1. How many subgraphs with at least one vertex does W3 have ?

   Solution: W3 is the same as K4. There are clearly just 4 subgraphs consisting of one vertex. If a subgraph is to have 2 vertices, then there are $C(4,2) = 6$ ways to choose the vertices, and then 2 ways in each case to decide whether or not to include the edge joining them. This gives us $6 \times 2 = 12$ subgraphs with 2 vertices. If a subgraph is to have three vertices, then there are $C(4,3) = 4$ ways to choose the vertices, and then $2^3 = 8$ ways to decide whether to include the edges. This gives us $4 \times 8 = 32$ subgraphs with three vertices. Finally, there are the subgraphs containing all four vertices. Here there are $2^6 = 64$ ways to decide the edges. So total is $4 + 12 + 32 + 64 = 112$.

2. Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

   Solution: Suppose the parts are of size k and $v - k$. Then the maximum number of edges the graph may have is $f(k) = k(v-k)$. Taking the derivative $f'(k) = v - 2k = 0$ we get $k = v/2$. So there are at most $f(v/2) = v^2/4$ edges.

3. What is the sum of the entries in a row of the adjacency matrix for an undirected graph? For a directed graph ?

   Solution: For an undirected graph, the sum of the entries in the ith row is the same as the corresponding column sum, namely the number of edges incident to vertex i, which is the same as the degree of vertex i minus the number of loops at vertex i.

   In a directed graph, the sum of the entries in the ith row is the number of edges that have vertex i as their initial vertex, i.e., the out-degree of vertex i.

4. Show that if G is a self-complementary simple graph with n vertices, then $n \equiv 0$ or $1(mod 4)$.

   Solution: If G is self-complementary, then the number of edges must equal the number of edges of G^c. But the sum of these two numbers is $n(n-1)/2$, where n is the number of vertices of G, since the union of the two graphs is K_n. Therefore, the number of edges of G must be $n(n-1)/4$. Since this number must be an integer, we get $n \equiv 0$ or $1 (mod 4)$.

5. Define isomorphism of directed graphs.

   Solution: We need only modify the definition of isomorphism of simple graph slightly. The directed graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function $f : V_1 \to V_2$ such that for all pairs of vertices a and b in V_1, $(a, b) \in E_1$ if and only if $(f(a), f(b)) \in E_2$.

6. What is the product of the incidence matrix and its transpose for an undirected graph ?

   Solution: Let the elements of the product matrix be a_{ij}. Then for nondiagonal element a_{ij}, it is the number of edges from vertex v_i to v_j. For diagonal element a_{ii}, it is the number of edges incident to v_i.

7. Prove that the any simple graph with 2 or more vertices contains at least two vertices with same degree.

   Solution: Let G be a simple graph with $n \geq 2$ vertices. Since G is simple, the degree of any vertex in G is at most $n-1$. Therefore, there are n possibilities for the degree of a vertex in G. However, if G has a vertex with degree 0 then it cannot have a vertex with degree $n-1$, and vice-versa. Therefore, there are in fact only $n-1$ different possibilities for the degrees. Thus, by the Pigeonhole Principle, at least two of the vertices in G must have the same degree.

8. Prove that any set of 7 distinct positive integers not exceeding 12 must contain at least one pair of integers whose difference is 6.

   Solution: Let the 7 distinct integers are $a_1, a_2, \cdots, a_7$. We know $a_i (mod 6)$ can take 6 values, but there are 7 integers. By Pigeonhole Principle, there are at least two integers (a_p and a_q) having the same remainder. Because a_i are distinct and not exceeding 12, we can conclude that $|a_p - a_q| = 6$.