O & W.

(1)

1.28. Two approaches

(1) If you want to prove that a system has a certain property you must prove it in general for all inputs.

(2) If you want to prove that it doesn't have a property provide a counter example.

If your initial guess is that property is not true and so you start to work on a counterexample.

If you have trouble with counterexample and you start to believe that system has that property you will have to switch tracks and start working on the general statement in approach (1)

Note: In using approach (1) it will be convenient to call a system T

so in (a) part $y[n] = x[-n]$

we may write $y[n] = T\{x[n]\} = x[-n]$.

1.28 a) $y[n] = x[-n]$

discrete time system (square bracket convention).

1) Memoryless.

try counterexample

Examine $y[1] = x[-1]$

Clearly output at time $t=1$ depends on input at another time $(t=-1)$

$\therefore$ NOT MEMORYLESS

2) Time Inv.

try counter ex.

define $x_1[n] = \delta[n]$

$\therefore \quad y_1[n] = \delta[-n] = \delta[n]$

Now $x_2[n] = x[n-4]$ ← delayed by 4.

$= \delta[n-4]$

$y_2[n] = \delta[-n-4] = \delta[n+4]$

delaying input by four caused output to be advanced by 4

$\therefore$ NOT T.I

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3) <u>Linear</u> (suspect true)

$$x_1[n] \xrightarrow{T} y_1[n]$$

$$x_2[n] \xrightarrow{T} y_2[n]$$

Examine $T\{x_1[n] + x_2[n]\}$

$$= x_1[-n] + x_2[-n]$$

$$= T\{x_1[n]\} + T\{x_2[n]\}$$

$\therefore$ additivity holds.

Check $T\{a\, x_1[n]\}$ $\qquad a \in \mathbb{C}$

$$= a\, x_1[-n]$$

$$= a\, T\{x_1[n]\}$$

$\therefore$ scalability holds

$\therefore$ system is linear

4) Causal

Counter example

$x_1[n] = \delta[n-1]$ ← impulse at 1

$y_1[n] = T\{x_1[n]\} = \delta[-n+1]$
$= \delta[n+1]$ — impulse at −1

$\therefore$ output at $t = -1$ depends on input at time 1

$\therefore$ is NOT ~~[illegible]~~ causal

5) Stable.

given $x[n]$ s.t $\exists\ M_x < \infty$, $M \in \mathbb{R}$

s.t. $|x[n]| < M_x \quad \forall\, n \in \mathbb{Z}$

i.e. input $x[n]$ bounded

Output is. $|y[n]| = |T\{x[n]\}| = |x[-n]| < M_x$

Clearly $y[n] < M_x \quad \forall\, n \in \mathbb{Z}$

$\therefore$ output is bounded

$\therefore$ system is BIBO Stable

d) $y[n] = x[n-2] - 2x[n-8]$

1). Memoryless

$x[n] = \delta[n]$

$y[n] = T\{x[n]\} = \delta[n-2] - 2\delta[n-8]$

Clearly output at time 2 & 8 depended on input at time 0

$\therefore$ NOT Memoryless

2) T.I.

$$y[n] = T\{x[n]\}$$

Examine

$$T\{x[n-n_0]\}$$

$$= x[n-n_0-2] - 2x[n-n_0-8]$$

$$= x[n-2-n_0] - 2x[n-8-n_0]$$

$$= y[n-n_0]$$

∴ system is T.I.

3) linear.

$$x_1[n] \xrightarrow{T} y_1[n]$$

$$x_2[n] \xrightarrow{T} y_2[n]$$

Check Additivity.

$$T\{x_1[n] + x_2[n]\}$$

$$= x_1[n-2] + x_2[n-2] - 2x_1[n-8] - 2x_2[n-8]$$

$$= x_1[n-2] - 2x_1[n-8] + x_2[n-2] - 2x_2[n-8]$$

$$= T\{x_1[n]\} + T\{x_2[n]\}$$

∴ Additivity holds

Scalability

Check $T\{a x_1[n]\}$ $a \in \mathbb{C}$

$$= a x_1[n-2] - 2a x_1[n-8]$$
$$= a\left(x_1[n-2] - 2x_1[n-8]\right)$$
$$= a\,T\{x_1[n]\}$$

$\therefore$ scalability hold

$\therefore$ system linear.

4). Causal.

Now $y[n] = x[n-2] - 2x[n-8]$

Clearly $\forall\, n \in \mathbb{Z}$ the output at n depends on input at $n-2$ and $n-8$ all in the past.

$\therefore$ system is causal

5) Stable.

Assume $\exists$ $M_x < \infty$ s.t.

$$|x[n]| < M_x \qquad \forall n \in \mathbb{Z}$$

Now
$$|y[n]| = |x[n-2] - 2x[n-8]|$$
$$\leq |x[n-2]| + 2\,|x[n-8]|$$

Δ inequality

$|y[n]| < M_x + 2M_x = 3M_x < \infty$

$\therefore$ $y[n]$ is bounded with bound $3M_x$

$\therefore$ system is BIBO stable.

d). $y[n] = n\,x[n]$

1) Memoryless.

Clearly for $\forall n \in \mathbb{Z}$ output at time n depends only on input at time n.

$\therefore$ Memoryless

2). T. I.

Counter example.

Let $x_1[n] = \delta[n]$

Then $y_1[n] = T\{x_1[n]\} = n\,\delta[n] = 0$

Now $x_2[n] = x_1[n-1] = \delta[n-1]$

$y_2[n] = T\{x_2[n]\} = T\{\delta[n-1]\}$

$= n\,\delta[n-1]$

$= 1$

$\therefore$ delaying input by 1 resulted in output which was not just a delay of original output

NOT T.I

3) Linear.

$$x_1[n] \xrightarrow{T} y_1[n]$$
$$x_2[n] \xrightarrow{T} y_2[n]$$

Additivity Check $T\{x_1[n] + x_2[n]\}$

$$= n(x_1[n] + x_2[n])$$
$$= n x_1[n] + n x_2[n]$$
$$= T\{x_1[n]\} + T\{x_2[n]\}$$

$\therefore$ additivity works

Scalability Check $T\{a x_1[n]\}$ $\quad a \in \mathbb{C}$

$$= a n x_1[n]$$
$$= a T\{x_1[n]\}$$

$\therefore$ scalability works

$\therefore$ system is linear.

4). Causal

Memoryless $\Rightarrow$ Causal

$\therefore$ <u>system is causal</u>

5) ∴ Stable.

Counterexample.

Let $x_1[n] = 1 \quad \forall n.$

Clearly $x_1[n]$ is bounded.
for example 2 is a bound

$$T\{x_1[n]\} = n\, x_1[n] = n.$$

which grows without limit as $n \uparrow$

∴ output not bounded

∴ NOT BIBO stable.

d) $y[n] = \mathcal{E}v\{x[n-1]\}$

i) Memoryless

Counterexample

$x_1[n] = \delta[n]$

$$T\{x_1[n]\} = \mathcal{E}v\{\delta[n-1]\}$$

$$= \tfrac{1}{2}\left(\delta[n-1] + \delta[n+1]\right)$$

∴ output at time $t = -1$ depends on input at time $t = 1$

∴ NOT Memoryless

4) Causal
use above counterexample again

out of order

since output at time $t = -1$ depends on input at $t = 0$ (future)

System is NOT causal

3). linear.

$$x_1[n] \xrightarrow{T} y_1[n]$$
$$x_2[n] \xrightarrow{T} y_2[n]$$

Additivity.
Check.

$$T\{x_1[n] + x_2[n]\}$$
$$= \mathcal{E}v\{x_1[n-1] + x_2[n-1]\}$$
$$= \tfrac{1}{2}\left[x_1[n-1] + x_2[n-1] + x_1[-n-1] + x_2[-n-1]\right]$$
$$= \tfrac{1}{2}\left(x_1[n-1] + x_1[-n-1]\right) + \tfrac{1}{2}\left[x_2[n-1] + x_2[-n-1]\right]$$
$$= \mathcal{E}v\{x_1[n-1]\} + \mathcal{E}v\{x_2[n-1]\}$$
$$= T\{x_1[n]\} + T\{x_2[n]\}$$

$\therefore$ additivity holds

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Scaling. Check

$$T\{a\, x_1[n]\} \qquad a \in \mathbb{C}$$

$$= \mathcal{E}v\{a x_1[n-1]\}$$

$$= \tfrac{1}{2}\left(a x_1[n-1] + a x_1[-n-1] \right)$$

$$= \tfrac{1}{2} a \left(x_1[n-1] + x_1[-n-1] \right)$$

$$= a\, \mathcal{E}v\{ x_1[n-1]\}$$

$$= a\, T\{x_1[n]\}$$

∴ scaling holds

∴ linear holds

2) use T.I.

we imagine that TI won't hold since output must always be even. but the delay of an even function is not even

Counter ex.

$$x_1[n] = \delta[n]$$

$$T\{x_1[n]\} = \mathcal{E}v\{\delta[n-1]\} = \tfrac{1}{2}(\delta[n-1] + \delta[-n-1])$$

Check $T\{x_1[n-1]\} = T\{\delta[n-1]\}$

$$= \mathcal{E}v\{\delta[n-2]\} = \tfrac{1}{2}(\delta[n-2] + \delta[-n-2])$$

of This is <u>not</u> just a delayed version original output.

$\therefore$ <u>NOT</u> TI.

5). Stable.

Assume $\exists \ M_x < \infty$ s.t.

$$|x[n]| < M_x \qquad \forall n \in \mathbb{Z}$$

Examine

$$\begin{aligned} |y[n]| &= |T\{x[n]\}| \\ &= |\mathcal{E}v\{x[n-1]\}| \\ &= \left| \tfrac{1}{2}\left[x[n-1] + x[-n-1]\right]\right| \\ &\leq \tfrac{1}{2}|x[n-1]| + \tfrac{1}{2}|x[-n-1]| \quad (\Delta \text{ ineq.}) \\ &< \tfrac{1}{2}M_x + \tfrac{1}{2}M_x \\ &= M_x \end{aligned}$$

$\therefore$ output is bdd. with same bound as input

$\therefore$ system is stable

$0 \frac{2}{i} W$ 1.28 (13)

i) $y[n] = \begin{cases} x[n] & n \geq 1 \\ 0 & n = 0 \\ x[n+1] & n \leq -1 \end{cases}$

clips out input signal at time t = 0 moves left side further left.

1). Memoryless.

Counter example.

$x_1[n] = \delta[n+1]$

$y_1[n] = T\{x_1[n]\} = \delta[n+2]$

Clearly output at time -2 depends on input at time 1.

$\therefore$ NOT Memoryless.

2) T.I.

counter ex.

$x_1[n] = \delta[n]$

$y_1[n] = T\{x_1[n]\} = \delta[n+1]$

consider. $x_1[n-1] = \delta[n-1]$

$T\{x_1[n-1]\} = T\{\delta[n-1]\} = \delta[n-1]$

not just a delayed version of orig output

$\therefore$ not T.I.

3) linear.

Examine

$$T\{x_1[n] + x_2[n]\}$$

$$= \begin{cases} x_1[n] + x_2[n] & n \geq 1 \\ 0 & n=0 \\ x_1[n+1] + x_2[n] & n \leq -1 \end{cases}$$

$$= \begin{cases} x_1[n] & n \geq 1 \\ 0 & n=0 \\ x_1[n+1] & n \leq -1 \end{cases} + \begin{cases} x_2[n] & n \geq 1 \\ 0 & n=0 \\ x_2[n+1] & n \leq -1 \end{cases}$$

$$= T\{x_1[n]\} + T\{x_2[n]\}$$

$\therefore$ additivity holds.

Scaling

Examine $T\{ax_1[n]\}$ $a \in \mathbb{C}$

$$= \begin{cases} ax_1[n] & n \geq 1 \\ 0 & n=0 \\ ax_1[n+1] & n \leq -1 \end{cases} = a \begin{cases} x_1[n] & n \geq 1 \\ 0 & n=0 \\ x_1[n+1] & n \leq -1 \end{cases}$$

$$= aT\{x_1[n]\}$$

$\therefore$ scaling holds

$\therefore$ system <u>linear</u>

4) Causal

Using counter example

$$x_1[n] = \delta[n]$$

$$y_1[n] = T\{x_1[n]\} = \delta[n+1]$$

$\therefore$ output at $t=-1$ depends on input at $t=0$ (future)

$\therefore$ not causal.

5) Stable

Assume $\exists$ $M_x < \infty$ s.t.

$$|x[n]| < M_x \qquad \forall n \in \mathbb{Z}$$

Examine

$$|y[n]| = \left| \begin{cases} x[n] & n \geq 1 \\ 0 & n=0 \\ x[n+1] & n \leq -1 \end{cases} \right|$$

$$= \begin{cases} |x[n]| & n \geq 1 \\ 0 & n=0 \\ |x[n+1]| & n \leq -1 \end{cases} = \begin{cases} M_x & n \geq 1 \\ 0 & n=0 \\ M_x & n \leq -1 \end{cases}$$

$|y[n]| < M_x$.

$\therefore$ $y[n]$ is bdd by same bound as $x[n]$

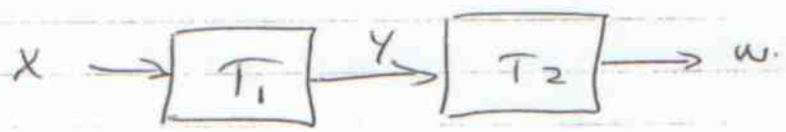
$\therefore$ system is stable.

O & W

1.30 a). $y(t) = x(t-4)$

clearly inverse operation is to advance by 4. (delay input by 4)

so using



(use this diagram for future parts of this question).

then. T_2 (inverse) is

$$w(t) = y(t+4).$$

b) $y(t) = \cos\left(x(t)\right)$

Not invertible.

let $x_1(t) = 0$ constant $\forall t \in \mathbb{R}$

$x_2(t) = 2\pi$

$y_1(t) = T_1\{x_1(t)\} = 1$

$y_2(t) = T_2\{x_2(t)\} = 1$

same output diff. inputs

$\therefore$ not 1-1 $\therefore$ not invertible.

c) $y[n] = n\,x[n]$.

↳ only problem pt. here is $n=0$

consider $x_1[n] = \delta[n]$

$x_2[n] = 2\,\delta[n]$.

$y_1[n] = T_1\{x_1[n]\} = 0$

$y_2[n] = T_1\{x_2[n]\} = 0$

∴ two diff inputs have same output

∴ not 1-1

∴ not invertible.

d) $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$.

Recall calculus formula for diff. integral functions

$$\frac{d}{du} \int_{\psi_1(u)}^{\psi_2(u)} f(u,v)\,dv$$

$$= \int_{\psi_1(u)}^{\psi_2(u)} f_u(u,v)\,dv - \psi_1'(u)\,f(u,\psi_1(u))$$

$$+ \psi_2'(u)\,f(u,\psi_2(u)))$$

where prime is deriv. wrt u.
and f_u is partial deriv. wrt u.

Using this formula we note:

$$\frac{d}{dt} y(t) = \frac{d}{dt} \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$= x(t)$$

Unsurprisingly since integrating then diff. should yield same result.

i.e. inverse system T_2 is

$$w(t) = \frac{d}{dt} y(t)$$

e) $y[n] = \begin{cases} x[n-1] & n \geq 1 \\ 0 & n = 0 \\ x[n] & n \leq -1 \end{cases}$

take $x[n]$ for $n \geq 0$ delay by 1
leave rest alone.
opens hole at $n = 0$

Clearly inverse system requires only that we move data back

1.30a
O & W

inverse system

(4)

$$w[n] = \begin{cases} y[n+1] & n \geq 0 \\ y[n] & n < 0 \end{cases}$$

(b) $y[n] = x[n]\,x[n-1]$.

$x_1[n] = 0 \Rightarrow y_1[n] = T_1\{x_1[n]\} = 0$

$x_2[n] = \delta[n] \Rightarrow y_2[n] = T_2\{x_1[n]\} = 0$

∴ two inputs give same output
∴ not 1-1.

∴ <u>not invertible.</u>

1.28 d) Proof of time variance:

$$y[n] = \mathcal{E}_v\{x[n-1]\} = \frac{1}{2}x[n-1] + \frac{1}{2}x[-n-1]$$

$$x_1[n] \longrightarrow y_1[n] = \frac{1}{2}x_1[n-1] + \frac{1}{2}x_1[-n-1]$$

$$x_2[n] = x_1[n-n_0] \longrightarrow y_2[n] = \frac{1}{2}x_2[n-1] + \frac{1}{2}x_2[-n-1]$$

$$= \frac{1}{2}x_1[n-1-n_0] + \frac{1}{2}x_1[-n-1-n_0]$$

$$y_1[n-n_0] = \frac{1}{2}x_1[n-1-n_0] + \frac{1}{2}x_1[-n-1+n_0]$$

therefor $y_1[n-n_0] \neq y_2[n]$ for all value of n

and system is time variant.

Note that this is another method, See also the method in the solution of Tutorial 3.