COMP 6361

Numerical Analysis of Nonlinear Equations

Assignment 2

Due Thursday February 18.

Consider again the Bratu boundary value problem on Pages 400-403 of the Background Notes, namely,

$$u''(x) \ + \ \lambda \ e^{u(x)} \ = \ 0 \ , \qquad \text{for} \quad x \in [0,1] \ , \quad \text{with} \quad u(0) \ = \ 0 \quad \text{and} \quad u(1) \ = \ 0 \ .$$

Use Collatz's "*Mehrstellenverfahren*" to discretize the Bratu equation, namely,

$$\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} \ + \ \frac{\lambda}{12}\Big(e^{u_{j-1}} + 10e^{u_j} + e^{u_{j+1}}\Big) \ = \ 0 \ , \qquad j = 1, 2, \cdots, N-1 \ ,$$

with $u_0 \ = \ u_N \ = \ 0$, and $h \ = \ 1/N$. Verify analytically that this local approximation is 4th-order accurate.

Solve the resulting nonlinear discrete equations by Newton's method, for each of a sequence of values of the parameter λ. The linear systems arising in Newton's method will be tridiagonal, so that you can re-use the tridiagonal solver from Assignment 1.

Specifically, first note that if $\lambda = 0$ then $u(x) = 0$ is a solution. Use this solution as initial guess in Newton's method for solving the discretized Bratu problem when $\lambda = \Delta\lambda$, where $\Delta\lambda$ is an appropriate small increment that you choose. Use an appropriate criterion for stopping the Newton iterations.

Upon convergence use the solution for $\lambda = \Delta\lambda$ as initial guess in Newton's method for $\lambda = 2\Delta\lambda$, and so on. Thus for each such value of λ an appropriate number of Newton iterations must be completed. This algorithm for computing such a sequence of solutions is known as "*parameter continuation*". Until what value of λ does this procedure give solutions? Try to locate the critical "final" value of λ as close as you can, by using smaller values of the increment $\Delta\lambda$ when approaching the critical value. Also keep track of the determinant that the tridiagonal matrix has upon convergence of Newton's method.

The Bratu problem has other solutions that were probably not found in Assigment 1, namely the solutions represented by the upper part of the curve in the left panel on Page 403 of the Background Notes. Can you also compute solutions along this upper part of the curve, using the parameter continuation method of the current assignment?

Give a concise summary and discussion of your overall findings.