1. How many nonisomorphic directed simple graphs are there with 2 vertices ?

   Solution: First, consider graphs that have no edges from one vertex to the other. There are 3 such graphs, depending on whether they have no, one, or two loops. Similiarly, there are 3 in which there is an edge from each vertex to the other (two edges). Finally, there are 4 graphs that have exactly one edge between the vertices, because now the vertices are distinguished, and there can be or fail to be a loop at each vertex.

2. How much storage is needed to represent a simple graph with v vertices and e edges using:

   Solution:

   (a) adjacency lists ?

   We need one adjacency list for each vertex, and the list needs some sort of name or header; this requires v storage locations. In addition, each edge will appear twice, once in the list of each of its endpoints; this will require $2e$

storage locations. Therefore, we need $v + 2e$ locations in all.

(b) an adjacency matrix ?

The adjacency matrix is a $v \times v$ matrix, so it requires v^2 bits of storage.

(c) an incidence matrix ?

The incidence matrix is a $v \times e$ matrix, so it requires ve bits of storage.

3. Find the number of paths of length 5 between any two adjacent vertices in $K_{3,3}$.

   Solution: Every move from one set to the other set has 3 choices, so for a path of length 5 we need 5 moves. The solution is 3^5. We can also use the power of the adjacency matrix to solve this.

4. Show that a connected graph with n vertices has at least $n - 1$ edges.

   Solution: We show this by induction on n. For $n = 1$ there is nothing to prove. Now assume the inductive hypothesis, and let G be a connected graph with $n+1$ vertices and fewer than n edges, where $n \geq 1$. Since the sum of the degrees of the vertices of G is equal to 2 times the number of the edges, we know that the sum of the degrees is less than $2n$, which is less than $2(n+1)$. Therefore, some vertex has degree less than 2. Since G is connected, this vertex is not isolated, so it must have degree 1. Remove this vertex and its edge. Clearly the result is still connected, and it has n vertices and fewer than $n-1$ edges, contracting the inductive hypothesis. Therefore, the statement holds for G.

5. Show that a vertex c in the connected simple graph G is a cut vertex if and only if there are vertices u and v, both different from c, such that every path between u and v passes through c.

   Solution: First, we show that if c is a cut vertex, then there exist vertices u and v such that every path between them passes through c. Since the removal of c increases the number of component, there must be two vertices in G that are in different components after the removal of c. Then every path between these two vertices has to pass c. Conversely, if u and v are as specified, then they must be in different components of with the graph with c removed. Therefore, the removal of c resulted in at least two components, so c is a cut vertex.

6. Explain how we can find the length of the shortest path from a vertex v to a vertex w in a graph.

   Solution: The length of the shortest path is the smallest k such that there is at least one path of length k from v to w. Therefore we can find the length by computing successively A^1, A^2, A^3, $\cdots$, until we find the first k such that the (i, j) entry of A^k is not 0, where v is the ith vertex and w is the jth.

7. Show that the existence of a simple circuit of length k, where k is a positive integer greater than 2, is an isomorphic invariant.

   Solution: Suppose that f is an isomorphism from graph G to graph H. If G has a simple circuit of length k, say $u_1, u_2, \cdots, u_k, u_1$, then we claim that $f(u_1), f(u_2), \cdots, f(u_k), f(u_1)$ is a simple circuit in H. Certainly, this is a circuit, since each edge $u_i u_{i+1}$ (and $u_k u_1$) in G corresponds to an edge $(f(u_i)f(u_{i+1})$ (and $(f(u_k)f(u_1))$ in H. Furthermore, since no edge was repeated in this circuit in G, no edge will be repeated when we use f to move over to H.

8. For which values of n do the following graphs have an Euler circuit ?

   Solution:

   (a) K_n: n is odd.

   (b) C_n: $n \geq 3$.

   (c) W_n: Since the degrees of the vertices around the rim are all odd, no wheel has an Euler circuit.

   (d) Q_n: The degrees of the vertices are all n. So n is even.

9. For which values of n do the following graphs have an Euler path but not a Euler circuit ?

Solution:

(a) K_n: n is 2.

(b) C_n: none.

(c) W_n: none.

(d) Q_n: n is 1.

10. For which values of m and n does the complete biparite graph $K_{m,n}$ have an

(a) Euler Circuit ?

m, n are even.

(b) Euler path ?

$K_{1,1}$ and $K_{n,2}$ (n is odd).

11. Prove that a simple graph is bipartite if and only if it does not contain a circuit with odd length.

Solution:

$\rightarrow$: Let G be a bipartite graph. Assume G contains a circuit with odd length. Let $v_1, v_2, \cdots, v_{2k+1}, v_1$ denote this circuit. Now, Since G is bipartite we can partition its vertices into two disjoint sets, V_1 and V_2, such that none of the vertices in V_1 are adjacent and none of the vertices in V_2 are adjacent. Assume v_1 is an element of V_1. Then, since v_2 is adjacent to v_1, v_2 must be an element of V_2. Similarly, since $v_2 \in V_2$ and v_3 is adjacent to v_2, v_3 must be an element of V_1. Continue in this manner, we find that $v_i \in V_1$ whenever i is odd, and $v_i \in V_2$ whenever i is even. In particular v_1 and v_{2k+1} are both elements of V_1. However, there is an edge between v_1 and v_{2k+1} which contradicts that fact that none of the vertices in V_1 are adjacent. Thus G cannot contain a circuit with odd length.

$\leftarrow$: Let G be a simple graph that does not contain a circuit with odd length. Pick any vertex u in G. Let V_1 denote the set of all vertices v in G for which there exists a path between u and v with odd length, and let V_2 denote the set of all vertices v in G for which there exists a path between u and v with even length. It will be shown that V_1 and V_2 are disjoint, none of the vertices in V_1 are adjacent, and none of the vertices in V_2 are adjacent. Assume V_1 and V_2 are not disjoint. That is, assume there exists a vertex v in which there exists a path P_1 between u and v with odd length, and a path P_2 between u and v with even length. This gives us a circuit with odd length (i.e. starting at u, follow P_1 to v and then follow P_2 back to u). This contradicts G not having such a circuit. Thus V_1 and V_2 are disjoint. Now, assume there exists vertices in V_1 that are adjacent. That is, assume there exists adjacent vertices v_1 and v_2 and paths P_1 and P_2, where P_1 is a path from u to v_1 with odd length and P_2 is a path from u to v_2 with odd length. This gives us a circuit with odd length (i.e. starting at u, follow path P_1 to v_1, then take the edge $\{v_1, v_2\}$ to v_2, then take the path P_2 back to u). Thus none of the vertices in V_1 are adjacent. A similar argument can be used to show none of the vertices in V_2 are adjacent. Therefore, the sets $V1$ and V_2 are disjoint, none of the vertices in V_1 are adjacent, and none of the vertices in V_2 are adjacent. In other words, G is bipartite.