Control Systems

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1 Syllabus

This is the syllabus for the exam that I required to take:

> Nodal and mesh analysis of linear, finite, passive circuits; equivalent networks. Steady state AC response of lumped constant, time-invariant networks. Time and frequency response of linear systems; impulse response and transfer functions, Laplace transform analysis, frequency response, including steady-state sinusoidal circuits. Models, transfer functions, and system response. Root locus analysis and design. Feedback and stability; Bode diagrams. Nyquist criterion, frequency domain design. State variable representation. Simple PID control systems.

2 Transforms

2.1 The Fourier Transform

A periodic function $f(t)$ with period T can be expressed as a sum of sine and cosine functions:

$$f(t) = \frac{a_0}{T} + \frac{2}{T}\sum_{n=1}^{\infty}(a_n \cos\omega_n t + b_n \sin\omega_n t)$$

where the angular frequency of each term is given by

$$\omega_n = \frac{2\pi n}{T}.$$

The coefficients a_n and b_n are defined by

$$\begin{aligned} a_n &= \int_{-T/2}^{T/2} f(t)\cos\omega_n t\,dt, \quad n = 0, 1, 2, 3, \ldots \\ b_n &= \int_{-T/2}^{T/2} f(t)\sin\omega_n t\,dt, \quad n = 1, 2, 3, \ldots \end{aligned}$$

Using the formulas $\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$ and $\sin\theta = \frac{j}{2}(e^{j\theta} - e^{-j\theta})$, we can express this sum using complex exponents as

$$f(t) = \frac{a_0}{T} + \frac{1}{T}\sum_{n=1}^{\infty}\left[(a_n - jb_n)e^{j\omega_n t} + (a_n + jb_n)e^{-j\omega_n t}\right] \quad (1)$$

where

$$\begin{aligned} a_n - jb_n &= \int_{-T/2}^{T/2} f(t)e^{-j\omega_n t}dt \\ a_n + jb_n &= \int_{-T/2}^{T/2} f(t)e^{j\omega_n t}dt \end{aligned}$$

Since $\omega_n = 2\pi n/T$, $\omega_{-n} = -\omega_n$ and (1) can be written

$$f(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty}\left\{\int_{-T/2}^{T/2} f(t)e^{-j\omega_n t}dt\right\}e^{j\omega_n t}.$$

We can split this into two parts:

$$f(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} C_n e^{j\omega_n t}$$

where

$$C_n = \int_{-T/2}^{T/2} f(t)e^{-j\omega_n t}dt.$$

Example 1: Fourier transform of a pulse train. Define a ***pulse*** by

$$p(t) = \begin{cases} A, & \text{if } -p/2 \le t \le p/2, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

and assume that $f(t)$ is this function extended periodically with period T — in other words, f is a pulse train. (Clearly, we require $p < 2T$.) Then

$$\begin{aligned} C_n &= \int_{-p/2}^{p/2} Ae^{-j\omega_n t}dt \\ &= -\frac{A}{j\omega_n}e^{-j\omega_n t}\Big|_{-p/2}^{p/2} \\ &= \frac{2A}{\omega_n}\sin\frac{\omega_n p}{2} \end{aligned}$$

and

$$\begin{aligned} f(t) &= \frac{1}{T}\sum_{n=0}^{\infty}\frac{2A}{\omega_n}\sin\left(\frac{\omega_n p}{2}\right)e^{j\omega_n t} \\ &= \frac{A}{\pi}\sum_{n=0}^{\infty}\frac{1}{n}\sin\left(\frac{\omega_n p}{2}\right)e^{j\omega_n t} \end{aligned}$$

since $\dfrac{2}{T\omega_n} = \dfrac{1}{n\pi}$.
□

If we let the period, T, tend to infinity, we obtain

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

which is the ***Fourier transform*** of an arbitrary (i.e., non-periodic) function of time, $f(t)$.

The Fourier transform has an important limitation: it is valid for a function $f(t)$ only if $\int_{-\infty}^{\infty}|f(t)|dt$ has a finite value. For example, the "step function", $u(t)$, has no Fourier transform. The Laplace transform introduces an additional "damping factor", e^{-st}, and consequently can be used for a greater variety of functions.

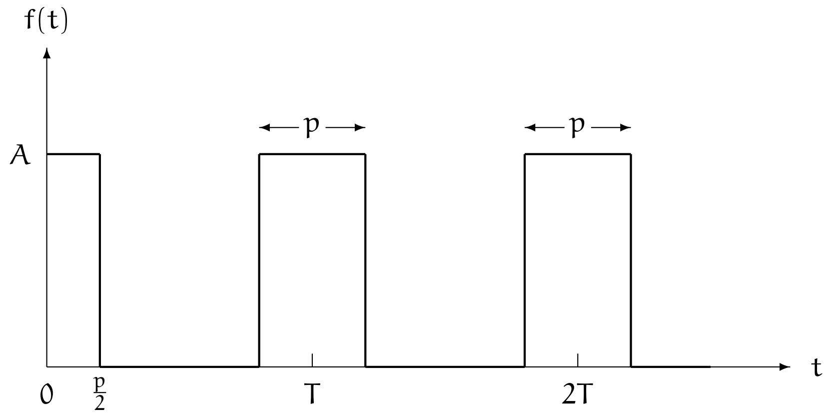


Figure 1: A periodic pulse train

2.2 The Laplace Transform

The Laplace transform of a function $f(t)$ is

$$\mathcal{L}(f(t)) = \int_0^\infty f(t)\, e^{-st}\, dt.$$

By convention, we write $F(s)$ for $\mathcal{L}(f(t))$.

The Laplace transform is linear:

$$\begin{aligned}\mathcal{L}(A\, f(t) + B\, g(t)) &= \int_0^\infty (A\, f(t) + B\, g(t))\, e^{-st}\, dt \\ &= A \int_0^\infty f(t)\, e^{-st}\, dt + B \int_0^\infty g(t)\, e^{-st}\, dt \\ &= A\, \mathcal{L}(f(t)) + B\, \mathcal{L}(g(t)).\end{aligned}$$

The transform of a derivative has a simple relationship to the transform of the original function. Using the formula for integration by parts

$$uv = \int u\, dv + \int v\, du \qquad (2)$$

with $u(t) = f(t)$ and $v(t) = e^{-st}$ gives

$$f(t)\, e^{-st}\Big|_0^\infty = -s \int_0^\infty f(t)\, e^{-st}\, dt + \int_0^\infty f'(t)\, e^{-st}\, dt. \qquad (3)$$

Assuming $\lim_{t\to\infty} f(t)\, e^{-st} = 0$, (3) leads to

$$\mathcal{L}(f'(t)) = s\, \mathcal{L}(f(t)) - f(0^+).$$

Differentiating again:

$$\begin{aligned}\mathcal{L}\left(f''(t)\right) &= s\mathcal{L}\left(f'(t)\right) - f'(0^+) \\ &= s^2F(s) - sf(0^+) - f'(0^+).\end{aligned}$$

The transform of an integral can be obtained in a similar way. Substituting

$$\begin{aligned}u(t) &= \int_0^t f(\tau)\,d\tau \\ v(t) &= e^{-st}\end{aligned}$$

in (2) gives

$$e^{-st}\int_0^t f(\tau)\,d\tau\bigg|_0^\infty = -s\int_0^\infty \left(\int_0^t f(\tau)\,d\tau\right) e^{-st}\,dt + \int_0^\infty f(t)\,e^{-st}\,dt.$$

The term on the left is zero since it contains the factor $\int_0^0 f(\tau)\,d\tau$ at $t = 0$ and the factor e^{-st} at $t = \infty$. Thus

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}\mathcal{L}\left(f(t)\right).$$

Suppose that $f(t)$ is a variable in a control system. Then typical operations on f scale it by a linear factor (e.g., amplifier), differentiate it (e.g., inductor), or integrate it (e.g., capacitor). Consequently, the Laplace transform of f is likely to be a rational function of s or, in other words, it will be a function of the form $M(s) = \dfrac{P(s)}{Q(s)}$ in which $P(s)$ and $Q(s)$ are polynomials in s. The roots of P are the ***zeroes*** of M and the roots of Q are the ***poles*** of M.

Theorem 2.1 ***Initial Value*** *Provided that the limits exist:*

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s).$$

Theorem 2.2 ***Final Value*** *Provided that the limits exist:*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s).$$

Example 2: Laplace transform of e^{-at}. Let $f(t) = e^{-at}$. Then

$$\begin{aligned}F(s) &= \int_0^\infty e^{-at}e^{-st}\,dt \\ &= \int_0^\infty e^{-(a+s)t}\,dt \\ &= \frac{e^{-(a+s)t}}{a+s}\bigg|_0^\infty \\ &= 0 - \left(-\frac{1}{a+s}\right) \\ &= \frac{1}{s+a}.\end{aligned}$$

Function	Transform	Remarks
$f(t)$	$F(s)$	General notation
$Af(t) + Bg(t)$	$AF(s) + BG(s)$	Linearity
$f(t-T)u(t-T)$	$e^{-sT}F(s)$	$u(t)$ is the unit step function
$e^{-at}f(t)$	$F(s+a)$	
$f'(t)$	$sF(s) - f(0^+)$	
$f''(t)$	$s^2F(s) - sf(0^+) - f'(0^+)$	
$\int_0^t f(\tau)d\tau$	$F(s)/s$	
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$	

Figure 2: General properties of the Laplace transform

□

Example 3: Laplace transforms of $\cos\omega t$ and $\sin\omega t$. Let $f(t) = \cos\omega t$. Then

$$\begin{aligned} F(s) &= \int_0^\infty \cos(\omega t)e^{-st}\,dt \\ &= \frac{1}{2}\int_0^\infty e^{-st+j\omega t}\,dt + \frac{1}{2}\int_0^\infty e^{-st-j\omega t}\,dt \\ &= \frac{1}{2}\frac{e^{-st+j\omega t}}{-s+j\omega}\bigg|_0^\infty + \frac{1}{2}\frac{e^{-st-j\omega t}}{-s-j\omega}\bigg|_0^\infty \\ &= -\frac{1}{2}\left(\frac{1}{-s+j\omega} + \frac{1}{-s-j\omega}\right) \\ &= -\frac{1}{2}\left(\frac{-s-j\omega}{s^2+\omega^2} + \frac{-s+j\omega}{s^2+\omega^2}\right) \\ &= \frac{s}{s^2+\omega^2}. \end{aligned}$$

Let

$$\begin{aligned} u &= \sin\omega t \\ du &= \omega\cos\omega t\,dt \\ v &= e^{-st} \\ dv &= -se^{-st}\,dt \end{aligned}$$

and apply the rule for integration by parts, $uv = \int u\,dv + \int v\,du$:

$$\sin(\omega t)e^{-st}\Big|_0^\infty = -s\int_0^\infty \sin(\omega t)e^{-st}\,dt + \omega\int_0^\infty \cos(\omega t)e^{-st}\,dt$$

Function	Transform	Remarks
$\delta(t)$	1	Dirac's δ-function
$u(t)$	$\frac{1}{s}$	Unit step function
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	
e^{-at}	$\frac{1}{s+a}$	
$\frac{1-e^{-at}}{a}$	$\frac{1}{s(s+a)}$	
$\cos at$	$\frac{s}{s^2+a^2}$	
$\cosh at$	$\frac{s}{s^2-a^2}$	
$\sin at$	$\frac{a}{s^2+a^2}$	
$\sinh at$	$\frac{a}{s^2-a^2}$	
$\frac{1-\cos at}{a^2}$	$\frac{1}{s(s^2+a^2)}$	
$\frac{at-\sin at}{a^3}$	$\frac{1}{s^2(s^2+a^2)}$	
te^{-at}	$\frac{1}{(s+a)^2}$	
$e^{-at}(1-at)$	$\frac{s}{(s+a)^2}$	
$\frac{e^{-at}\sin bt}{b}$	$\frac{1}{s^2+2\zeta\omega+\omega^2}$	$a=\zeta\omega,\ b=\omega\sqrt{1-\zeta^2}$, and $\zeta<1$

Figure 3: Particular Laplace transforms

which is equivalent to

$$0 = -s\mathcal{L}(\sin\omega t) + \omega\mathcal{L}(\cos\omega t)$$

and therefore

$$\begin{aligned}\mathcal{L}(\sin\omega t) &= \frac{\omega}{s}\mathcal{L}(\cos\omega t)\\ &= \frac{\omega}{s^2+\omega^2}.\end{aligned}$$

□

2.2.1 Partial Fractions

The last step of a solution that employs Laplace transforms usually requires the inverse transformation of an expression of the form $\frac{P(s)}{Q(s)}$ in which P and Q are polynomials in s. Since the inverse transform of $\frac{1}{s+a}$ is straightforward, we need to express such rational functions in the form

$$\frac{A_1}{s+s_1} + \frac{A_2}{s+s_2} + \frac{A_3}{s+s_3} + \cdots$$

If $Q(s)$ has only simple roots, we use the following rule: if

$$Q(s) = (s+s_1)(s+s_2)\cdots(s+s_n)$$

then

$$\frac{P(s)}{Q(s)} = \sum_{i=1}^{n} \frac{A_i}{s+s_i}$$

where

$$A_i = (s+s_i)\left.\frac{P(s)}{Q(s)}\right|_{s=-s_i}.$$

The situation is slightly more complicated when there are repeated roots. If

$$Q(s) = \cdots(s+s_i)^r\cdots$$

then the partial fraction expansion of $\frac{P(s)}{Q(s)}$ will include the terms

$$\frac{B_1}{s+s_i} + \frac{B_2}{(s+s_i)^2} + \cdots + \frac{B_r}{(s+s_i)^r}$$

in which

$$\begin{aligned}
B_r &= (s+s_i)^r \left.\frac{P(s)}{Q(s)}\right|_{s=-s_i} \\
B_{r-1} &= \frac{1}{1!}\frac{d}{ds}\left.\left\{(s+s_i)^r \frac{P(s)}{Q(s)}\right\}\right|_{s=-s_i} \\
B_{r-2} &= \frac{1}{2!}\frac{d^2}{ds^2}\left.\left\{(s+s_i)^r \frac{P(s)}{Q(s)}\right\}\right|_{s=-s_i} \\
&\vdots \\
B_1 &= \frac{1}{(r-1)!}\frac{d^{r-1}}{ds^{r-1}}\left.\left\{(s+s_i)^r \frac{P(s)}{Q(s)}\right\}\right|_{s=-s_i}
\end{aligned}$$

2.3 The Z-Transform

A ***sampling system*** takes "samples" of a continuous input at discrete points in time. Each sample can be represented by Dirac's δ function with an appropriate amplitude. Thus if $e(t)$ is a continuous signal sampled at times $0, T, 2T, 3T, \ldots$, the sampled signal is given by

$$e^*(t) = \sum_{n=0}^{\infty} e(nT)\delta(t - nT)$$

where T is the ***sampling period***.

The Laplace transform of $e^*(t)$ is

$$E^*(s) = \sum_{n=0}^{\infty} e(nT)e^{-nTs}. \quad (4)$$

In order to avoid the non-algebraic terms that would be introduced by e^{-nTs}, we change the variables by defining

$$z = e^{Ts}.$$

With this change, we have $s = \frac{\ln z}{T}$, $e^{-nTs} = \left(e^{Ts}\right)^{-n} = z^{-n}$, and (4) becomes

$$\begin{aligned} E^*(s) &= E^*\left(\frac{\ln z}{T}\right) \\ &= \sum_{n=0}^{\infty} e(nT)z^{-n} \\ &= E(z) \end{aligned}$$

and we say that $E(z)$ ***is the z-transform of*** $e(t)$. Figure 4 shows some common z-transforms.

In general, $E(z) = e(0)\,z^0 + e(T)\,z^{-1} + e(2T)\,z^{-2} + \cdots + e(nT)\,z^{-n} + \cdots$ and we can think of the nth coefficient as representing the function at time nT.

Example 4: z-transform of unit step. For the unit step function, $r(t) = 1$ for $t \geq 0$ and therefore $e(nT) = 1$ for all $n \geq 0$. Thus

$$\begin{aligned} R(z) &= \sum_{n=0}^{\infty} z^{-n} \\ &= \frac{z}{z-1}. \end{aligned}$$

□

Example 5: z-transform of unit ramp. For the unit ramp function, $r(t) = t$ for $t \geq 0$ and therefore $e(nT) = nT$ for $n \geq 0$.

$$\begin{aligned} R(z) &= \sum_{n=0}^{\infty} nTz^{-n} \\ &= T(z^{-1} + 2z^{-2} + \cdots) \\ &= T\frac{z}{(z-1)^2}. \end{aligned}$$

Time function	z-transform	Remark
$\delta(t)$	1	Impulse
$u(t)$	$\frac{z}{z-1}$	Unit step
$\delta_T(t)$	$\frac{z}{z-1}$	$\delta_T(t) = \sum_{n=0}^{\infty} \delta(t-nT)$
t	$\frac{Tz}{(z-1)^2}$	
$\frac{1}{2}t^2$	$\frac{T^2 z(z+1)}{2(z-1)^3}$	
e^{-at}	$\frac{z}{z-e{-}aT}$	
$1-e^{-at}$	$\frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$	
$\sin \omega t$	$\frac{z \sin \omega T}{z^2 - 2z \cos \omega t + 1}$	

Figure 4: Common z-transforms

□

Example 6: z-transform of exponential decay. For the exponential decay function, $r(t) = e^{-at}$ for $t \geq 0$ and $e(nT) = e^{-aTn}$ for $n \geq 0$.

$$R(z) = \sum_{n=0}^{\infty} e^{-aTn} z^{-n}.$$

We note that

$$\begin{aligned} R(z) &= & +1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + \cdots \\ R(z)ze^{aT} &= ze^{aT} & +1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + \cdots \end{aligned}$$

and therefore

$$R(z)ze^{aT} - R(z) = ze^{aT}$$

from which

$$\begin{aligned} R(z) &= \frac{ze^{aT}}{ze^{aT} - 1} \\ &= \frac{z}{z - e^{-aT}}. \end{aligned}$$

□

3 A Simple Control System

In the simple control system shown in Figure 5:

$$R(s) = \text{System input, or stimulus signal}$$

$$
\begin{aligned}
E(s) &= \text{Actuating signal} \\
G(s) &= \text{Forward path, or open-loop, transfer function} \\
C(s) &= \text{System output, or controlled signal} \\
H(s) &= \text{Feedback path transfer function} \\
B(s) &= \text{Feedback signal} \\
G(s)H(s) &= \text{Loop transfer function}
\end{aligned}
$$

All functions are expressed as Laplace transforms of the corresponding time-domain functions.

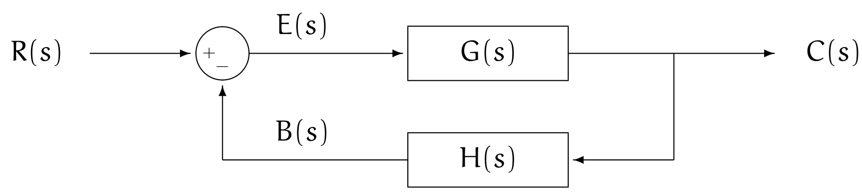


Figure 5: A simple control system

Inspecting Figure 5, we see that:

$$
\begin{aligned}
E(s) &= R(s) - B(s) \\
C(s) &= E(s)\,G(s) \\
B(s) &= C(s)\,H(s)
\end{aligned}
$$

From these equations,

$$
\begin{aligned}
C(s) &= E(s)\,G(s) \\
&= [R(s) - B(s)]\,G(s) \\
&= [R(s) - C(s)\,H(s)]\,G(s)
\end{aligned}
$$

and therefore

$$C(s) = \frac{R(s)\,G(s)}{1 + G(s)\,H(s)}$$

and the ***transfer function***, $M(s)$, is given by

$$M(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)}.$$

Since the transfer function without feedback is simply $G(s)$, the denominator $1 + G(s)H(s)$ represents the effect of feedback. Note that the sign of the feedback (negative) is assumed: $B(s)$ is ***subtracted*** from $R(s)$.

The ***characteristic equation*** is obtained by equating the denominator of the transfer function to zero. For the system in Figure 5, the characteristic equation is $R(s) = 0$. In many cases — for example, when $G(s)$ is a simple polynomial — it will be

$$1 + G(s)H(s) = 0.$$

From the form of the characteristic equation, we note that G and H have ***reciprocal units***. In the following example, the units of G are radians/volt and the units of H must therefore be volts/radian.

4 Characteristics of Electrical and Mechanical Systems

Kirchoff's Laws

Current Law (KCL). The algebraic sum of the currents leaving/entering a node is zero.

Voltage Law (KVL). The algebraic sum of the potential differences around a loop is equal to the algebraic sum of the e.m.f.'s acting around the loop in the same direction.

Definition 4.1 ***Mesh Analysis*** *Assume an unknown current in every loop of the circuit; apply KVL to every loop.*

Definition 4.2 ***Node Analysis*** *Assume zero potential at a selected node and an unknown potential difference (voltage) between this node and each other node; apply KCL at each node.*

4.1 Passive Electrical Components

For any circuit component, we can write:

- an equation that gives the relationship of the voltage difference across the component and the current flowing through it as a function of time;
- the relationship between phasors (see below) when voltage and current vary sinusoidally; and
- the Laplace transform of the time function.

Resistance. For resistance, the time function (and hence the other functions) is very simple: $v(t) = R\,i(t)$.

Capacitance. A capacitance stores charge, giving

$$i = C\frac{dv}{dt}$$

or

$$v = \frac{1}{C}\int_{\tau=0}^{t} i\,d\tau.$$

Inductance. An inductance has the inverse effect:

$$v = L\frac{di}{dt}.$$

Component	Time	Frequency	Laplace
	$v = f(i)$	$V(\omega)/I(\omega)$	$V(s)/I(s)$
Resistance	Ri	R	R
Capacitance	$\frac{1}{C}\int i\,dt$	$1/j\omega C$	$1/sC$
Inductance	$L\dfrac{di}{dt}$	$j\omega L$	sL

Figure 6: Formulas for passive components

Phasors are derived as follows: we use inductance as an example. Assume, in the equation above, that $i = Ae^{j\omega t}$. Then

$$\begin{aligned} v &= L\frac{di}{dt} \\ &= L\frac{d}{dt}\left(Ae^{j\omega t}\right) \\ &= LAj\omega e^{j\omega t} \\ &= j\omega Li \end{aligned}$$

and therefore

$$\frac{v}{i} = j\omega L$$

in which $j\omega L$ is a ***phasor*** that expresses the impedance of the inductance. In general, we can forget about the term $e^{j\omega t}$ in our calculations, because it cancels out in all the equations. We can ignore that amplitude, A, for the same reason.

The three forms are summarized in Figure 6.

It is useful to note that there are several ways of combining components to give terms of particular dimensions. For example, RC has the dimensions of time and $RC\omega$ is therefore a pure number. Similarly, LC has dimension $\sec^2$ and $LC\omega$ is therefore a pure number. See Section 7 for more examples.

4.2 Phase Control

Figure 7 is a general circuit for phase control. If i is the current flowing through Z_1 and Z_2, we have

$$\begin{aligned} V_{in} &= i(Z_1 + Z_2) \\ V_{out} &= iZ_2 \end{aligned}$$

and so the response of the circuit is

$$\frac{V_{out}}{V_{in}} = \frac{Z_2}{Z_1 + Z_2}.$$

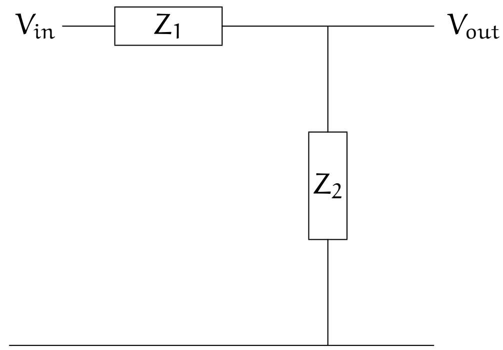


Figure 7: Diagram for phase modification circuits

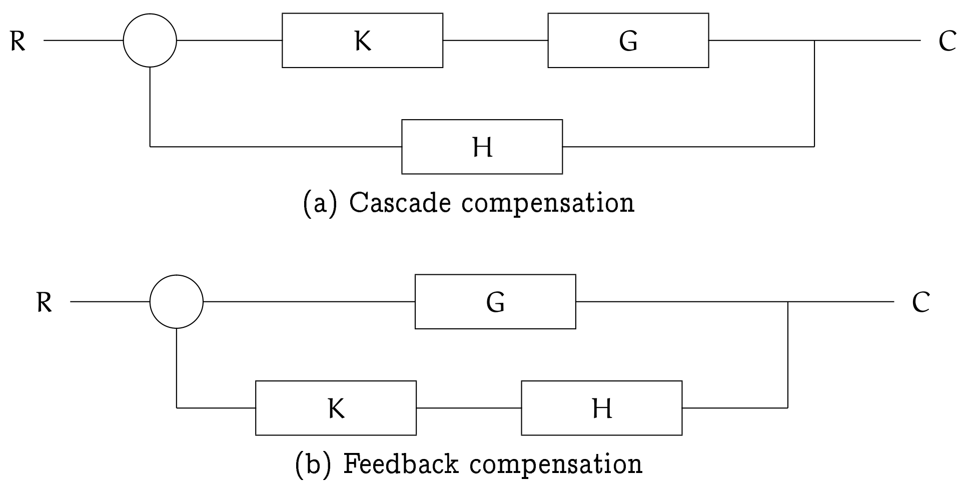


Figure 8: Kinds of compensation: K is the compensation circuit

The effect of compensation depends on where the compensation circuit is placed. Figure 8 shows the two most common topologies: cascade compensation is placed in series with the forward control network and feedback compensation is placed in series with the feedback network.

4.2.1 Lead Compensation

The transfer function for a ***lead compensator*** is $\frac{s+a}{s+b}$, which has a zero at $s=-a$ and a pole at $s=-b$. For lead compensation, $a<b$, and the phase shift is always positive:

$$\begin{aligned}\Delta\phi &= \arg\frac{j\omega+a}{j\omega+b}\\ &= \arg(j\omega+a)-\arg(j\omega+b)\\ &= \tan^{-1}\frac{\omega}{a}-\tan^{-1}\frac{\omega}{b}\end{aligned}$$

which is positive because $\tan^{-1}$ is a monotonically increasing function.

We can make the circuit of Figure 7 a lead compensator by putting an RC shunt as Z_1 and a resistance as Z_2. Then

$$\begin{aligned} Z_1 &= \frac{R_1/Cs}{R_1 + 1/Cs} \\ &= \frac{R_1}{1 + R_1 Cs} \\ Z_2 &= R_2 \end{aligned}$$

The response is

$$\begin{aligned} \frac{V_o}{V_i} &= \frac{Z_1}{Z_1 + Z_2} \\ &= \frac{R_2(1 + R_1 Cs)}{R_1 + R_2 + R_1 R_2 Cs} \\ &= \frac{s + \frac{1}{R_1 C}}{s + \frac{1}{R_1 C} + \frac{1}{R_2 C}} \\ &= \frac{s + a}{s + b} \end{aligned}$$

with $a = 1/R_1C$ and $b = 1/R_1C + 1/R_2C$, so that $a < b$, as required.

The phase-lead compensator is a form of high-pass filter. The compensator introduces gain at high frequencies, which may increase instability, and phase lead, which tends to be stabilizing. The pole and zero are typically placed in the high frequency region.

4.2.2 Lag Compensation

The transfer function for a ***lag compensator*** is $\frac{s+b}{s+a}$, which has a zero at $s = -b$ and a pole at $s = -a$. For lag compensation, $a < b$, and the phase shift is always negative:

$$\begin{aligned} \Delta\phi &= \arg \frac{j\omega + b}{j\omega + a} \\ &= \arg(j\omega + b) - \arg(j\omega + a) \\ &= \tan^{-1} \frac{\omega}{b} - \tan^{-1} \frac{\omega}{a} \end{aligned}$$

which is negative because $\tan^{-1}$ is a monotonically increasing function.

We can make the circuit of Figure 7 a lag compensator by putting a resistance as Z_1 and a resistance and a capacitance in series as Z_2. Then

$$\begin{aligned} Z_1 &= R_1 \\ Z_2 &= R_2 + \frac{1}{Cs} \\ Z_1 + Z_2 &= R_1 + R_2 + \frac{1}{Cs} \\ \frac{Z_2}{Z_1 + Z_2} &= \frac{R_2 + 1/Cs}{R_1 + R_2 + 1/Cs} \end{aligned}$$

$$
\begin{aligned}
&= \frac{s + 1/R_2C}{s + 1/(R_1 + R_2)C} \\
&= \frac{s + b}{s + a}
\end{aligned}
$$

where $a = 1/(R_1 + R_2)C$ and $b = 1/R_2C$, so that $a < b$, as required.

The phase-lag compensator is a simple form of low-pass filter. It reduces the gain at high frequencies, which tends to stabilize the system, and introduces phase lag, which tends to destabilize the system. The pole and zero of the compensator are usually placed in the low frequency region.

4.2.3 Lag/Lead Compensation

The transfer function for a lag/lead compensator is $\dfrac{(s + a_1)(s + b_2)}{(s + b_1)(s + a_2)}$ with $a_1 < b_1$ and $a_2 < b_2$.

With an RC parallel circuit as Z_1 and an RC serial circuit as Z_2, we have

$$
\begin{aligned}
Z_1 &= \frac{R_1}{1 + R_1C_1s} \\
Z_2 &= R_2 + \frac{1}{C_2s} \\
\frac{Z_2}{Z_1 + Z_2} &= \frac{R_2 + 1/C_2s}{R_1/(1 + R_1C_1s) + R_2 + 1/C_2s} \\
&= \frac{(1 + R_1C_1s)(1 + R_2C_2s)}{R_1C_2s + (1 + R_1C_1s)(R_2C_2s) + 1 + R_1C_1s} \\
&= \frac{\left(s + \frac{1}{R_1C1}\right)\left(s + \frac{1}{R_2C_2}\right)}{s^2 + \left(\frac{1}{R_2C_2} + \frac{1}{R_2C_1} + \frac{1}{R_1C_1}\right)s + \frac{1}{R_1C_1R_2C_2}}
\end{aligned}
$$

which meets the requirements for lag/lead compensation with

$$
\begin{aligned}
a_1 &= 1/R_1C_1 \\
b_2 &= 1/R_2C_2 \\
a_1b_2 &= a_2b_1 \\
a_2 + b_1 &= a_1 + b_2 + 1/R_2C_1.
\end{aligned}
$$

Table 9 summarizes the advantages and disadvantages of the various kinds of compensation.

4.3 Operational Amplifiers

Figure 10 shows the circuit symbol for an operational amplifier ("op amp") and the equivalent circuit. In designing with op amps, we assume:

- The input impedance is large ($Z_{in} \approx \infty$)
- The output impedance is small ($Z_{out} \approx 0$)

Compensation	Advantages	Disadvantages
Phase lag	LF characteristics improved	at least one slow term in transient reponse
	stability marrgins maintained or improved	
	bandwidth reduced (useful if HF noise is a problem)	reduced bandwidth may be a disadvantage
Phase lead	improved stability margins	
	improved HF response	possible HF noise problems
	may be required for stability	may generate large signals, out of linear range of system

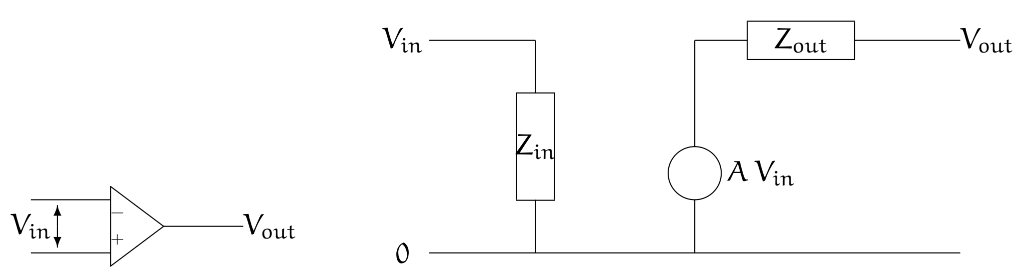
Figure 9: Compensation: summary

- The amplifier is linear with large gain ($V_{out} = A\, V_{in}$ and $A \gg 1$)

Figure 11 shows op amps in two typical configurations. Each circuit uses two impedances, Z_1 and Z_2, in a feedback loop. The input signal for the left circuit (a) goes to the $-$ input of the op amp, and the circuit has negative gain. The input signal for the right circuit (b) goes to the $+$ input of the op amp, and the circuit has positive gain.

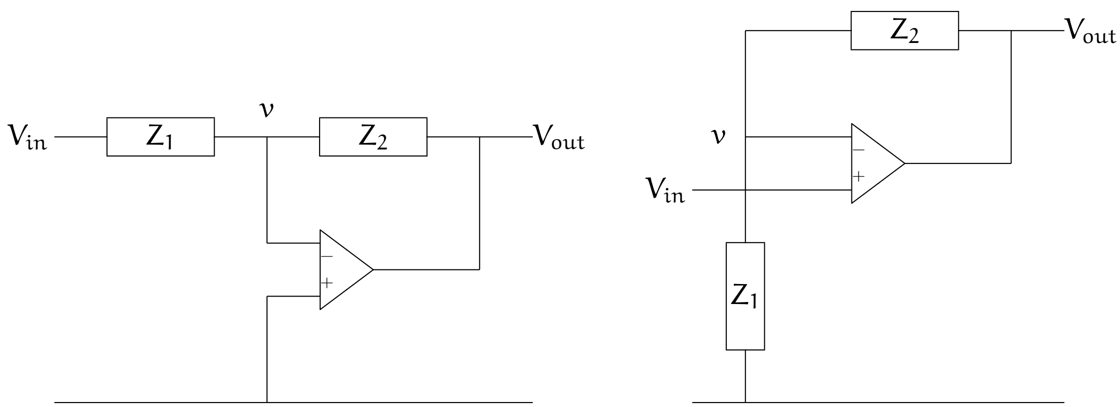
To analyze Figure 11(a), we assume that the input impedance of the op amp is high enough to ensure that the current flowing into its $-$ input is negligible. Consequently, we can assume that the same current flows through Z_1 and Z_2, and we will call it i. It follows that:

$$\begin{aligned} v - V_{in} &= i\, Z_1 \\ V_{out} - v &= i\, Z_2 \\ V_{out} &= -A\, v \end{aligned}$$



(a) Circuit Symbol (b) Equivalent circuit

Figure 10: Operational amplifier



(a) Inverting amplifier with feedback (b) Non-inverting amplifier with feedback

Figure 11: Typical applications of operational amplifiers

in which v is the voltage at the junction of Z_1 and Z_2 (as shown in the diagram), and A is the gain of the amplifier. Eliminating i, we obtain

$$\frac{v - V_{in}}{Z_1} = \frac{V_{out} - v}{Z_2}$$

and therefore

$$\frac{V_{out}}{V_{in}} = -\frac{Z_2}{Z_1 + \frac{Z_1 + Z_2}{A}}.$$

If we assume that A is large, this simplifies to

$$\frac{V_{out}}{V_{in}} \approx -\frac{Z_2}{Z_1}.$$

We can analyze Figure 11(b) in a similar way. Again, we assume that the currents flowing into the inputs of the op amps are negligible and that the current flowing through Z_1 and Z_2 is i. Thus

$$\begin{aligned} v &= i\,Z_1 \\ V_{out} &= i(Z_1 + Z_2) \end{aligned}$$

and, eliminating i,

$$\frac{v}{Z_1} = \frac{V_out}{Z_1 + Z_2}$$

or

$$v = \frac{V_{out}\,Z_1}{Z_1 + Z_2}.$$

The op amp ensures that

$$V_{out} = A(v - V_{in}).$$

We can eliminate v, giving

$$\frac{V_{out}}{V_{in}} = \frac{Z_1}{Z_1 + Z_2} - \frac{1}{A}.$$

For large A, we have

$$\frac{V_{out}}{V_{in}} \approx \frac{Z_1}{Z_1 + Z_2}.$$

If we replace Z_1 and Z_2 by the Laplace transforms of the corresponding impedances, we have, as the response function for the op amp circuit:

$$T(s) = -\frac{Z_2}{Z_1}$$

for the inverting circuit and

$$T(s) = \frac{Z_1}{Z_1 + Z_2}$$

for the non-inverting circuit.

Example 7: Amplifier with proportional feedback. If the impedances are both resistive, $Z_1 = R_1$ and $Z_2 = R_2$, say, then

$$T(s) = -\frac{R_2}{R_1}$$

and the circuit behaves as an (inverting) linear amplifier with gain R_2/R_1. □

Example 8: Differentiation. If $Z_1 = 1/Cs$ is a capacitor and $Z_2 = R$ is a resistor,

$$T(s) = -RCs$$

and the circuit is an inverting differentiator. □

Example 9: Filtering. The impedance of a resistor R in parallel with a capacitor C is $\dfrac{R}{1 + RCs}$. If we use parallel circuits for the impedances, we have

$$\begin{aligned} Z_1 &= \frac{R_1}{1 + R_1 C_1 s} \\ Z_2 &= \frac{R_2}{1 + R_2 C_2 s} \\ T(s) &= -\frac{Z_2}{Z_1} \\ &= -\frac{R_2}{1 + R_2 C_2 s} \cdot \frac{1 + R_1 C_1 s}{R_1} \\ &= -\frac{R_2}{R_1} \cdot \frac{1 + R_1 C_1 s}{1 + R_2 C_2 s} \end{aligned}$$

For a d.c. signal, $s = j\omega = 0$ and $T(s) = -R_2/R_1$, as we would expect. For a.c. signals, the gain is

$$\begin{aligned} g &= \left| \frac{R_2}{R_1} \cdot \frac{1 + R_1 C_1 j\omega}{1 + R_2 C_2 j\omega} \right| \\ &= \sqrt{\left(\frac{R_2}{R_1 \left(1 + R_2^2 C_2^2 \omega^2\right)} + \frac{R_2^2 C_1 C_2 \omega^2}{1 + R_2^2 C_2^2 \omega^2} \right)^2 + \left(\frac{R_2 C_1 \omega}{1 + R_2^2 C_2^2 \omega^2} - \frac{R_2^2 C_2 \omega}{R_1 \left(1 + R_2^2 C_2^2 \omega^2\right)} \right)^2}. \end{aligned}$$

For large frequencies, the gain is determined by the capacitances:

$$\lim_{\omega\to\infty} g = C_1/C_2.$$

□

Example 10: Simulating an inductance.

Figure 12 shows a circuit that is intended to simulate an inductance. To demonstrate its action, we compute the effective impedance between the two input connections at the left of the diagram. To obtain the impedance, we apply a voltage V to these terminals and calculate the resulting current. Let i_1 be the current flowing through R_1 and i_2 be the current flowing through C and R_2. We have

$$u = V + i_1 R_1, \quad (5)$$

$$v = V + \frac{i_2}{sC}, \quad (6)$$

$$V = i_2 R_2 + \frac{i_2}{sC}. \quad (7)$$

The effect of the op amp is to keep the voltages at points u and v approximately equal. Assuming $u = v$, we obtain from (5) and (6):

$$V + i_1 R_1 = V + \frac{i_2}{sC}$$

and therefore

$$i_1 = \frac{i_2}{sR_1C}.$$

From (7),

$$i_2 = \frac{V\,sC}{1 + sR_2C}$$

and so

$$\begin{aligned} i_1 + i_2 &= \frac{V\,sC + V\,s^2R_1C^2}{sR_1C(1 + sR_2C)} \\ &= \frac{V\,sC}{sR_1C} \cdot \frac{1 + sR_1C}{1 + sR_2C} \end{aligned}$$

The effective impedance of the circuit is therefore

$$\begin{aligned} Z &= \frac{V}{i_1 + i_2} \\ &= R_1 \cdot \frac{1 + sR_1C}{1 + sR_2C}. \end{aligned}$$

Setting $s = j\omega$ and separating into real and imaginary parts gives

$$\begin{aligned} Z &= \frac{R_1(1 + \omega^2 R_1 R_2 C^2)}{1 + \omega^2 R_1^2 C^2} + \frac{j\omega R_1 C(R_2 - R_1)}{1 + \omega^2 R_1^2 C^2} \\ &= R' + j\omega L'. \end{aligned}$$

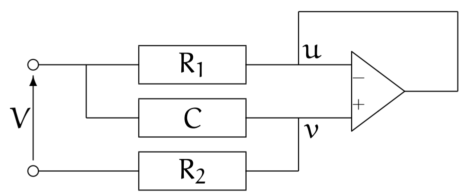


Figure 12: Simulating an inductance

The justification for writing the imaginary part as $\omega L'$ is that $R_1(R_2 - R_1)C$ has the dimensions of an inductance, as shown in Section 7.

The quality factor (see Section 4.5) of the simulated inductor is

$$\begin{aligned} Q &= \omega L'/R' \\ &= \frac{\omega C(R_2 - R_1)}{1 + \omega^2 R_1 R_2 C^2}. \end{aligned}$$

In some situations, these inequalities will hold:

$$\omega^2 R_1 R_2 C^2 \gg 1 \quad (8)$$

$$\omega^2 R_1^2 C^2 \gg 1 \quad (9)$$

$$R_2 \gg R_1 \quad (10)$$

In these circumstances,

$$\begin{aligned} R' &\approx R_2 \\ L' &\approx \frac{R_2}{\omega^2 R_1 C} \\ Q &\approx \frac{1}{\omega R_1 C} \end{aligned}$$

It appears from these approximations that we can obtain large values of both L' and Q by using a very small value of C. Small values of C, however, may invalidate the inequalities (8)–(10).

In principle, we can get some quite impressive inductors. For example, suppose that we use the values $R_1 = 10\,\text{K}\Omega$, $R_2 = 1\,\text{M}\Omega$, and $C = 1\,\text{nF}$. At $f = 1\,\text{KHz}$, the values of R', L', and Q are shown below. The values given by the approximate formulas, shown in parentheses, are not accurate at this frequency.

$$\begin{aligned} R' &= 13,893\,\Omega \quad (1\,\text{M}\Omega) \\ L' &= 9.86\,\text{H} \quad (2533\,\text{H}) \\ Q &= 4.46 \quad (15.9) \end{aligned}$$

At a higher frequency, $f = 1\,\text{MHz}$, the approximations are much better:

$$\begin{aligned} R' &= 999,749\,\Omega \quad (1\,\text{M}\Omega) \\ L' &= 2.507\,\text{mH} \quad (2.533\,\text{mH}) \\ Q &= 0.0157 \quad (0.0159) \end{aligned}$$

These results suggest that when the approximations are useful, the inductor is not useful, and *vice versa*. *C'est la vie.* □

4.4 Mechanical Systems

Equations for mechanical systems enable us to calculate the motion caused by a force.

A ***mass*** obeys Newton's law: force = mass × acceleration or

$$\begin{aligned} f(t) &= m\frac{d^2x}{dt^2} \\ &= m\frac{dv}{dt}. \end{aligned}$$

In an electrical system in which e is the potential, q is the charge, i is the current, and L is an inductance, we have the analogous equations

$$\begin{aligned} e(t) &= L\frac{d^2q}{dt^2} \\ &= L\frac{di}{dt}. \end{aligned}$$

A ***spring*** changes its length in approximate proportion to the applied force:

$$f(t) = kx.$$

This is analogous to a capacitance:

$$e(t) = \frac{q}{C}.$$

The analogy is also apparent in the following equations:

$$\begin{aligned} \frac{1}{k}\frac{df}{dt} &= v \\ C\frac{de}{dt} &= i. \end{aligned}$$

Friction is non-linear in general. However, we can approximate viscous friction as a force proportional to velocity:

$$f(t) = \mu v$$

and, in this form, friction is analogous to resistance:

$$e(t) = Ri.$$

Rotating systems are similar but, instead of mass, force, and velocity, we have torque, moment of inertia, and angular velocity. Figure 13 summarizes the analogies between mechanical and electrical systems.

Linear	Symbol	Rotating	Symbol	Electrical	Symbol
Position	x	Angle	θ	Charge	q
Mass	m	Moment of Inertia	I	Inductance	L
Velocity	$v = \frac{dx}{dt}$	Angular Velocity	$\omega = \frac{d\theta}{dt}$	Current	$i = \frac{dq}{dt}$
Force	$f = m\frac{d^2x}{dt^2}$	Torque	$T = I\frac{d^2\theta}{dt^2}$	Potential	$e = L\frac{d^2q}{dt^2}$
Spring	$k\frac{df}{dt} = v$	Clockspring	$\lambda\frac{dT}{dt} = \omega$	Capacitance	$C\frac{de}{dt} = i$
Friction	$\mu v = f$	Angular friction	$\delta\omega = T$	Resistance	$Ri = e$

Figure 13: Analogies between mechanical and electrical systems

4.5 Quality Factor

The ***quality factor***, Q, of a resonant circuit is a measure of the height of the "spike" in the response function. The Q of an LC circuit is typically determined largely by the resistance, R, of the inductor L, and

$$Q = \frac{\omega L}{R}.$$

Quality does not apply only to electronic circuits, however, and the general definition of Q is given in terms of the bandwidth of the "spike".

Definition 4.3 ***Quality Factor*** *Let S be a resonant system with resonant frequency (that is, maximum response) at ω_0. Let ω_1 and ω_2, where $\omega_1 < \omega_0 < \omega_2$, be the "half-power points", where the amplitude is 3 dB (or $1/\sqrt{2}$) down. Then the quality factor for S is*

$$Q = \frac{\omega_0}{\omega_2 - \omega_1}.$$

Comparable quality factors are: $\frac{m\omega}{\mu}$ for a linear mechanical oscillator and $\frac{I\omega}{\delta}$ for a rotating mechanical oscillator, where μ and δ are linear and angular friction as described in Figure 13, and $\frac{\omega}{2\pi}$ is the frequency of oscillation.

4.6 Second-Order Systems

For the ***canonical second order system***

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

The response of this system to a unit step is

$$\mathcal{L}^{-1}\left(\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}\right) = 1-\frac{1}{\beta}e^{-\zeta\omega_n t}\sin(\beta\omega_n t+\theta)$$

where

$$\beta = \sqrt{1-\zeta^2}$$
$$\theta = \tan^{-1}(\beta/\zeta)$$

The time to the first peak is $T_p = \dfrac{\pi}{\beta\omega_n}$ and the amplitude at this time is $1+e^{-\pi\zeta/\beta}$. Consequently, the ***overshoot*** is $e^{-\pi\zeta/\sqrt{1-\zeta^2}}$.

5 Stability Criteria

Definition 5.1 ***Stability*** *A system is* ***stable*** *if, for every bounded input, the output remains bounded with increasing time.*

The characteristic equation (CE) is

$$1+G(s)H(s) = 0.$$

In most case, $G(s)$ and $H(s)$ are rational functions of s and we can write

$$1+G(s)H(s) = \frac{(s-z_1)(s-z_2)(s-z_3)\cdots}{(s-p_1)(s-p_2)(s-p_3)\cdots} \tag{11}$$

in which the factors in the numerator are (mis)named the ***zeroes*** and the factors in the denominator are (mis)named the ***poles*** of the CE.

Recalling that the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \tag{12}$$

we see that the zeros of (11) are the poles of (12).

For a given input $R(s)$, the output is

$$\begin{aligned} C(s) &= \frac{R(s)G(s)}{1+G(s)H(s)} \\ &= \frac{k_1}{s-p_1}+\frac{k_2}{s-p_2}+\cdots+\frac{k_n}{s-p_n}+C_r(s) \end{aligned}$$

and consequently

$$c(t) = k_1e^{p_1t}+k_2e^{p_2t}+\cdots+k_ne^{p_nt}+c_r(t)$$

Since the transfer function is a polynomial, $c_r(t)$ will be bounded for bounded inputs $c(t)$. The other terms will be bounded ***if all of the*** p_i ***are negative***. If there are quadratic factors, some of the p_i will be complex, and must have negative real parts. It follows that, for stability, the roots of the CE must have ***negative real parts***. If the CE has roots on the imaginary axis, the system is ***marginally stable*** and, in practice, will oscillate.

Consequently, the stability analysis depends on being able to find the zeros of a polynomial. We begin with some general considerations.

The equation $(s - r_1)(s - r_2)\cdots(s - r_n) = 0$ has roots at $s = r_1,\ s = r_2, \ldots\ s = r_n$ and expands to

$$s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0 \quad = \quad 0.$$

We have:

$$\begin{aligned}
a_{n-1} &= -\text{sum of all roots} \\
a_{n-2} &= +\text{sum of products of pairs of roots} \\
a_{n-3} &= -\text{sum of products of triples of roots} \\
&\ldots \\
a_0 &= (-1)^n \times \text{product of all roots}
\end{aligned}$$

If all of the roots real and negative, then $r_i < 0$ for $i = 1, 2, \ldots, n$ and it follows that $a_j > 0$ for $j = n-1, n-2, \ldots, 0$. If the roots are complex, they must occur in conjugate pairs. Since the a_i are real, the imaginary parts will cancel, and we have the same result: if the roots have negative real parts, then $a_i > 0$.

In summary:

1. If any $a_i = 0$, then there are roots not in the left half-plane.
2. If any $a_i < 0$, then at least one root is in the right half-plane.

5.1 Routh-Hurwitz

The Routh-Hurwitz method uses the ideas above to determine whether a given polynomial has roots in the right half-plane. Given the equation

$$a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0 \quad = \quad 0$$

the ***Routh-Hurwitz array*** is set up like this:

$$\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \ldots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \ldots \\
s^{n-2} & b_1 & b_2 & b_3 & \ldots & \\
s^{n-3} & c_1 & c_2 & \ldots & & \\
\ldots & & & & &
\end{array}$$

where

$$
\begin{aligned}
b_1 &= -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} \\
b_2 &= -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-4} & a_{n-5} \end{vmatrix} \\
\ldots & \\
c_1 &= -\frac{1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} \\
c_2 &= -\frac{1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix} \\
\ldots &
\end{aligned}
$$

If an entry does not exist, it is assumed to be zero.

The terms in the third and successive rows have the form $-\frac{d_i}{c}$, in which d_i is a determinant formed from the two preceding rows and k is the first term of the immediately preceding row. The columns of d_i consist of the first and $(i+1)$'th terms in the two preceding rows.

Definition 5.2 ***Routh-Hurwitz Criterion*** *The number of zeroes in the right half-plane of a polynomial* P *is equal to the number of sign changes in the first column of its Routh-Hurwitz array.*

Three cases arise in practice:

1. All entries in the first column are non-zero: the criterion can be applied directly.
2. The first entry of a row is zero.

   We can proceed by assuming that the value of the zero term is actually ε and completing the calculations. Then we let $\varepsilon \to 0^+$ and $\varepsilon \to 0^-$. However, there will invariably be sign changes in one or both cases, and consequently there are zeroes in the right half-plane.
3. There is an entire row of zeroes. In this case, the polynomial has an even polynomial as a factor. (An even polynomial in s has only even powers of s.) The coefficients of this factor are the entries in the line above the line of zeroes. We can divide the original polynomial by this polynomial and proceed.

The Routh-Hurwitz array for the quadratic $as^2 + bs + c$ is

$$
\begin{array}{c|cc}
s^2 & a & c \\
s^1 & b & \\
s^0 & c &
\end{array}
$$

showing that all three coefficients must be positive, which is alredy obvious from the stadnard solution. For the cubic $a_3s^3 + a_2s^2 + a_1s + a_0$, the Routh-Hurwitz array is

$$
\begin{array}{c|cc}
s^3 & a_3 & a_1 \\
s^2 & a_2 & a_0 \\
s^1 & a_1 - a_3a_0/a_2 & \\
s^0 & a_0 &
\end{array}
$$

showing that the coeefficients must be positive and, in addition, $a_1a_2 > a_3a_0$, for stability.

Advantage Routh-Hurwitz provides a simple way of deciding whether all roots of the CE lie in the negative half-plane.

Disadvantage Routh-Hurwitz does not provide any other information about the location of the roots: it is therefore not useful for system design.

5.2 Nyquist

The Nyquist criterion is based on a polar map of the open-loop transfer function, $G(j\omega)H(j\omega)$.

The ***Nyquist diagram*** is obtained by the mapping $s \mapsto 1 + G(s)H(s)$ where s has the following locus:

- s starts at $-j\infty$ and moves up the imaginary axis to $j\infty$. If there are poles of $G(s)H(s)$ on the imaginary axis, the locus follows a small semicircle, with positive real part, to go around them.
- The locus is closed by an infinite semicircle defined by $s = u + jv$ where $u > 0$, $r = \sqrt{u^2 + v^2}$, and $r \to \infty$.

The locus is infinite only for generality; all that matters is that it must enclose all poles and zeros that lie in the right half-plane.

The Nyquist criterion makes use of a theorem of Cauchy:

> If $F(s)$ is analytic within a closed contour C, then $N = Z - P$, where N is the number of times $F(s)$ encircles the origin as s moves around C, Z is the number of zeroes within C, and P is the number of poles within C.

The number of times that $1 + G(s)H(s)$ encircles the origin is clearly the same as the number of times $G(s)H(s)$ encircles the point $-1 + j0$. Since the contour we have chosen encloses the right half-plane, Z is the number of zeros with positive real parts which, for stability, must be zero. Also, P is the number of poles which, since the open-loop transfer function should be stable, is also zero.

Definition 5.3 ***Nyquist Criterion*** *As s moves along the contour described above, the locus of $G(s)H(s)$ must not encircle the point $-1 + j0$.*

The importance of the Nyquist criterion is that it tells us not only whether a system is stable, but how far it is from instability. Suppose that, for some value of s, $G(s)H(s) = \alpha$, where α is real and $\alpha > -1$. Then the open-loop gain could be increased by a factor of $\frac{1}{\alpha}$ before instability occurs; the ***gain margin*** is

$$20 \log_{10} \frac{1}{\alpha} \, \text{db}$$

and an acceptable value for it is $12\,\text{db}$ corresponding to $\alpha = 1/4$.

Let γ be the angle between the negative real axis and the point where $|G(s)H(s)| = 1$. Then γ is the ***phase margin*** of the system and an acceptable value for it is $\pi/3 = 60°$.

5.3 Bode

The Bode criterion is similar to the Nyquist criterion but uses approximations to simplify the calculations needed. A ***Bode plot*** of a response function $R(s)$ has two parts, the ***amplitude plot*** showing $|R(j\omega)|$ and the ***phase plot*** showing $\arg R(j\omega)$. The horizontal axis shows $\log \omega$ (or, more often, $\log_{10} \omega$) and the vertical axis uses decibels for $|R(j\omega)|$ and degrees or radians for $\arg R(j\omega)$. Figure 14 summarizes the results for first-order functions.

The log-log scales enable typical response functions to be approximated as straight lines, as follows:

Constant: K. The magnitude plot is constant at $20 \log_{10} K$ and the phase plot is constant at $0°$ for $K \geq 0$ or $180°$ for $K < 0$.

Integration: $1/j\omega$. The amplitude plot is $-20 \log_{10} \omega$ which is a line with slope -20 db/decade passing through 0 db at $\omega = 1$. The phase plot is constant at $-\pi/2$.

Differentiation: $j\omega$. The amplitude plot is $20 \log_{10} \omega$ which is a line with slope 20 db/decade passing through 0 db at $\omega = 1$. The phase plot is constant at $\pi/2$.

Phase lag: $\dfrac{a}{a + j\omega}$. The amplitude plot is

$$\begin{aligned} 20 \log_{10} \left| \frac{a}{a + j\omega} \right| &= 20 \log_{10} \frac{1}{\sqrt{\frac{\omega^2}{a^2} + 1}} \\ &= -20 \log_{10} \sqrt{\frac{\omega^2}{a^2} + 1}. \end{aligned}$$

We can approximate this in two parts. When $\omega \ll a$, the amplitude is approximately $\log_{10} 1 = 0$ db. When $\omega \gg a$

$$-20 \log_{10} \sqrt{\frac{\omega^2}{a^2} + 1} \quad \approx \quad -20 \log_{10} \frac{\omega}{a}$$

which is a straight line with a slope of -20 db/decade crossing 0 db at $\omega = a$. The two lines (asymptotes) join at $\omega = a$ and the approximate Bode plot consists of two lines.

The phase is given by $\arg \frac{a}{a+j\omega} = \tan^{-1} \frac{\omega}{a}$. For $\omega \ll a$, this is approximately zero and, for $\omega \gg a$, it tends to $\pi/2$. We approximate this as

$$\begin{aligned} \omega < 0.1a &: 0 \\ 0.1a \leq \omega \leq 10a &: \frac{\pi}{2} \cdot \frac{\omega - 0.1a}{10a - 0.1a} \\ \omega > 10a &: \pi/2 \end{aligned}$$

To sketch the phase plot, we can draw the first and the third components, and then draw a straight line between their end points.

Phase lead: $\dfrac{a + j\omega}{a}$. By similar reasoning, the Bode plot is constant at 0 db for $\omega \leq a$ and increases at -20 db/decade for $\omega > a$. The phase is approximated as

$$\begin{aligned} \omega < 0.1a &: 0 \\ 0.1a \leq \omega \leq 10a &: -\frac{\pi}{2} \cdot \frac{\omega - 0.1a}{10a - 0.1a} \\ \omega > 10a &: -\pi/2 \end{aligned}$$

Function			Amplitude	Point ω	Point db	Phase rule		
K			$20\log_{10}\lvert K\rvert$					0
$j\omega$			$20\log_{10}\omega$	1	0			$\pi/2$
$1/j\omega$			$-20\log_{10}\omega$	1	0			$-\pi/2$
$1+j\omega/\omega_n$	$\omega \le \omega_n$	:	0	ω_n	0	$\omega < 0.1\omega_n$	:	0
	$\omega > \omega_n$	:	$20(\log_{10}\omega - \log_{10}\omega_n)$			$\omega > 10\omega_n$	:	$\pi/2$
$\frac{1}{1+j\omega/\omega_n}$	$\omega \le \omega_n$	:	0	ω_n	0	$\omega < 0.1\omega_n$	:	0
	$\omega > \omega_n$	:	$-20(\log_{10}\omega - \log_{10}\omega_n)$			$\omega > 10\omega_n$	:	$-\pi/2$

Figure 14: Rules for Bode diagrams

Let $G(\omega)$ denote the gain of the system and $\phi(\omega)$ the phase shift.

The ***phase crossover frequency***, ω_p, is defined by

$$\phi(\omega_p) = -\pi$$

and the ***gain margin*** is

$$\left|\frac{1}{G(\omega_p)}\right|.$$

The ***gain crossover frequency***, ω_g, is defined by

$$G(\omega_g) = 1$$

or

$$\log G(\omega_g) = 0.$$

and the ***phase margin*** is

$$\arg G(\omega_g) + \pi.$$

5.4 Root-Locus

Root-locus techniques provide a way of studying the response of a system as one real parameter is varied. In principle, any parameter can be varied; in practice, it is often the gain that is varied, because this makes it easy to sketch the loci.

The ***root-locus*** is the set of paths traced by the roots of the equation $1 + KG(s) = 0$ on the complex plane with $0 \le K \le \infty$. K is the ***gain*** of the system. The equation is equivalent to the two real equations

$$\begin{aligned} |KG(s)| &= 1 \\ \arg KG(s) &= -\pi \end{aligned}$$

which are called the ***magnitude criterion*** and the ***phase criterion*** respectively. The first equation does not tell us much (since $0 \le K \le \infty$, it can be satisfied by arbitrary vlues of s).

The second is more useful: it says that $KG(s)$ is a negative real number. The root locus can be estimated by using the following rules (Shiners, pages 146–151, Philips & Harbor, pages 211ff.). Figure 15 summarizes the rules for sketching the root locus: detailed notes follow.

1. There is one locus for each pole of $1 + KG(s)$. In other words, the number of loci is the same as the order of the characteristic equation.

2. If the coefficients of the CE are real, as is usually the case, roots are either real or complex conjugate pairs. Consequently, each locus is symmetrical with respect to the real axis.

3. In general, $KG(s)$ has the form $K\frac{b_m s^m + \cdots}{s^n + \cdots}$ with $\alpha = n - m > 0$. To make the number of zeros match the number of poles, we say that $KG(s)$ has α ***zeros at infinity***. We can construct asymptotes to the zeros at infinity, as shown below. Since the number of poles and zeros is then equal, we have the following rule:

   As $K \to 0$, $G(s) \to \infty$ and as $K \to \infty$, $G(s) \to 0$. Consequently, each locus starts at a pole of $G(s)$ and ends at a zero of $G(s)$.

4. A point on the real axis is on a locus if the sum of poles and zeroes to its right is odd.

   Since complex conjugate poles cancel in the imaginary direction, we need to consider only real poles and zeroes. Let $x + j0$ be the given point on the real axis. The angular contribution of a pole or zero $> x$ is π while the angular contribution of a pole or zero $< x$ is 0. Thus the effect of all the poles is $n\pi$, where n is the number of poles or zeroes $> x$. The angle criterion is satisfied if n is odd.

5. For large values of s, the equation $1 + KG(s) = 0$ becomes

   $$1 + \frac{Kb_m}{s^\alpha} = 0$$

   and the roots satisfy

   $$s^\alpha + Kb_m = 0.$$

   The angles of these roots are $\theta = r\pi/\alpha$ for $r = \pm 1, \pm 2, \ldots$.

6. The intersection of the root locus and the real axis can be determined by applying the Routh-Hurwitz criterion to the formula $1 + KG(s)$. In some situatiopns, we can do better than this.

   A ***breakaway point*** is a point at which the root locus leaves the real axis and becomes complex. At a breakaway point, the CE has multiple roots and, consequently, it shares roots with its derivative. Since $\frac{d}{ds}(1 + KG(s)) = K\frac{d}{ds}G(s)$, we need to consider only $\frac{d}{ds}G(s)$. The rule is:

   Breakaway points occur where $1 + KG(s)$ and $\frac{d}{ds}G(s) = 0$ have a common real root.

7. Let δ_i be the angle at which a locus departs from pole p_i and let α_j be the angle at which a locus arrives at zero z_j. Then:

   $$\begin{aligned} \delta_j &= \sum_i \alpha_i + \sum_{i \neq j} \delta_i + r\pi \\ \alpha_j &= \sum_i \delta_i + \sum_{i \neq j} \alpha_i + r\pi \end{aligned}$$

Rule	Note
#loci = #roots of $1 + KG = 0$	1
Loci are symmetrical about the real axis	2
Loci start at poles and end at zeros	3
Loci includes all real points to the left of an odd number of real critical frequencies	4
Loci are asymptotic to zeros at infinity	5
Breakaway points occur at multiple roots	6
Angles sum to 180°	7
Asymptotes intersect real axis	8

Figure 15: Summary of root locus rules

where $r = \pm 1, \pm 2, \ldots$.

8. The intersection of the asymptotic lines and the real axis occurs at $x + j0$ where

$$x = \frac{\sum \text{poles} - \sum \text{zeros}}{\#\text{poles} - \#\text{zeros}}$$

in which ***finite*** poles and zeros only are considered.

6 Case Study: Motor Controller

The motor controller is used as a running example throughout these notes. The purpose of the system is to control the angular position of a shaft, $\theta(t)$, with an electric motor. The motor is driven by a voltage, $u(t)$, and its equation of motion is

$$I\frac{d^2\theta}{dt^2} = -\mu\frac{d\theta}{dt} + ku$$

where I is the moment of inertia of the motor, μ is a frictional factor, and k is the coupling constant relating voltage to torque. The equation is in Joules (kg m^2 s^{-2}). The units for individual variables are:

$$\begin{aligned} I &: \text{kg m}^2 \\ \mu &: \text{kg m}^2\ \text{s}^{-1} \\ k &: \text{Joules/volt} = \text{coulombs} = \text{A s} \\ u &: \text{volts} = \text{kg m}^2\ \text{s}^{-3}\ \text{A}^{-1} \end{aligned}$$

The Laplace transform of this equation is

$$I\Theta s^2 = -\mu\Theta s + kU$$

and hence

$$G = \frac{\Theta}{U} = \frac{k}{s(Is + \mu)} \text{ radians/volt.}$$

Feedback is provided by a potentiometer driven by the output shaft. This provides a voltage signal proportional to the angular position, $v = f\theta$. The transfer function of the feedback loop is therefore $H = f$ volts/radian and the closed-loop transfer function of the system is

$$\begin{aligned} M(s) &= \frac{G(s)}{1 + G(s)H(s)} \\ &= \frac{k/s(Is + \mu)}{1 + fk/s(Is + \mu)} \\ &= \frac{k}{Is^2 + \mu s + fk} \text{ radians/volt} \end{aligned}$$

and the characteristic equation is

$$Is^2 + \mu s + fk = 0. \tag{13}$$

The open-loop transfer function is

$$GH = \frac{fk}{s(Is + \mu)}. \tag{14}$$

Suppose the input to the system is a step function of V volts. Then $U(s) = V/s$ and

$$\begin{aligned} \Theta(s) &= U(s)M(s) \\ &= \frac{Vk}{s(Is^2 + \mu s + fk)} \end{aligned}$$

Let

$$s^2 + \mu s/I + fk/I = (s + A)(s + B). \tag{15}$$

Then

$$\Theta(s) = \frac{Vk}{I}\left(\frac{1}{ABs} + \frac{1}{A(A-B)(s+A)} + \frac{1}{B(B-A)(s+B)}\right)$$

and

$$\theta(t) = \frac{Vk}{I}\left(\frac{1}{AB} + \frac{e^{-At}}{A(A-B)} + \frac{e^{-Bt}}{B(B-A)}\right).$$

We can obtain the values of A and B from (15):

$$\begin{aligned} A &= \frac{1}{2}\left(\mu/I + \sqrt{(\mu/I)^2 - 4fk/I}\right), \\ B &= \frac{1}{2}\left(\mu/I - \sqrt{(\mu/I)^2 - 4fk/I}\right). \end{aligned}$$

Since $\mathcal{R}(A) > 0$ and $\mathcal{R}(B) > 0$, the zeroes of the characteristic equation have negative real parts and the system is stable. There will be oscillation if $\left(\frac{\mu}{I}\right)^2 < \frac{4fk}{I}$, that is, if $\mu < 2\sqrt{fkI}$.

We have

$$\theta(0) = \frac{Vk}{I}\left(\frac{1}{AB} + \frac{1}{A(A-B)} + \frac{1}{B(B-A)}\right) = 0$$

and, since $AB = fk/I$, we have

$$\lim_{t\to\infty} \theta(t) = \frac{Vk}{IAB} = \frac{V}{a}.$$

6.1 Zero Input

Consider the case when there is no input voltage. Then $u = 0$ and the equation of motion for the motor shaft is

$$I\frac{d^2\theta}{dt^2} + \mu\frac{d\theta}{dt} = 0. \tag{16}$$

The Laplace transform of (16) is

$$I(s^2\Theta(s) - s\theta(0^+) - \frac{d\theta}{d(}0^+)) + \mu(s\Theta(s) - \theta(0^+)) = 0.$$

If we assume $\theta(0^+) = 0$ and $\dot{\theta}(0^+) = \omega_0$, this simplifies to

$$\Theta(s) = \frac{I\omega_0}{s(Is + \mu)}$$

To split the right side into partial fractions, assume

$$\frac{1}{s(Is+\mu)} = \frac{A}{s} + \frac{B}{Is+\mu}.$$

This equation holds for all values of s. Setting $s = 0$ gives $A = 1/\mu$ and setting $s = -\mu/I$ gives $B = -I/\mu$. Since I/μ has the dimensions of time, we also define $\tau = I/\mu$. Then:

$$\begin{aligned}\Theta(s) &= \frac{I\omega_0}{\mu s} - \frac{I\omega_0}{\mu(s+\mu/I)} \\ &= \frac{\omega_0\tau}{s} - \frac{\omega_0\tau}{s+1/\tau}.\end{aligned}$$

The inverse Laplace transform of $\Theta(s)$ gives the shaft position as a function of time

$$\begin{aligned}\theta(t) &= \omega_0\tau\left(\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+\mu/I}\right)\right) \\ &= \omega_0\tau\left(1 - e^{t/\tau}\right)\end{aligned}$$

and we can differentiate it to give

$$\omega = \omega_0 e^{-t/\tau}.$$

6.2 Stability

Routh-Hurwitz The characteristic equation (13) is

$$Is^2 + \mu s + fk = 0.$$

The coefficients of this equation are

$$\begin{aligned} a_2 &= I, \\ a_1 &= \mu, \\ a_0 &= fk \end{aligned}$$

and the Routh-Hurwitz array is

$$\begin{array}{c|cc} s^2 & I & fk \\ s^1 & \mu & \\ s^0 & fk. & \end{array}$$

The system is stable if all of the constants are positive.

Nyquist The open loop transfer function (14) is

$$G(s)H(s) = \frac{fk}{s(Is + \mu)}$$

and the frequency response is

$$G(j\omega)H(j\omega) = \frac{fk}{-I\omega^2 + \mu j\omega}.$$

The locus does not encircle the point $-1 + j0$ and the system is therefore stable.

6.3 Compensation

Applying compensation $\frac{s+a}{s+b}$ gives

$$G = \frac{k(s+a)}{s(Is+\mu)(s+b)}$$

and so

$$GH = \frac{fk(s+a)}{s(Is+\mu)(s+b)}$$

The CE is $GH + 1 = 0$ which is equivalent to

$$Is^3 + (Ib + \mu)s^2 + (\mu b + fk)s + fka = 0.$$

The Routh-Hurwitz array is

$$\begin{array}{c|cc} s^3 & I & \mu b + fk \\ s^2 & Ib + \mu & fka \\ s^1 & B & \\ s^0 & fka & \end{array}$$

where

$$
\begin{aligned}
B &= -\frac{1}{Ib+\mu}\begin{vmatrix} I & \mu b + fk \\ Ib+\mu & fka \end{vmatrix} \\
&= \frac{(Ib+\mu)(\mu b+fk) - Ifka}{IB+\mu}
\end{aligned}
$$

The system is stable if $B > 0$ or $(Ib+\mu)(\mu b+fk) > Ifka$. Since this is equivalent to the condition

$$
\mu(Ib^2 + \mu b + fk) + Ifk(b-a) \quad > \quad 0 \tag{17}
$$

a sufficient condition for stability is $a < b$. This is lead compensation (Section 4.2.1 on page 15) and generally improves stability. However, we could use lag compensation provided that (17) is satisfied.

6.4 Normal Form

The closed-loop transfer function $\dfrac{k}{Is^2+\mu s+fk}$ can be written in the form $\dfrac{k}{I}\cdot\dfrac{1}{s^2+\frac{\mu}{I}s+\frac{fk}{I}}$. Comparing this to the canonical form $\dfrac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$, we see that

$$
\begin{aligned}
\omega_n &= \sqrt{\frac{fk}{I}} \\
\zeta &= \frac{\mu}{2\sqrt{Ifk}}
\end{aligned}
$$

The time constant of the system is

$$
\begin{aligned}
\tau &= \frac{1}{\zeta\omega_n} \\
&= \frac{2\sqrt{Ifk}}{\mu}
\end{aligned}
$$

Substituting some plausible values

$$
\begin{aligned}
I &= 5\times 10^{-6} \\
k &= 0.1 \\
f &= 0.5 \\
\mu &= 0.1
\end{aligned}
$$

gives

$$
\begin{aligned}
\tau &= \frac{2\sqrt{5\times 10^{-6}\times 0.5\times 0.1}}{0.1} \\
&= 0.01 \text{ s.}
\end{aligned}
$$

If we put these values and $b = 10$ (corresponding to a time constant of 0.1 s) into (17), we obtain the approximate condition

$$
0.1 + 25\times 10^{-8}(10-a) \quad > \quad 0
$$

which will be satisfied for reasonable values of a.

7 SI Units

Figure 16 shows the basic SI units and Figure 17 shows units derived from the basic units. We have

$$\begin{aligned} RF &= V\ A^{-1}\ C\ V^{-1} \\ &= A\ s\ A^{-1} \\ &= s \end{aligned}$$

showing that the product of resistance and capacitance is a time. Thus, in circuit analysis, RC is a time and $RC\omega$ is a pure number.

Similarly,

$$\begin{aligned} HF &= Wb\ A^{-1}\ C\ V^{-1} \\ &= V\ s\ A^{-1}\ A\ s\ V^{-1} \\ &= s^2 \end{aligned}$$

showing that the product of an inductance and a capacitance is a squared time. In circuit analysis, LC has dimension T^2 and $LC\omega^2$ is a pure number.

Furthermore, the unit of inductance is the product of resistance and time:

$$\begin{aligned} H &= Wb\ A^{-1} \\ &= V\ A^{-1}\ s \\ &= \Omega\ s \end{aligned}$$

This means, for example, that a circuit with impedance $j\omega R_1 R_2 C$ behaves like an inductance. In practice, we cannot construct a circuit from resistors and capacitors with pure imaginary conductance, but we could obtain an impedance of the form $R + j\omega R_1 R_2 C$, corresponding to a resistor R in series with an inductor $L = R_1 R_2 C$.

Quantity	Unit	Symbol
Length	metre	m
Mass	kilogram	kg
Time	second	s
Current	ampere	A
Temperature	kelvin	K
Luminosity	candela	cd
Amount	mole	mol

Figure 16: Basic units

The following expressions were generated by the program `dimensions.cpp`. The dimension of each formula is noted in the heading. Common abbreviations are used to denote values in the expressions: C = capacitance, I = current, ℓ = length, L = inductance, μ = friction, R = resistance, t = time, V = voltage.

Quantity	Unit	Symbol	SI	SI (basic)			
				kg	m	s	A
Frequency	hertz	Hz	s^{-1}			−1	
Force	newton	N	kg m s^{-2}	1	1	−2	
Friction			kg s^{-1}	1	−1		
Torque	newton-metre		kg m^2s^{-2}	1	2	−2	
Moment of inertia			kg m^2	1	2		
Angular friction			kg m^2s^{-1}	1	2	−1	
Pressure	pascal	Pa	N m^{-2}	1	−1	−2	
Work, energy, heat	joule	J	N m	1	2	−2	
Power	watt	W	J s^{-1}	1	2	−3	
Charge	coulomb	C	A s			1	1
Potential	volt	V	J C^{-1}	1	2	−3	−1
Capacitance	farad	F	C V^{-1}	−1	−2	4	2
Resistance	ohm	Ω	V A^{-1}	1	2	−3	−2
Conductance	siemens	S	A V^{-1}	−1	−2	3	2
Magnetic flux	weber	Wb	V s	1	2	−2	−1
Flux density	tesla	T	Wb m^{-2}	1		−2	−1
Inductance	henry	H	Wb A^{-1}	1	2	−2	−2

Figure 17: Derived units

Dimensions: pure number.

$$\begin{array}{ccccc}
\frac{\ell}{m\mu} & \frac{m\mu}{\ell} & \frac{1}{t\omega} & t\omega & \frac{V}{IR} \\
\frac{IR}{V} & \frac{Rt}{L} & \frac{R}{L\omega} & \frac{L}{Rt} & \frac{L\omega}{R} \\
\frac{t^2}{CL} & \frac{1}{CL\omega^2} & \frac{L}{CR^2} & \frac{t}{CR} & \frac{1}{CR\omega} \\
\frac{CR^2}{L} & \frac{CL}{t^2} & CL\omega^2 & \frac{CR}{t} & CR\omega
\end{array}$$

Dimensions: T.

$$\begin{array}{ccccc}
\frac{1}{t\omega^2} & t & t^2\omega & \frac{IL}{V} & \frac{1}{\omega} \\
\frac{Rt^2}{L} & \frac{R}{L\omega^2} & \frac{L}{R} & \frac{L^2}{R^2 t} & \frac{L^2\omega}{R^2} \\
\frac{t^2}{CR} & \frac{1}{CR\omega^2} & \frac{CV}{I} & \frac{CL}{t} & CL\omega \\
CR & \frac{C^2R^2}{t} & C^2R^2\omega & &
\end{array}$$

Dimensions: A.

$$\begin{array}{ccccc}
\frac{V^2}{IR^2} & I & \frac{I}{t^2\omega^2} & \frac{I}{t\omega} & It\omega \\
It^2\omega^2 & \frac{I^2R}{V} & \frac{Vt}{L} & \frac{V}{L\omega} & \frac{V}{R} \\
\frac{CV}{t} & CV\omega & & &
\end{array}$$

Dimensions: $LM^2T^{-3}A^{-1}$ (volts).

$$\begin{array}{ccccc}
\frac{V^2}{IR} & \frac{IL}{t} & IL\omega & IR & \frac{I^2R^2}{V} \\
V & \frac{V}{t^2\omega^2} & \frac{V}{t\omega} & Vt\omega & Vt^2\omega^2 \\
\frac{It}{C} & \frac{I}{C\omega} & & &
\end{array}$$

Dimensions: $LM^2T^{-3}A^{-2}$ (ohms).

$$
\begin{array}{ccccc}
\frac{V^2}{I^2R} & \frac{V}{I} & \frac{IR^2}{V} & \frac{R^2t}{L} & \frac{R^2}{L\omega} \\[2ex]
\frac{L}{t^2\omega} & \frac{L}{t} & Lt\omega^2 & L\omega & \frac{L^2}{Rt^2} \\[2ex]
\frac{L^2\omega^2}{R} & R & \frac{R}{t^2\omega^2} & \frac{R}{t\omega} & Rt\omega \\[2ex]
Rt^2\omega^2 & \frac{t^2}{C^2R} & \frac{1}{C^2R\omega^2} & \frac{1}{Ct\omega^2} & \frac{t}{C} \\[2ex]
\frac{t^2\omega}{C} & \frac{1}{C\omega} & \frac{L}{CR} & \frac{CR^2}{t} & CR^2\omega
\end{array}
$$

8 Useful Formulas

Function	Transform	Remarks
$f(t)$	$F(s)$	General notation
$Af(t)+Bg(t)$	$AF(s)+BG(s)$	Linearity
$f(t-T)u(t-T)$	$e^{-sT}F(s)$	$u(t)$ is the unit step function
$e^{-at}f(t)$	$F(s+a)$	
$f'(t)$	$sF(s)-f(0^+)$	
$f''(t)$	$s^2F(s)-sf(0^+)-f'(0^+)$	
$\int_0^t f(\tau)d\tau$	$F(s)/s$	
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$	

Figure 18: Laplace: rules

Express $\frac{P(s)}{Q(s)}$ in the form

$$\frac{A_1}{s+s_1}+\frac{A_2}{s+s_2}+\frac{A_3}{s+s_3}+\cdots$$

$Q(s)$ has only simple roots: if

$$Q(s) = (s+s_1)(s+s_2)\cdots(s+s_n)$$

then

$$\frac{P(s)}{Q(s)} = \sum_{i=1}^{n}\frac{A_i}{s+s_i}$$

where

$$A_i = (s+s_i)\left.\frac{P(s)}{Q(s)}\right|_{s=-s_i}.$$

Repeated roots: if $Q(s)=\cdots(s+s_i)^r\cdots$ then

$$\frac{P(s)}{Q(s)} = \cdots+\frac{B_1}{s+s_i}+\frac{B_2}{(s+s_i)^2}+\cdots+\frac{B_r}{(s+s_i)^r}+\cdots$$

where

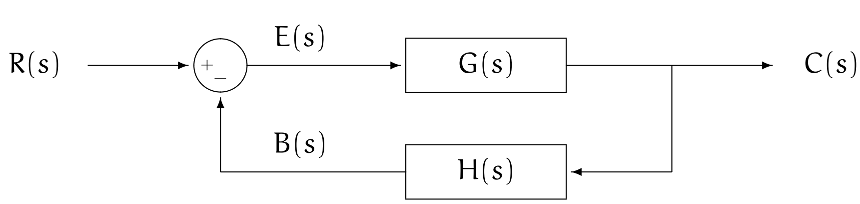
$$B_r = (s+s_i)^r\left.\frac{P(s)}{Q(s)}\right|_{s=-s_i}$$

$$B_{r-1} = \frac{1}{1!}\frac{d}{ds}\left.\left\{(s+s_i)^r\frac{P(s)}{Q(s)}\right\}\right|_{s=-s_i}$$

Function	Transform	Remarks
$\delta(t)$	1	Dirac's δ-function
$u(t)$	$\frac{1}{s}$	Unit step function
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	
e^{-at}	$\frac{1}{s+a}$	
$\frac{1-e^{-at}}{a}$	$\frac{1}{s(s+a)}$	
$\cos at$	$\frac{s}{s^2+a^2}$	
$\cosh at$	$\frac{s}{s^2-a^2}$	
$\sin at$	$\frac{a}{s^2+a^2}$	
$\sinh at$	$\frac{a}{s^2-a^2}$	
$\frac{1-\cos at}{a^2}$	$\frac{1}{s(s^2+a^2)}$	
$\frac{at-\sin at}{a^3}$	$\frac{1}{s^2(s^2+a^2)}$	
te^{-at}	$\frac{1}{(s+a)^2}$	
$e^{-at}(1-at)$	$\frac{s}{(s+a)^2}$	
$\frac{e^{-at}\sin bt}{b}$	$\frac{1}{s^2+2\zeta\omega+\omega^2}$	$a=\zeta\omega$, $b=\omega\sqrt{1-\zeta^2}$, and $\zeta<1$

Figure 19: Laplace: special cases

$$
\begin{aligned}
B_{r-2} &= \frac{1}{2!}\frac{d^2}{ds^2}\left\{(s+s_i)^r\frac{P(s)}{Q(s)}\right\}\Bigg|_{s=-s_i} \\
&\vdots \\
B_1 &= \frac{1}{(r-1)!}\frac{d^{r-1}}{ds^{r-1}}\left\{(s+s_i)^r\frac{P(s)}{Q(s)}\right\}\Bigg|_{s=-s_i}
\end{aligned}
$$

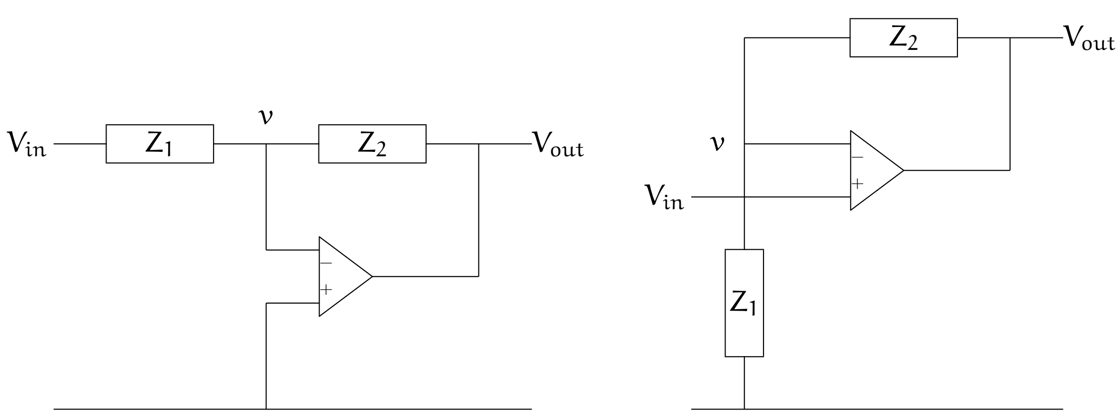


Open-loop response	$G(s)\,H(s)$
Closed-loop response	$\dfrac{G(s)}{1+G(s)\,H(s)}$
Characteristic equation	$1+G(s)\,H(s)=0$

The phase-lead compensator is a form of high-pass filter. The compensator introduces gain at high frequencies, which may increase instability, and phase lead, which tends to be stabilizing. The pole and zero are typically placed in the high frequency region.

The phase-lag compensator is a simple form of low-pass filter. It reduces the gain at high frequencies, which tends to stabilize the system, and introduces phase lag, which tends to destabilize the system. The pole and zero of the compensator are usually placed in the low frequency region.

Compensation	Advantages	Disadvantages
Phase lag	LF characteristics improved	at least one slow term in transient reponse
	stability marrgins maintained or improved	
	bandwidth reduced (useful if HF noise is a problem)	reduced bandwidth may be a disadvantage
Phase lead	improved stability margins	
	improved HF response	possible HF noise problems
	may be required for stability	may generate large signals, out of linear range of system



(a) $T(s) = -\frac{Z_2}{Z_1}$ (b) $T(s) = \frac{Z_1}{Z_1+Z_2}$

For the ***canonical second order system***

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Linear	Symbol	Rotating	Symbol	Electrical	Symbol
Position	x	Angle	θ	Charge	q
Mass	m	Moment of Inertia	I	Inductance	L
Velocity	$v = \frac{dx}{dt}$	Angular Velocity	$\omega = \frac{d\theta}{dt}$	Current	$i = \frac{dq}{dt}$
Force	$f = m\frac{d^2x}{dt^2}$	Torque	$T = I\frac{d^2\theta}{dt^2}$	Potential	$e = L\frac{d^2q}{dt^2}$
Spring	$k\frac{df}{dt} = v$	Clockspring	$\lambda\frac{dT}{dt} = \omega$	Capacitance	$C\frac{de}{dt} = i$
Friction	$\mu v = f$	Angular friction	$\delta\omega = T$	Resistance	$Ri = e$

Function			Amplitude	Point ω	Point db	Phase rule		
K			$20\log_{10}\|K\|$					0
$j\omega$			$20\log_{10}\omega$	1	0			$\pi/2$
$1/j\omega$			$-20\log_{10}\omega$	1	0			$-\pi/2$
$1 + j\omega/\omega_n$	$\omega \le \omega_n$	:	0	ω_n	0	$\omega < 0.1\omega_n$	:	0
	$\omega > \omega_n$	:	$20(\log_{10}\omega - \log_{10}\omega_n)$			$\omega > 10\omega_n$	:	$\pi/2$
$\frac{1}{1+j\omega/\omega_n}$	$\omega \le \omega_n$	:	0	ω_n	0	$\omega < 0.1\omega_n$	:	0
	$\omega > \omega_n$	:	$-20(\log_{10}\omega - \log_{10}\omega_n)$			$\omega > 10\omega_n$	:	$-\pi/2$

Figure 20: Bode diagrams

Routh-Hurwitz array for $a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0 = 0$:

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \ldots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \ldots \\
s^{n-2} & b_1 & b_2 & b_3 & \ldots & \\
s^{n-3} & c_1 & c_2 & \ldots & & \\
\ldots & & & & &
\end{array}
$$

where

$$
\begin{aligned}
b_1 &= -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} \\
b_2 &= -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-4} & a_{n-5} \end{vmatrix} \\
\ldots \\
c_1 &= -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} \\
c_2 &= -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix} \\
\ldots
\end{aligned}
$$