieuw Archief voor Wiskunde* 15(1927), 212-216.