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CONDITIONAL INDEPENDENCES AMONG FOUR RANDOM VARIABLES II.[1]

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Numerous new properties of stochastic conditional independence are introduced. They are aimed, together with two surprisingly trivial examples, at a further reduction of the problem of probabilistic representability for four–element sets, i.e. of the problem which conditional independences within a system of four random variables can occur simultaneously. Proofs are based on fundamental properties of conditional independence and, in discrete case, on use of I–divergence and algebraic manipulations with marginal probabilities. A duality question is answered in negative.[3]

1. Introduction

Let $N = \{1, 2, 3, 4\}$ and $\mathcal{S}$ be the family of all 56 couples $(ij|K)$, where $K \subset N$ and ij is the union of two singletons i and j of $N - K$. Elements and singletons of N are not distinguished and the sign for union between subsets of N is omitted.

A relation $\mathcal{L} \subset \mathcal{S}$ is called *probabilistically (p–) representable* if there exists a system of (four) random variables $\xi = (\xi_i)_{i\in N}$ such that

$$\mathcal{L} = [\![\xi]\!] = \{(ij|K) \in \mathcal{S};\ \xi : i \perp j|K\}.$$

The expression $\xi : i \perp j|K$ shortens the phrase "ξ_i is conditionally independent of ξ_j given ξ_K", where $\xi_K = (\xi_k)_{k\in K}$, $K \subset N$, is a subsystem of ξ ($\xi_\emptyset$ is taken as a constant). The system ξ, called *p–representation* of $\mathcal{L}$, is assumed to take only finite number of values. We reserve the symbol $\mathbf{P}$ for the class of all p–representable relations.

This paper presents further advances in the problem which relation $\mathcal{L} \subset \mathcal{S}$ is p–representable. To formulate them we have to describe a reduction of the problem attained in [10]. This task demands an extensive notation.

Let $\mathbf{S}_{12}^{(34|\emptyset)}$ denote the class of all relations $\mathcal{L} \subset \mathcal{S}$ which contain

$$\mathcal{M}_{12}^{(34|\emptyset)} = \{(34|1), (34|2), (12|\emptyset)\},$$

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which are contained in $\mathcal{M}_{12} \cup \mathcal{N}_{12}$, where

$$\mathcal{M}_{12} = \{(34|12), (12|\emptyset), (12|3), (12|4), (34|1), (34|2), (13|4), (23|4), (14|3), (24|3)\}$$

and

$$\mathcal{N}_{12} = \{(3|12), (4|12), (1|234), (2|134), (3|124), (4|123)\},$$

and which satisfy

$$(k|ij) \in \mathcal{L} \;\Leftrightarrow\; \{(kl|ij), (k|ijl)\} \subset \mathcal{L}\,, \qquad i,j,k,l \in N \;\text{ distinct.}$$

Similarly, $\mathbf{S}_{12}^{(24|\emptyset)}$, $\mathbf{S}_{12}^{(13|2)}$, $\mathbf{S}_{12}^{(12|34)}$ and $\mathbf{S}_{12}^{(13|24)}$ will be the classes defined as above with the relations

$$\begin{aligned} \mathcal{M}_{12}^{(24|\emptyset)} &= \{(34|1), (24|3), (12|\emptyset)\}, & \mathcal{M}_{12}^{(13|2)} &= \{(12|3), (13|4), (34|2)\}, \\ \mathcal{M}_{12}^{(12|34)} &= \{(12|3), (12|4), (34|12)\}, & \mathcal{M}_{12}^{(13|24)} &= \{(12|3), (13|4), (34|12)\}, \end{aligned}$$

playing the role of $\mathcal{M}_{12}^{(34|\emptyset)}$, respectively. Every class $\mathbf{S}_{12}^{(\bullet)}$ has the cardinality 2^{11}. Elements of the union

$$\mathbf{S}_{12} = \mathbf{S}_{12}^{(34|\emptyset)} \cup \mathbf{S}_{12}^{(24|\emptyset)} \cup \mathbf{S}_{12}^{(13|2)} \cup \mathbf{S}_{12}^{(12|34)} \cup \mathbf{S}_{12}^{(13|24)}$$

will be called *semimatroids*[4] here.

We have reduced (see [10]) the problem of p–representability for four-element sets to the question which semimatroid $\mathcal{L} \in \mathbf{S}_{12}$ is p–representable. The following two theorems exhibit in a compressed form all results of the present paper and reduce the problem, roughly speaking, to the last less than $3 \cdot 2^5$ cases.

Theorem 1.

A semimatroid $\mathcal{L}$ from $\mathbf{S}_{12}^{(34|\emptyset)}$ ($\mathbf{S}_{12}^{(12|34)}$) is p–representable if and only if

$$\mathcal{L} \subset \mathcal{L}_{12}^{(34|\emptyset)} = \mathcal{M}_{12}^{(34|\emptyset)} \cup \{(34|12)\} \cup \mathcal{N}_{12} \qquad (\; \mathcal{L} \subset \mathcal{L}_{12}^{(12|34)} = \mathcal{M}_{12}^{(12|34)} \cup \{(34|1), (34|2)\}\;)\,.$$

Theorem 2.

1. $\mathcal{L} \in \mathbf{P} \cap \mathbf{S}_{12}^{(\bullet)} \quad \Rightarrow \quad \mathcal{L} \subset \mathcal{M}_{12}^{(\bullet)} \cup \{(34|12)\} \cup \mathcal{N}_{12}\,, \qquad (\bullet) \in \{(24|\emptyset), (13|2)\}\,,$
2. $\mathcal{L} \in \mathbf{P} \cap \mathbf{S}_{12}^{(13|24)} \quad \Rightarrow \quad \mathcal{L} \subset \mathcal{M}_{12}^{(13|24)} \cup \{(12|\emptyset), (34|2), (24|3)\} \cup \{(2|134), (4|123)\}\,.$

Unlike in our previous paper, where all results on the problem were consequences of a few fundamental properties of Shannon entropies of subsystems and where the core was a convex analysis of special cones, proofs of these two theorems necessitate varied genuinely probabilistic arguments. Beside examples witnessing the p–representability of semimatroids from Theorem 1, a considerable number of subtle assertions of the type

$$(ij|K) \in [\![\xi]\!],\; \mathcal{M}_{12}^{(\bullet)} \subset [\![\xi]\!] \quad \Rightarrow \quad (\bullet) \in [\![\xi]\!]$$

will be demonstrated. Here ξ is any system of four random variables, $(ij|K)$ a couple from $(\mathcal{M}_{12} \cup \mathcal{N}_{12}) - \mathcal{M}_{12}^{(\bullet)}$ forbidden in one of the theorems and $(\bullet)$ a couple admissible

[4]In the language of [10], the isomorphic images of all $\mathcal{L} \in \mathbf{S}_{12}$ exhaust the class of all semimatroids which are not Ingleton. For simplicity, we use here the word "semimatroid" in more narrow sense that in [10].

in the upper index position (which is always out of $\mathcal{M}_{12}$ and $\mathcal{N}_{12}$). Every assertion of this type entails that any p–representable semimatroid $\mathcal{L} \in \mathbf{S}_{12}^{(\bullet)}$ does not contain the very couple $(ij|K)$.

The proofs of both theorems result from a series of propositions organized as follows. First, we observe in Section 2 that p–representable semimatroids from $\mathbf{S}_{12}$ cannot contain a couple $(ij|k)$ along with the couple $(ik|j)$, i, j, k distinct. This fact ensues from ten visually transparent schemes and the arguments behind them work even for four σ–algebras. Second, we deal with p–representable semimatroids from $\mathbf{S}_{12}^{(\bullet)}$, where $(\bullet)$ is one of the couples $(34|\emptyset)$, $(24|\emptyset)$ and $(13|2)$. As stated in Section 3 they can contain at most one couple $(34|12)$ from $\mathcal{M}_{12} - \mathcal{M}_{12}^{(\bullet)}$. Third, we continue with two classes $\mathbf{S}_{12}^{(12|34)}$ and $\mathbf{S}_{12}^{(13|24)}$ including functional dependence considerations (Section 4). A summary of the achieved arguments together with two examples of Section 5 will provide the desired two proofs. At the end a duality conjecture of [8] is refuted. With the exception of a few concluding remarks, this paper should be readable without any knowledge of the results and notations from [10].

2. The couples $(ij|k)$ and $(ik|j)$

Having four sub–σ–algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and $\mathcal{A}_4$ in a probability space, the fundamental property of the conditional independence relation (see [1], [4], [5], [6], [12]) has the form

$$\mathcal{A}_1 \perp \mathcal{A}_2 \vee \mathcal{A}_3|\mathcal{A}_4 \quad \Leftrightarrow \quad \mathcal{A}_1 \perp \mathcal{A}_2|\mathcal{A}_3 \vee \mathcal{A}_4 \ \text{ and } \ \mathcal{A}_1 \perp \mathcal{A}_3|\mathcal{A}_4 \ ,$$

where $\mathcal{A}_1 \vee \mathcal{A}_2$ is the smallest sub-σ-algebra containing $\mathcal{A}_1 \cup \mathcal{A}_2$. The projection technique of [11] and [14] provides also

$$\mathcal{A}_1 \perp \mathcal{A}_2|\mathcal{A}_3 \ \text{ and } \ \mathcal{A}_1 \perp \mathcal{A}_3|\mathcal{A}_2 \quad \Leftrightarrow \quad \mathcal{A}_1 \perp \mathcal{A}_2 \vee \mathcal{A}_3|\mathcal{A}_2 \cap \mathcal{A}_3$$

and

$$\mathcal{A}_2 \perp \mathcal{A}_2|\mathcal{A}_1 \quad \Leftrightarrow \quad \mathcal{A}_2 \subset \mathcal{A}_1$$

for complete σ–algebras (the completion of σ–algebras does not affect the conditional independences, cf. [10]). As a consequence of these three assertions we obtain the following result.

Proposition 1. *If a semimatroid $\mathcal{L} \in \mathbf{S}_{12}$ contains two couples $(ij|k)$ and $(ik|j)$, where $i, j, k \in N$ are distinct, then it is not p–representable.*

Proof. To begin with an example of our argumentation, we shall show that the relation $\mathcal{A}_3 \perp \mathcal{A}_4|\mathcal{A}_\emptyset$ ($\mathcal{A}_\emptyset$ will be always the smallest sub–σ–algebra; this statement means that $\mathcal{A}_3$ is unconditionally independent of $\mathcal{A}_4$) ensues from

$$\mathcal{A}_3 \perp \mathcal{A}_1|\mathcal{A}_4 \ , \quad \mathcal{A}_3 \perp \mathcal{A}_4|\mathcal{A}_1 \ , \quad \mathcal{A}_3 \perp \mathcal{A}_4|\mathcal{A}_2 \ \text{ and } \ \mathcal{A}_1 \perp \mathcal{A}_2|\mathcal{A}_\emptyset \ .$$

In virtue of this claim, any p–representable $\mathcal{L} \in \mathbf{S}_{12}^{(34|\emptyset)}$ cannot contain $(13|4)$ and, by permutations, the proof of the case $\mathbf{S}_{12}^{(34|\emptyset)} \subset \mathbf{S}_{12}$ will be over. In fact, first two independences imply $\mathcal{A}_3 \perp \mathcal{A}_1 \vee \mathcal{A}_4|\mathcal{A}_1 \cap \mathcal{A}_4$ and thus $\mathcal{A}_3 \perp \mathcal{A}_4|\mathcal{A}_1 \cap \mathcal{A}_4$. Second two ones yield $\mathcal{A}_1 \cap \mathcal{A}_4 \perp \mathcal{A}_3|\mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{A}_4 \perp \mathcal{A}_2|\mathcal{A}_\emptyset$, respectively, which gives $\mathcal{A}_1 \cap \mathcal{A}_4 \perp \mathcal{A}_2 \vee \mathcal{A}_3|\mathcal{A}_\emptyset$ and then $\mathcal{A}_3 \perp \mathcal{A}_1 \cap \mathcal{A}_4|\mathcal{A}_\emptyset$. Finally, we obtain $\mathcal{A}_3 \perp \mathcal{A}_4 \vee (\mathcal{A}_1 \cap \mathcal{A}_4)|\mathcal{A}_\emptyset$ meaning the desired $\mathcal{A}_3 \perp \mathcal{A}_4|\mathcal{A}_\emptyset$.

Before proceeding further we rewrite the above reasoning in a visually more tidy fashion.

$$
(1)\qquad
\begin{tikzcd}[column sep=small]
(3-1|4) \arrow[r] & (3-14|\underline{14}) \arrow[r] & (3-4|\underline{14}) \arrow[rr] & & (3-4|\emptyset) \\
(3-4|1) \arrow[ur] & & & & \\
(3-4|2) \arrow[r] & (\underline{14}-3|2) \arrow[r] & (\underline{14}-23|\emptyset) \arrow[r] & (3-\underline{14}|\emptyset) \arrow[uur] & \\
(1-2|\emptyset) \arrow[r] & (\underline{14}-2|\emptyset) \arrow[ur] & & &
\end{tikzcd}
$$

The whole proof will consist of ten schemes of the above form.

We continue with the case $\mathcal{L} \in \mathbf{S}_{12}^{(24|\emptyset)}$ supposing $(34|2) \notin \mathcal{L}$ (otherwise $\mathcal{L} \in \mathbf{S}_{12}^{(34|\emptyset)}$). Thus,

$$
(2)\qquad
\begin{tikzcd}[column sep=small]
(2-3|4) \arrow[r] & (2-34|\underline{34}) \arrow[r] & (2-4|\underline{34}) \arrow[r] & (2-4|\emptyset) \\
(2-4|3) \arrow[ur] & & & \\
(3-4|1) \arrow[r] & (\underline{34}-\underline{34}|1) \arrow[r] & (2-\underline{34}|\emptyset) \arrow[uur] & \\
(1-2|\emptyset) \arrow[urr] & & &
\end{tikzcd}
$$

using $\mathcal{A}_3 \cap \mathcal{A}_4 \subset \mathcal{A}_1$ on the third and fourth line (this argument is employed only here and in (5)). The assertions (1) and (2) almost coincide with $(B1)$ and $(B2)$ of [13].[5] Next,

$$
(3)\qquad
\begin{tikzcd}[column sep=small]
(3-1|4) \arrow[r] & (4-3|\underline{14}) \arrow[r] & (4-2|\underline{14}) \arrow[r] & (2-4|\emptyset) \\
(3-4|1) \arrow[ur] & & & \\
(2-4|3) \arrow[r] & (4-2|3\underline{14}) \arrow[uur] & & \\
(1-2|\emptyset) \arrow[r] & (2-\underline{14}|\emptyset) \arrow[uuurr] & &
\end{tikzcd}
$$

$$
(4)\qquad
\begin{tikzcd}[column sep=small]
(4-1|3) \arrow[r] & (4-3|\underline{13}) \arrow[r] & (4-2|\underline{13}) \arrow[r] & (2-4|\emptyset) \\
(4-3|1) \arrow[ur] & & & \\
(2-4|3) \arrow[r] & (4-2|3\underline{13}) \arrow[uur] & & \\
(1-2|\emptyset) \arrow[r] & (2-\underline{13}|\emptyset) \arrow[uuurr] & &
\end{tikzcd}
$$

what closes the proof for $\mathbf{S}_{12}^{(24|\emptyset)}$ (the difference between (3) and (4) is only seemingly small; note $\underline{13} \leftrightarrow \underline{14}$).

To cover the case of semimatroids from $\mathbf{S}_{12}^{(13|2)}$ we need four very similar schemes

$$
(5)\qquad
\begin{tikzcd}[column sep=small]
(1-4|3) \arrow[r] & (1-3|\underline{34}) \arrow[r] & (1-23|\underline{34}) \arrow[r] & (1-3|2\underline{34}) \arrow[r] & (1-3|2) \\
(1-3|4) \arrow[ur] & & & & \\
(1-2|3) \arrow[r] & (1-2|3\underline{34}) \arrow[uur] & & & \\
(3-4|2) \arrow[r] & (\underline{34}-\underline{34}|2) \arrow[uuurrr] & & &
\end{tikzcd}
$$

$$
(6)\qquad
\begin{tikzcd}[column sep=small]
(3-4|1) \arrow[r] & (1-3|\underline{14}) \arrow[r] & (1-23|\underline{14}) \arrow[r] & (3-1|\underline{14}2) \arrow[r] & (1-3|2) \\
(3-1|4) \arrow[ur] & & & & \\
(1-2|3) \arrow[r] & (1-2|3\underline{14}) \arrow[uur] & & & \\
(3-4|2) \arrow[r] & (3-\underline{14}|2) \arrow[uuurrr] & & &
\end{tikzcd}
$$

[5] I would like to thank to my colleague M. Studený for providing me manuscripts of his rather long proofs of $(B1), (B2)$ which settled the discrete setting.

$$
(7)\qquad
\begin{tikzcd}[row sep=small]
(3-2|4) \arrow[r] & (3-4|\underline{24}) \arrow[r] & (3-14|\underline{24}) \arrow[r] & (1-3|\underline{24}) \arrow[r] & (1-3|2) \\
(3-4|2) \arrow[ur] & & & & \\
(1-3|4) \arrow[r] & (3-1|4\underline{24}) \arrow[uur] & & & \\
(1-2|3) \arrow[r] & (1-2|3\underline{24}) \arrow[uuurrr] & & &
\end{tikzcd}
$$

$$
(8)\qquad
\begin{tikzcd}[row sep=small]
(4-2|3) \arrow[r] & (3-4|\underline{23}) \arrow[r] & (3-14|\underline{23}) \arrow[r] & (1-3|\underline{23}) \arrow[r] & (1-3|2) \\
(4-3|2) \arrow[ur] & & & & \\
(1-3|4) \arrow[r] & (3-1|4\underline{23}) \arrow[uur] & & & \\
(1-2|3) \arrow[r] & (1-2|3\underline{23}) \arrow[uuurrr] & & &
\end{tikzcd}
$$

Further, we concentrate on $\mathbf{S}_{12}^{(13|24)}$. If $(34|2) \in \mathcal{L}$ we have a former case as $\mathcal{L} \in \mathbf{S}_{12}^{(13|2)}$. On account of (5) and (6), respectively,

$$
(9)\qquad
\begin{tikzcd}[row sep=small]
(1-4|3) \arrow[dr] & & & \\
(1-3|4) \arrow[r] & (3-1|2\underline{34}) \arrow[r] & (3-14|2\underline{34}) \arrow[r] & (1-3|24) \\
(1-2|3) \arrow[ur] & & & \\
(3-4|12) \arrow[r] & (3-4|12\underline{34}) \arrow[uur] & &
\end{tikzcd}
$$

$$
(10)\qquad
\begin{tikzcd}[row sep=small]
(3-4|1) \arrow[dr] & & & \\
(3-1|4) \arrow[r] & (3-1|2\underline{14}) \arrow[r] & (3-14|2\underline{14}) \arrow[r] & (1-3|24) \\
(1-2|3) \arrow[ur] & & & \\
(3-4|12) \arrow[r] & (3-4|12\underline{14}) \arrow[uur] & &
\end{tikzcd}
$$

It remains to consider a p–representable semimatroid $\mathcal{L} \in \mathbf{S}_{12}^{(13|24)}$ containing the two couples $(23|4)$ and $(24|3)$. However, permuting $1 \leftrightarrow 2$ in (9) we arrive at $(23|14) \in \mathcal{L}$, a contradiction with $\mathcal{L} \subset \mathcal{M}_{12} \cup \mathcal{N}_{12}$.

Finally, for $\mathcal{L} \in \mathbf{S}_{12}^{(12|34)}$ it suffices to exclude $(13|4) \in \mathcal{L}$ as then $\mathcal{L} \in \mathbf{S}_{12}^{(13|24)}$ and by permutations also $(23|4), (14|3), (24|3) \in \mathcal{L}$. ∎

3. Semimatroids from $\mathbf{S}_{12}^{(34|\emptyset)}$, $\mathbf{S}_{12}^{(24|\emptyset)}$ and $\mathbf{S}_{12}^{(13|2)}$

Though the results of the previous section concerned the most general situations, from now on we are not able to keep further this level and start to assume discrete systems $\xi = (\xi_i)_{i \in N}$. It is, however, a challenge to find respective proofs for σ–algebras.

The notation is fixed as follows. Every random variable ξ_i takes its values in a finite set X_i, $i \in N$, and these sets are chosen to be minimal, i.e. every value $x_i \in X_i$ has a positive probability. The subsystems ξ_I range over the Cartesian products $X_I = \Pi_{i \in I} X_i$, $I \subset N$ ($X_\emptyset = \{\emptyset\}$). Marginal probabilities like $\Pr(\xi_1 = x_1, \xi_3 = x_3)$ are shortened to $(x_1 x_3)$ and, occasionally, to $p_{13}(x_1, x_3)$.

Proposition 2. *If $\mathcal{L} \in \mathbf{S}_{12}^{(\bullet)}$ is a p–representable semimatroid and $(\bullet)$ one of the couples $(34|\emptyset), (24|\emptyset)$ and $(13|2)$ then*

$$\mathcal{L} \subset \mathcal{M}_{12}^{(\bullet)} \cup \{(34|12)\} \cup \mathcal{N}_{12}\ .$$

Proof. To perform the demonstration for $\mathcal{L} \in \mathbf{S}_{12}^{(34|\emptyset)}$ is suffices (by Proposition 1) to establish

$$\{(12|3),(34|1),(34|2),(12|\emptyset)\} \subset [\![\xi]\!] \quad \Rightarrow \quad (34|\emptyset) \in [\![\xi]\!] \,.$$

The first three assumed conditional independences afford

$$(x_1x_3x_4)(x_2x_3x_4)(x_1)(x_2) = (x_1x_3)(x_1x_4)(x_2x_3)(x_2x_4) = (x_1x_2x_3)(x_3)(x_1x_4)(x_2x_4)$$

(we have been strictly sticking to the context notation $x_1 \in X_1$, $x_2 \in X_2$, etc.). This being so the nonnegative function q_{34} defined on X_{34} by

$$q_{34}(x_3,x_4) = \frac{(x_3x_4)^2}{(x_3)(x_4)} = \sum_{x_1\in X_1}\sum_{x_2\in X_2} \frac{(x_1x_2x_3)(x_1x_4)(x_2x_4)}{(x_1)(x_2)(x_4)}$$

is due to $\xi : 1 \perp 2|\emptyset$ a probability distribution

$$\sum_{x_4\in X_4}\sum_{x_3\in X_3} q_{34}(x_3,x_4) = \sum_{x_4\in X_4}(x_4) = 1 \ .$$

The marginal distribution p_{34} of ξ_3 and ξ_4 is absolutely continuous w.r.t. the distribution q_{34} and then the I–divergence (see [3])

$$I(p_{34}\|q_{34}) = \sum_{x_3\in X_3}\sum_{x_4\in X_4} p_{34}(x_3,x_4)\ln\frac{p_{34}(x_3,x_4)}{q_{34}(x_3,x_4)} = \sum_{x_3\in X_3}\sum_{x_4\in X_4}(x_3x_4)\ln\frac{(x_3)(x_4)}{(x_3x_4)}$$

is a finite nonnegative number vanishing if and only if $p_{34} = q_{34}$. Therefore

$$0 \le I(p_{34}\|q_{34}) = h_\xi(34) - h_\xi(3) - h_\xi(4) \,,$$

where $h_\xi(I), I \subset N$, is the Shannon entropy of the subsystem ξ_I. The opposite inequality is, however, well-known (consider $I(p_{34}\|r_{34}) \ge 0$ with $r_{34}(x_3,x_4) = (x_3)(x_4)$). The equality entails $\xi : 3 \perp 4|\emptyset$.

Second, we have to prove that any p–representable semimatroid $\mathcal{L} = [\![\xi]\!] \in \mathbf{S}_{12}^{(24|\emptyset)}$ cannot contain the couples $(12|3)$ and $(12|4)$. Namely, this would cause $(24|\emptyset) \in \mathcal{L}$ as will be shown in the following two minisections.

If $(12|3) \in \mathcal{L}$ then

$$(x_1x_3x_4)(x_2x_3x_4)(x_1)(x_3) = (x_1x_3)(x_1x_4)(x_2x_3)(x_3x_4) = (x_1x_2x_3)(x_3)(x_1x_4)(x_3x_4)$$

from which

$$(x_3x_4)(x_2x_3x_4) = (x_3x_4)\sum_{x_1\in X_1}\frac{(x_1x_2x_3)(x_1x_4)}{(x_1)} \,.$$

For $(x_3x_4) = 0$ we have $(x_2x_3x_4) = 0$, $x_2 \in X_2$, and also

$$0 = (x_1x_3x_4)(x_1)(x_2x_3) = (x_1x_3)(x_1x_4)(x_2x_3) = (x_1x_2x_3)(x_3)(x_1x_4),\ \ x_1 \in X_1,$$

what implies that the sum equals zero. Anyway, after cancellation (if needed)

$$(x_2x_3x_4) = \sum_{x_1\in X_1}\frac{(x_1x_2x_3)(x_1x_4)}{(x_1)} \,.$$

The summation over $x_3 \in X_3$ together with $(12|\emptyset) \in \mathcal{L}$ yield $(24|\emptyset) \in \mathcal{L}$.

If $(12|4) \in \mathcal{L}$ then owing to $(34|1) \in \mathcal{L}$

$$(x_1x_2x_4)(x_4)(x_1x_3) = (x_1x_3x_4)(x_1)(x_2x_4)$$

and using also $(24|3) \in \mathcal{L}$ we obtain

$$(x_2x_3)(x_4) \sum_{x_1 \in X_1} \frac{(x_1x_2x_4)(x_1x_3)}{(x_1)} = (x_2x_3)(x_3x_4)(x_2x_4) = (x_2x_3x_4)(x_3)(x_2x_4)\,.$$

Thus,

$$(x_2x_3) \sum_{x_4 \in X_4} \sum_{x_1 \in X_1} \frac{(x_1x_2x_4)(x_1x_3)}{(x_1)} = (x_3) \sum_{x_4 \in X_4} \frac{(x_2x_3x_4)(x_2x_4)}{(x_4)}$$

and $(12|\emptyset) \in \mathcal{L}$ affords

$$(x_2x_3) = \sum_{x_4 \in X_4} \frac{(x_2x_3x_4)(x_2x_4)}{(x_2)(x_4)}\,.$$

On account of the last formula we see immediately that the function q_{24} defined in the first part of the proof ($2 \leftrightarrow 3$) is a probability distribution on X_{24} and becose of it $(24|\emptyset) \in \mathcal{L}$ similarly as before.

Third, we step to the case $\mathcal{L} = [\![\xi]\!] \in \mathbf{S}_{12}^{(13|2)}$. If $(12|\emptyset) \in \mathcal{L}$ then the permutation $1 \leftrightarrow 2$, $3 \leftrightarrow 4$ bring over $\mathcal{L}$ to the previous case and thus this incidence has been yet settled. It remains to prove $(12|4) \notin \mathcal{L}$ pursuant to Proposition 1. In fact, the opposite combined with $(13|4), (34|2) \in \mathcal{L}$ ensures

$$(x_1x_2x_4)(x_3x_4)(x_2x_3) = (x_1x_3x_4)(x_2x_4)(x_2x_3) = (x_1x_3x_4)(x_2x_3x_4)(x_2)$$

and then

$$(x_3x_4) \sum_{x_2 \in X_2} \frac{(x_1x_2x_4)(x_2x_3)}{(x_2)} = (x_3x_4)(x_1x_3x_4)\,.$$

Here we can formally cancel (x_3x_4) as from $(x_3x_4) = 0$ one appoints $0 = (x_2x_3x_4)(x_2) = = (x_2x_3)(x_2x_4)$ and thus $0 = (x_2x_3)(x_1x_2x_4)$ for any $x_2 \in X_2$. But then the probability distribution

$$q_{123}(x_1, x_2, x_3) = \frac{(x_1x_2)(x_2x_3)}{(x_2)}$$

on X_{123} has the marginal

$$\sum_{x_2 \in X_2} q_{123}(x_1, x_2, x_3) = (x_1x_3)\,.$$

Due to $(12|3) \in \mathcal{L}$ it is absolutely continuous w.r.t. the marginal distribution p_{123} (note that $(x_1x_2x_3) = 0$ implies $0 = (x_1x_3)(x_2x_3) \geq (x_2x_3)q_{123}(x_1, x_2, x_3)$) and, moreover,

$$0 = h_\xi(13) + h_\xi(23) - h_\xi(123) - h_\xi(3) \leq \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} \sum_{x_3 \in X_3} \frac{(x_1x_2)(x_2x_3)}{(x_2)} \ln \frac{(x_1x_2)(x_3)}{(x_1x_3)(x_2)}$$

(on the right-hand side is the I–divergence $I(q_{123} \| p_{123})$). This inequality can be casted to

$$h_\xi(23) + h_\xi(12) \leq h_\xi(123) + h_\xi(2)\,.$$

The semimodularity of h_ξ forces the equality, whence $(13|2) \in \mathcal{L}$. ∎

4. Semimatroids from $\mathbf{S}_{12}^{(12|34)}$ and $\mathbf{S}_{12}^{(13|24)}$

We continue with the notation of the previous section.

Proposition 3. *The intersection* $\mathcal{L} \cap \{(12|\emptyset), (13|4), (23|4), (14|3), (24|3)\}$ *is empty as soon as* $\mathcal{L} \in \mathbf{S}_{12}^{(12|34)}$ *is p–representable. If* $\mathcal{L} \in \mathbf{P} \cap \mathbf{S}_{12}^{(13|24)}$ *then* $(23|4) \notin \mathcal{L}$.

Proof. Let ξ be a p–representation of a semimatroid $\mathcal{L} \supset \mathcal{M}_{12}^{(12|34)}$ and $(12|\emptyset) \in \mathcal{L}$. Then

$$(x_1x_2x_3x_4) = \frac{(x_1x_2x_3)(x_1x_2x_4)}{(x_1)(x_2)} = \frac{(x_1x_3)(x_2x_3)(x_1x_4)(x_2x_4)}{(x_1)(x_2)(x_3)(x_4)}$$

and it is easy to see

$$(x_1x_2x_3x_4)(y_1y_2x_3x_4) = (y_1x_2x_3x_4)(x_1y_2x_3x_4), \quad y_1 \in X_1, y_2 \in X_2,$$

i.e. $(12|34) \in \mathcal{L}$ (cf. also [6], the equivalent definition of the conditional independence by a factorization property). Hence, a semimatroid $\mathcal{L} \in \mathbf{P} \cap \mathbf{S}_{12}^{(12|34)}$ cannot contain $(12|\emptyset)$.

If $(13|4) \in \mathcal{L}$ we fix $x_4 \in X_4$ and denote by $Y_i = \{x_i \in X_i; (x_ix_4) > 0\}$, $i = 1, 2, 3$. Let us consider these two probability distributions on $Y_{123} = Y_1 \times Y_2 \times Y_3$

$$q(y_1, y_2, y_3) = \frac{(y_1y_2y_3x_4)}{(x_4)} \quad \text{and} \quad r(y_1, y_2, y_3) = \frac{(y_1x_4)(y_2y_3x_4)}{(x_4)^2} .$$

In other words (since $(12|4) \in \mathcal{L}$ the probability (y_1y_2) is positive for $y_1 \in Y_1$ and $y_2 \in Y_2$)

$$q(y_1, y_2, y_3) = \frac{(y_1y_3)(y_2y_3)(y_1x_4)(y_2x_4)}{(y_1y_2)(y_3)(x_4)^2} .$$

The distribution r is absolutely continuous w.r.t. q as $(y_1y_3) > 0$ is valid due to $(13|4) \in \mathcal{L}$. It is, however, clearly the other way round what offers possibility to write

$$I(q\|r) + I(r\|q) = \sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} \sum_{y_3 \in Y_3} [q(y_1, y_2, y_3) - r(y_1, y_2, y_3)] \ln \frac{(y_1y_3)(y_2y_3)(y_2x_4)}{(y_1y_2)(y_3)(y_2y_3x_4)} = 0 .$$

The point is that both q and r have the same all three two–dimensional marginals. Thus $q = r$ whence $(12|34) \in \mathcal{L}$ and we have proved the first assertion. (Another reasoning might us the observation that q and r are factorizable w.r.t. the marginals and then they are both maximum entropy extensions from them. Uniqueness of the extension brings $q = r$, see [2], [6].)

Let ξ be a p–representation of a semimatroid $\mathcal{L} \supset \mathcal{M}_{12}^{(13|24)}$ and $(23|4) \in \mathcal{L}$. We proceed as above with the distributions

$$q(y_1, y_2, y_3) = \frac{(y_1y_2y_3x_4)}{(x_4)} = \frac{(y_1y_3)(y_2y_3)(y_1y_2x_4)}{(y_1y_2)(y_3)(x_4)} \quad \text{and} \quad r(y_1, y_2, y_3) = \frac{(y_3x_4)(y_1y_2x_4)}{(x_4)^2} .$$

Note that if $(y_1y_2) = 0$ for some $y_1 \in Y_1$ and $y_2 \in Y_2$ then by $(12|3) \in \mathcal{L}$ also the expression $(y_1y_3)(y_2y_3)$ vanishes for any $y_3 \in Y_3$ and by $(13|4), (24|3) \in \mathcal{L}$ also $(y_1x_4)(y_3x_4)(y_2x_4) = 0$, a contradiction. The distributions q and r have the same supports ($(y_1y_3)(y_2y_3) > 0$ owing to $(13|4), (23|4) \in \mathcal{L}$) and along the same guidelines as before we conclude $q = r$; i.e. the desired $(13|24) \in \mathcal{L}$. ∎

Our last proposition concerns the functional dependence claims (cf. [7]). Note that e.g. $(1|23) \in [\![\xi]\!]$ means that the variable ξ_1 is a function of ξ_{23} (in the σ–algebras environment $\mathcal{A}_1 \subset \mathcal{A}_2 \vee \mathcal{A}_3$, for $\mathcal{A}_2$ or $\mathcal{A}_3$ complete).

Proposition 4. *Every p–representable semimatroid from* $\mathbf{S}_{12}^{(12|34)}$ *has empty intersection with the class* $\mathcal{N}_{12} \subset \mathcal{S}$. *If* $\mathcal{L} \in \mathbf{P} \cap \mathbf{S}_{12}^{(13|24)}$ *then the couples* $(1|234)$ *and* $(3|124)$ *are not elements of* $\mathcal{L}$.

Proof. Let $\mathcal{L} = [\![\xi]\!] \supset \mathcal{M}_{12}^{(12|34)} \cup \{(1|234)\}$. Suppose that $(x_1x_3x_4)(y_1x_3x_4) > 0$ and chose $x_2, y_2 \in X_2$ such that $(x_1x_2x_3x_4)(y_1y_2x_3x_4) > 0$. Then $(y_1x_3)(x_2x_3) > 0$ and by $(12|3) \in \mathcal{L}$ also $(y_1x_2x_3) > 0$; similarly $(y_1x_2x_4) > 0$ using $(12|4) \in \mathcal{L}$. The incidence $(34|12) \in \mathcal{L}$ brings thus $(y_1x_2x_3x_4) > 0$. Since ξ_1 is a function of ξ_{234} the inequality $(x_1x_2x_3x_4)(y_1x_2x_3x_4) > 0$ implies $x_1 = y_1$. This means that ξ_1 is even a function of ξ_{34} and consequently $(12|34) \in \mathcal{L}$ (cf. [11], [14]).

Let $\mathcal{L} = [\![\xi]\!] \supset \mathcal{M}_{12}^{(12|34)} \cup \{(3|124)\}$. Then

$$(x_1x_2x_3x_4)(x_1x_2)\,(y_1y_2x_3x_4)(y_1y_2) = (y_1x_2x_3x_4)(y_1x_2)\,(x_1y_2x_3x_4)(x_1y_2)$$

and if $(x_1x_2x_3x_4)(y_1y_2x_3x_4) > 0$ we can drop out x_3 throughout this equation. Further casting leads to

$$(x_1x_4)(x_2x_4)(y_1x_4)(y_2x_4)\,[(x_1x_2)(y_1y_2) - (y_1x_2)(x_1y_2)] = 0$$

what affords

$$(x_1x_2x_3x_4)(y_1y_2x_3x_4) = (y_1x_2x_3x_4)(x_1y_2x_3x_4)\ .$$

However, if the left-hand side is equal to zero here then $(x_1x_3)(x_2x_3)(x_1x_4)(x_2x_4) = 0$ or $(y_1x_3)(y_1x_4)(y_2x_3)(y_2x_4) = 0$ and thus also the right-hand side vanishes. Consequently, $(12|34) \in \mathcal{L}$ and we finished the proof for $\mathbf{S}_{12}^{(12|34)}$.

Let $\mathcal{L} = [\![\xi]\!] \supset \mathcal{M}_{12}^{(13|24)} \cup \{(1|234)\}$. We shall demonstrate that ξ_1 is a function of ξ_{24} similarly as at the beginning. If $(x_1x_2x_4)(y_1x_2x_4) > 0$ then for some $x_3, y_3 \in X_3$ also $(x_1x_2x_3x_4)(y_1x_2y_3x_4) > 0$. Owing to $(13|4) \in \mathcal{L}$ we realize $(x_1y_3x_4) > 0$ whence $(x_1y_3)(x_2y_3) > 0$. By $(12|3) \in \mathcal{L}$ we get $(x_1x_2y_3) > 0$ and knowing that $(x_1x_2x_4)$ is positive the incidence $(34|12) \in \mathcal{L}$ implies $(x_1x_2y_3x_4) > 0$. As ξ_1 is a function of ξ_{234} and $(y_1x_2y_3x_4)(x_1x_2y_3x_4) > 0$ we arrive at $x_1 = y_1$, i.e. ξ_1 is a function of ξ_{24}, too.

Let $\mathcal{L} = [\![\xi]\!] \supset \mathcal{M}_{12}^{(13|24)} \cup \{(3|124)\}$ and thus $(3|12) \in \mathcal{L}$. Starting with the assumptons $(x_2x_3x_4)(x_2y_3x_4) > 0$ and $(x_1x_2x_3x_4)(y_1x_2y_3x_4) > 0$ for properly chosen $x_1, y_1 \in X_1$, we infer $(x_1y_3x_4) > 0$ from $(13|4) \in \mathcal{L}$ and then $(x_1x_2y_3) > 0$ from $(12|3) \in \mathcal{L}$. Plainly, $x_3 = y_3$ what reads "ξ_3 is a function of ξ_{24}" and hence $(13|24) \in \mathcal{L}$. ∎

5. Examples and conclusions

Collecting assertions of Propositions 2, 3 and 4 we recognize immediately that Theorem 2 is their consequence. Also the implication ⇒ of Theorem 1 follows from them. To prove the converse one we present two simple examples and refer the reader to Consequence 2 of [10] which designates that a semimatroid from $\mathbf{S}_{12}$ contained in a p–representable semimatroid from $\mathbf{S}_{12}$ is p–representable, too.

Example 1. Let $\Omega = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}\}$ be a four–element probability space with the uniform distribution and $\xi_1 = (\mathsf{ab})(\mathsf{cd})$, $\xi_2 = (\mathsf{ac})(\mathsf{bd})$, $\xi_3 = (\mathsf{a})(\mathsf{bcd})$ and $\xi_4 = (\mathsf{abc})(\mathsf{d})$ be four random variables specified as partitions. We claim

$$[\![\xi]\!] = \mathcal{L}_{12}^{(34|\emptyset)} = \mathcal{M}_{12}^{(34|\emptyset)} \cup \{(34|12)\} \cup \mathcal{N}_{12}\ .$$

To see it one has to verify the inclusion $\supset$ and $(34|\emptyset) \notin [\![\xi]\!]$. Then Theorem 2 of [10] forces $[\![\xi]\!]$ to be contained in $\mathcal{M}_{12} \cup \mathcal{N}_{12}$ and the "only if" part of Theorem 1 entails the desired equality. ∎

Example 2. Let Ω have seven elements $\mathsf{a}, \mathsf{b}, \ldots, \mathsf{g}$ and be endowed with the uniform distribution. The system ξ of four random variables $\xi_1 = (\mathsf{aceg})(\mathsf{bdf})$, $\xi_2 = (\mathsf{abef})(\mathsf{cdg})$, $\xi_3 = (\mathsf{abcdeg})(\mathsf{f})$ and $\xi_4 = (\mathsf{abcdef})(\mathsf{g})$ is a p–representation of the semimatroid

$$[\![\xi]\!] = \mathcal{L}_{12}^{(12|34)} = \mathcal{M}_{12}^{(12|34)} \cup \{(34|1)(34|2)\}$$

(the argumentation is as before). ∎

In [8] we found an interesting symmetry on the family $\mathcal{R} = \{(ij|K) \in \mathcal{S}; i \neq j\}$, which was motivated by the matroid duality (see [8], [15])

$$(ij|K) \to (ij|K)^{\neg} = (ij|N - ijK)\ .$$

A natural open question was whether the dual $\mathcal{L}^{\neg} = \{(ij|K); (ij|K)^{\neg} \in \mathcal{L}\}$ of a p–representable relation $\mathcal{L} \subset \mathcal{R}$ is necessarily p–representable. In spite of some positive indications the answer is negative. A counterexample is provided by this semimatroid (Theorem 1)

$$\mathcal{L} = \mathcal{L}_{12}^{(34|\emptyset)} \cap \mathcal{R} = \{(34|1), (34|2), (12|\emptyset), (34|12)\} \in \mathbf{P}$$

with its dual

$$\mathcal{L}^{\neg} = \{(34|1), (34|2), (12|34), (34|\emptyset)\}$$

(permuting $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ we see from Theorem 1

$$\{(12|3), (12|4), (12|\emptyset), (34|12)\} \in \mathbf{S}_{12}^{(12|34)} - \mathbf{P}\ ).$$

The same question for matroids remains, however, open.

Speaking precisely in terms of [10] the elements of $\mathbf{S}_{12}$ are, up to permutations, the semimatroids which are not Ingleton. The above two examples show for the first time representatives of types that are irreducible in $\mathbf{P}$ and not Ingleton (cf. Consequence 4 of [10]). Hence, there are at least thirteen irreducible types in the lattice $\mathbf{P}$. The final aim in the problem of p–representability for four-elements sets is to find all these irreducible types.

References

[1] CARNAL, E. (1980). Independence conditionnelle permutable. *Ann. Inst. Henri Poincaré* **16** 39–47.

[2] CSISZÁR, I. (1975). I–divergence geometry of probability distributions and minimization problems. *The Annals of Probability* **3** 146–158.

[3] CSISZÁR, I. and KÖRNER, J. (1981). *Information Theory (Coding Theorems for discrete memoryless systems).* Akadémiai Kiadó, Budapest.

[4] DAWID, A.P. (1979) Conditional independence in statistical theory (with discussion). *J. Roy. Statist. Soc.* B **41** 1–31.

[5] DÖHLER, R. (1980). Zur bedingten Unabhängigkeit zufälliger Ereignisse (On the conditional independence of random events). *Theory Probability Appl.* **25** 628–634.

[6] LAURITZEN, S.L. (1989). *Lectures on Contingency Tables.* University of Aalborg Press (3rd edn.), Aalborg.

[7] MATÚŠ, F. (1991). Abstract functional dependency structures. *Theoretical Computer Science* **81** 117–126.

[8] MATÚŠ, F. (1992). Ascending and descending conditional independence relations. *Transactions of the Eleventh Prague Conference on Information Theory, Statistical Decision Functions and Random Processes,* Academia, Prague, Vol. B, 181–200.

[9] MATÚŠ, F. (1994). Probabilistic conditional independence structures and matroid theory: background. *International Journal of General Systems* **22** 185–196.

[10] MATÚŠ, F. and STUDENÝ, M. (1994). Conditional independences among four random variables I. (to appear in *Combinatorics, Probability and Computation*)

[11] MOUCHART, M. and ROLIN, J.M. (1984). A note on conditional independence with statistical applications. *Statistica* **44** 557–584.

[12] PEARL, J. (1988). *Probabilistic Reasoning in Intelligent Systems.* Morgan Kaufman, San Meteo, California.

[13] STUDENÝ, M. (1992). Structural semigraphoids. *International Journal of General Systems* **22** 207–217.

[14] VAN PUTTEN, C. and VAN SCHUPPEN, J.H. (1985). Invariance properties of the conditional independence relation. *The Annals of Probability* **13** 934–945.

[15] WELSH, D.J.A. (1976). *Matroid Theory.* Academic Press, London.