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Conditional Independences among Four Random Variables III: Final Conclusion

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The problem of probabilistic representability of semimatroids over a four-element set is solved. In this problem, one looks for all combinations of conditional independences within four random variables which can occur simultaneously. New properties of the stochastic conditional independences are deduced from conditional information inequalities. Examples of four-tuples of random variables are presented to show the probabilistic representability of three non-Ingleton semimatroids.

1. Introduction

Let N be a finite set and $\mathcal{S}$ be the family of all ordered couples $(ij|K)$ where $K \subset N$ and ij is the union of two, possibly equal, singletons i and j of $N-K$. Where $\xi = (\xi_i)_{i\in N}$ is a system of random variables, we pick up all conditional independences within ξ by means of the emphasized bracket

$$[\![\xi]\!] = \{(ij|K) \in \mathcal{S};\ \xi\colon i \perp j|K\}\ .$$

The notation $\xi\colon i \perp j|K$ stands for 'ξ_i is conditionally independent of ξ_j given the subsystem $\xi_K = (\xi_k)_{k\in K}$', $K \subset N$. The subsystem $\xi_\emptyset$ is treated as a constant. The special case $i = j$ amounts the functional dependence of ξ_i on ξ_K and the case $K = \emptyset$ the unconditional independence. We suppose that ξ takes only a finite number of values.

A relation $\mathcal{L} \subset \mathcal{S}$ (over N) is called *probabilistically (p-) representable* if $\mathcal{L} = [\![\xi]\!]$ for some ξ. The main goal of this paper is to describe all p-representable relations over a four-element set. An easy necessary condition for $\mathcal{L}$ to be p-representable is that it must be a *semimatroid*, see [6, 10]. Here we speak about p-representability of semimatroids exclusively.

For $N = \{1, 2, 3, 4\}$ the notion of Ingleton semimatroid was introduced and all Ingleton semimatroids were proved to be p-representable, cf. [10]. Using permutation symmetries, the problem of p-representability of semimatroids can be reduced to considering a special class $\mathbf{S}_{12}$ of semimatroids which are not Ingleton. The class $\mathbf{S}_{12}$ was written as a union of five subclasses and the p-representability in two of them, $\mathbf{S}_{12}^{(34|\emptyset)}$ and $\mathbf{S}_{12}^{(12|34)}$, was analyzed completely in [7]. The problem of p-representability on the remaining three classes, $\mathbf{S}_{12}^{(13|24)}$, $\mathbf{S}_{12}^{(13|2)}$ and $\mathbf{S}_{12}^{(24|\emptyset)}$, was further reduced. Relying on this reduction, the semimatroids $\mathcal{L}$ to be examined here are restricted by one of the three conditions

$$\mathcal{M}_{12}^{(13|24)} = \{(34|12),(12|3),(13|4)\} \subset \mathcal{L} \subset \mathcal{M}_{12}^{(13|24)} \cup \{(34|2),(12|\emptyset),(24|3)\} \cup \mathcal{N}_{12}^{2,4}\,,$$

$$\mathcal{M}_{12}^{(13|2)} = \{(12|3),(13|4),(34|2)\} \subset \mathcal{L} \subset \mathcal{M}_{12}^{(13|2)} \cup \{(34|12)\} \cup \mathcal{N}_{12}$$

and

$$\mathcal{M}_{12}^{(24|\emptyset)} = \{(34|1),(24|3),(12|\emptyset)\} \subset \mathcal{L} \subset \mathcal{M}_{12}^{(24|\emptyset)} \cup \{(34|12)\} \cup \mathcal{N}_{12} = \mathcal{L}_{12}^{(24|\emptyset)}$$

where

$$\mathcal{N}_{12}^{2,4} = \{(2|134),(4|123),(4|12)\}$$

and

$$\mathcal{N}_{12} = \{(3|12),(4|12),(1|234),(2|134),(3|124),(4|123)\}\,.$$

They must in addition satisfy

$$(k|ij) \in \mathcal{L} \;\Leftrightarrow\; \{(kl|ij),(k|ijl)\} \subset \mathcal{L}\,, \quad i,j,k,l \in N \text{ distinct},$$

cf. Theorem 1.1. and Theorem 1.2. in [7].[1] Accordingly, we consider three subclasses $\mathbf{T}_{12}^{(13|24)}$, $\mathbf{T}_{12}^{(13|2)}$ and $\mathbf{T}_{12}^{(24|\emptyset)}$ of the classes $\mathbf{S}_{12}^{(13|24)}$, $\mathbf{S}_{12}^{(13|2)}$ and $\mathbf{S}_{12}^{(24|\emptyset)}$ of non-Ingleton semimatroids. All three classes have the cardinality 2^5.

Here we are able to recognize which semimatroids from $\mathbf{T}_{12}^{(13|24)}$, $\mathbf{T}_{12}^{(13|2)}$ and $\mathbf{T}_{12}^{(24|\emptyset)}$ are p-representable. This is expressed in the following three assertions.

Theorem 1.1. *A semimatroid $\mathcal{L}$ from $\mathbf{T}_{12}^{(13|24)}$ is p-representable if and only if*

$$\mathcal{L} \subset \mathcal{M}_{12}^{(13|24)} \cup \{(34|2),(4|123),(4|12)\} = \mathcal{L}_{12}^{(13|24)}\,.$$

Theorem 1.2. *A semimatroid $\mathcal{L}$ from $\mathbf{T}_{12}^{(13|2)}$ is p-representable if and only if it is contained in $\mathcal{L}_{12}^{(13|24)}$ or*

$$\mathcal{L} \subset \mathcal{M}_{12}^{(13|2)} \cup \{(1|234),(2|134),(3|124),(4|123)\} = \mathcal{L}_{12}^{(13|2)}\,.$$

Theorem 1.3. *Every semimatroid from $\mathbf{T}_{12}^{(24|\emptyset)}$ is p-representable.*

Since the class $\mathbf{P}$ of all p-representable semimatroids is intersection-closed and has therefore a natural lattice structure, these theorems combined with our previous results enable us to find explicitly all intersection-irreducible semimatroids in the lattice $\mathbf{P}$ over $N = \{1,2,3,4\}$. As a consequence, knowing that some conditional independences among a four-tuple of random variables (taking a finite number of values) hold, one can easily derive all conditional independences holding necessarily among the four variables. This kind of inference on the stochastic conditional independence is of interest in multivariate statistics, see [5], and in its applications in artificial intelligence, see [11]. For other recent work on stochastic conditional independence we refer the reader to [1], [3] and [4].

[1] We apologize the readers of [7] for omitting the couple (4|12) in the second line of Theorem 1.2.

This paper is organized as follows. In Section 2 the forward implication of Theorem 1.1. is proved. Section 3 contains three crucial examples completing the proofs of our three theorems. Finally, we present the main theorem of this series of three papers and a challenging conjecture on the p-representability in Section 4.

2. Semimatroids from $\mathbf{T}_{12}^{(13|24)}$

A few new properties of the p-representable semimatroids are derived from conditional information inequalities below. The first inequality of this kind was found by Zhang and Yeung [13]. For further progress on the problem whether a given polymatroid has its rank function equal to the entropy function of a system of random variables see [14] and [12].

We adopt the notation of Section 3 from [7], p. 412, assuming all the time that our variables take only finite number of values. We remind the reader of the crucial role of Ingleton inequality in our analysis of the cone of polymatroids, cf. [10], p. 273.

Proposition 2.1. *Let $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ be four random variables such that ξ_1 is conditionally independent of ξ_2 given ξ_3, $(12|3) \in [\![\xi]\!]$. If $(12|\emptyset) \in [\![\xi]\!]$ or $(24|3) \in [\![\xi]\!]$ then the Ingleton inequality $\square h_\xi(12) \geq 0$ holds for the entropy function h_ξ of ξ.*

Proof. Under the condition $\{(12|3), (12|\emptyset)\} \subset [\![\xi]\!]$ the expression

$$r_{1234}(x_1x_2x_3x_4) = \frac{(x_1x_3)(x_2x_3)}{(x_1)(x_2)(x_3)} \cdot \frac{(x_1x_4)(x_2x_4)}{(x_4)}$$

defines a probability distribution on the state space X_{1234} (to see it sum first over $x_3 \in X_3$ and then use $(x_1x_2) = (x_1)(x_2)$). The distribution r_{1234} dominates the conditional product distribution $q_{1234}(x_1x_2x_3x_4) = (x_1x_3x_4)(x_2x_3x_4)/(x_3x_4)$ and hence the I-divergence $I(q_{1234}\|r_{1234})$ is a nonnegative number. Rewriting the I-divergence we have immediately

$$h_\xi(13)+h_\xi(23)+h_\xi(14)+h_\xi(24)+h_\xi(34)-h_\xi(1)-h_\xi(2)-h_\xi(3)-h_\xi(4)-h_\xi(134)-h_\xi(234) \geq 0\,.$$

This results in $\square h_\xi(12) \geq 0$ due to $h_\xi(12) = h_\xi(1) + h_\xi(2)$.

In the second case, the expression, with the convention $0/0 = 0$,

$$s_{1234}(x_1x_2x_3x_4) = c \cdot \frac{(x_1x_2x_3)}{(x_1x_2)} \cdot \frac{(x_1x_4)(x_2x_4)}{(x_4)}$$

defines, for an appropriate number $c \geq 1$, a probability distribution s_{1234} on X_{1234}. (To see it, sum first over $x_3 \in X_3$ again.) Owing to $(12|3) \in [\![\xi]\!]$ the distribution s_{1234} dominates q_{1234} and by $(24|3) \in [\![\xi]\!]$ the marginal q_{123} of q_{1234} is

$$q_{123}(x_1x_2x_3) = \sum_{x_4 \in X_4} \frac{(x_1x_3x_4)(x_2x_3)}{(x_3)} = (x_1x_2x_3)\ .$$

The inequality $I(q_{1234}\|s_{1234}) \geq 0$ rewrites therefore to

$$h_\xi(123) + h_\xi(14) + h_\xi(24) + h_\xi(34) - h_\xi(12) - h_\xi(4) - h_\xi(134) - h_\xi(234) - \log c \geq 0$$

implying that $\square h_\xi(12) \geq 0$. ∎

A look at the first and the fourth mask of $\square(12)$ on p. 274 in [10] results in

$$\triangle h_\xi(34|1) + \triangle h_\xi(34|2) \geq \triangle h_\xi(34|\emptyset)$$

and

$$\triangle h_\xi(12|4) + \triangle h_\xi(34|12) \geq \triangle h_\xi(12|34)$$

once $\{(12|3),(12|\emptyset)\} \subset [\![\xi]\!]$. These are the two (equivalent) conditional information inequalities of [13]. They can be employed to get alternative argumentations in the proofs of Proposition 3.1. and Proposition 4.1. of [7].

A look at the fifth mask of $\square(12)$ provides

$$\triangle h_\xi(13|4) + \triangle h_\xi(34|12) \geq \triangle h_\xi(13|24)$$

under the assumptions of Proposition 2.1. If in addition $\{(13|4),(34|12)\} \subset [\![\xi]\!]$, then necessarily $(13|24) \in [\![\xi]\!]$. Hence, a p-representable semimatroid from $\mathbf{T}_{12}^{(13|24)}$ cannot contain $(12|\emptyset)$ and $(24|3)$. This conclusion together with the following lemma proves one implication of Theorem 1.1.

Lemma 2.2. *P-representable semimatroids from* $\mathbf{T}_{12}^{(13|24)}$ *do not contain* $(2|134)$.

Proof. Let a semimatroid $\mathcal{L} \supset \mathcal{M}_{12}^{(13|24)} \cup \{(2|134)\}$ be p-representable by ξ, $\mathcal{L} = [\![\xi]\!]$. Then we have the factorization

$$(x_1x_2x_3x_4) = \mathrm{sg}(x_2x_3) \cdot \mathrm{sg}(x_1x_2x_4) \cdot (x_1x_3x_4)$$

where e.g. $\mathrm{sg}(x_2x_3)$ equals zero or one according to (x_2x_3) is zero or a positive number. In fact, if $(x_1x_2x_3x_4) = 0$ then $(x_1x_2x_4) = 0$ or $(x_1x_2x_3) = 0$. The second equality implies $(x_1x_3) = 0$ or $(x_2x_3) = 0$ and thus the right-hand side in the factorization is equal to zero as well. If $(x_1x_2x_3x_4) > 0$ then obviously $(x_2x_3) > 0$ and $(x_1x_2x_4) > 0$ and the desired equality follows from $(2|134) \in [\![\xi]\!]$. Using this special factorization one can realize that the expression

$$(x_1x_2x_3x_4)(y_1x_2y_3x_4) - (y_1x_2x_3x_4)(x_1x_2y_3x_4)$$

equals

$$\mathrm{sg}(x_2x_3)\mathrm{sg}(x_2y_3) \cdot \mathrm{sg}(x_1x_2x_4)\mathrm{sg}(y_1x_2x_4) \cdot \big[(x_1x_3x_4)(y_1y_3x_4) - (y_1x_3x_4)(x_1y_3x_4)\big]$$

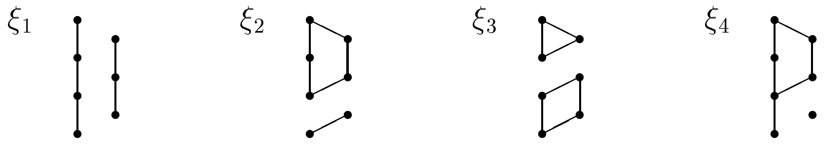
and this equals zero because the bracket vanishes. Thus $(13|24) \in \mathcal{L}$ and $\mathcal{L}$ cannot belong to $\mathbf{T}_{12}^{(13|24)}$. ∎

3. Examples of p-representations

To finish the proofs of our three theorems we must show that the semimatroids $\mathcal{L}_{12}^{(13|24)}$, $\mathcal{L}_{12}^{(13|2)}$ and $\mathcal{L}_{12}^{(24|\emptyset)}$ are p-representable, by Consequence 3 from [10] on p. 277.

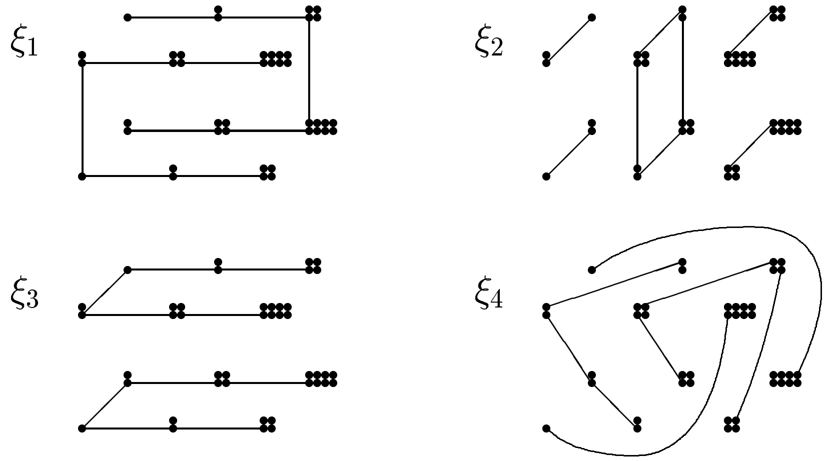
Example 1. Let $\Omega = \{\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d},\mathsf{e},\mathsf{f},\mathsf{g}\}$ be a seven-element probability space endowed with the uniform distribution and four random variables be defined as the partitions (or equivalently, as their factor-mappings) $\xi_1 = (\mathsf{abcd})(\mathsf{efg})$, $\xi_2 = (\mathsf{abcef})(\mathsf{dg})$, $\xi_3 = (\mathsf{abe})(\mathsf{cdfg})$ and $\xi_4 = (\mathsf{abcdef})(\mathsf{g})$. It is not difficult to establish the equality $[\![\xi]\!] = \mathcal{L}_{12}^{(13|24)}$. To this end, it suffices to verify the inclusion $[\![\xi]\!] \supset \mathcal{L}_{12}^{(13|24)}$ and $(13|24) \notin [\![\xi]\!]$. The argumentation justifying the equality follows similarly to Example 1 from [7].

The next two examples have the same structure. Their probability spaces Ω possess uniform distributions and all random variables are given as partitions of Ω. Since the spaces have larger cardinalities every such partition will be specified visually via a simple nonoriented graph with the vertex set Ω. The blocks of the partition are taken as the connectivity components of its defining graph. Labelings of vertices can be omitted without confusion. A visualization of this kind for Example 1 can have the following form.



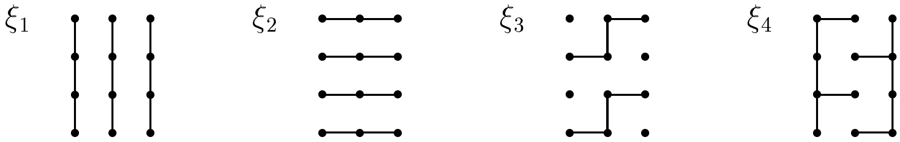
The conditional independences can be determined from the graphs with practice.

Example 2. Let Ω have 42 elements and four variables be given by (the aggregated groups of vertices are considered for cliques)



Then $[\![\xi]\!] = \mathcal{L}_{12}^{(13|2)}$ arguing as before.

Example 3. Let Ω have 12 elements. The four variables



defined on Ω provide a p-representation of $\mathcal{L}_{12}^{(24|\emptyset)}$.

Let us remark that in these three examples and also in the two examples of [7] the Ingleton inequality $\square h_\xi(12) \geq 0$ is violated. This is clear from the masks; the only nonzero $\triangle$'s have there the minus sign. Another complicated example of the violation was found in [14] using a pair of projective planes. In [2] p. 436, a variation on Example 1 of [7] was presented along with a conjecture on new linear inequality for entropy functions.

4. Final conclusions

The structure of the lattice $\mathbf{P}$ of all p-representable semimatroids on a four-element set is described by this final assertion.

Theorem 4.1. *There are 120* $\mathbf{P}$*-irreducible semimatroids of sixteen types in the lattice* $\mathbf{P}$ *over a four-element set* N*. They can be divided into the Ingleton semimatroids:* $[\![0]\!]$, $[\![r_1^{N-i}]\!]$ *for* $i \in N$, $[\![r_1^{ij}]\!]$ *for* $i, j \in N$ *distinct,* $[\![r_1^i]\!]$ *for* $i \in N$, $[\![r_1]\!]$, $[\![r_2^i]\!]$ *for* $i \in N$, $[\![r_2^{i\|j}]\!]$ *for* $i, j \in N$ *distinct,* $[\![r_2]\!]$, $[\![r_3]\!]$, $[\![g_i^{(2)}]\!]$, *for* $i \in N$, $[\![g_i^{(3)}]\!]$, *for* $i \in N$ *(cf. Example in* [10] *on p. 275), and the non-Ingleton semimatroids:*

$$\mathcal{L}_{ij}^{(kl|\emptyset)} = \{(kl|i),(kl|j),(ij|\emptyset)\} \cup \{(kl|ij)\} \cup \{(k|ij),(l|ij),(i|jkl),(j|ikl),(k|ijl),(l|ijk)\}$$

$$\mathcal{L}_{ij}^{(ij|kl)} = \{(ij|k),(ij|l),(kl|ij)\} \cup \{(kl|i),(kl|j)\}$$

(cf. Example 1 and Example 2 in [7] *on p. 416, correspondingly),*

$$\mathcal{L}_{ij}^{(ik|jl)} = \{(kl|ij),(ij|k),(ik|l)\} \cup \{(kl|j)\} \cup \{(l|ij),(l|ijk)\}$$

$$\mathcal{L}_{ij}^{(ik|j)} = \{(ij|k),(ik|l),(kl|j)\} \cup \{(i|jkl),(j|ikl),(k|ijl),(l|ijk)\}$$

and

$$\mathcal{L}_{ij}^{(jl|\emptyset)} = \{(kl|i),(jl|k),(ij|\emptyset)\} \cup \{(kl|ij)\} \cup \{(k|ij),(l|ij),(i|jkl),(j|ikl),(k|ijl),(l|ijk)\}$$

for $i, j, k, l \in N$ *distinct (cf. Example 1, Example 2 and Example 3 in the previous section, respectively). Every p-representable semimatroid is equal to intersection of some of the listed 120 semimatroids.*

We refer the reader to [8] for a visualization of the p-representable semimatroids containing only the couples $(ij|K)$ with $i \neq j$ and for the approach to the problem of p-representability through minors.

A p-representable semimatroid $\mathcal{L}$ on a finite set N is *rational* if it has at least one p-representation $\xi = (\xi_i)_{i \in N}$, $\mathcal{L} = [\![\xi]\!]$, which can be defined on a finite probability space with the uniform probability distribution. The equivalent requirement is that the distribution of ξ, living on its sample (state) space, takes only the rational values. Such a p-representation will be called *rational* as well. It is not difficult to recognize that every semimatroid on a set with at most three elements is rational. Since our examples of p-representations for all the irreducible p-representable semimatroids on a four-element set were rational and the intersections of rational semimatroids are obviously rational, we conclude by Theorem 4.1. that all p-representable semimatroids on a four-element set are rational. In [6] we found that all p-representable matroids are rational, even more, all their p-representations are rational. Therefore the smallest counterexample to the following conjecture must include five random variables.

Conjecture. Every p-representable semimatroid is rational.

The validity of this conjecture would have conceptual consequences for the p-representability of semimatroids. It would change the probabilistic nature of the problem to the study of the combinatorial structure of families of partitions. On matroids one can go even further to a purely algebraic formulation of the problem. Namely, a matroid is p-representable if and only if a system of generalized quasigroup equations has a nontrivial solution, see [9].

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