1. How many positive integers less than 1000 are divisible by either 7 or 11?

   Solution: We need to use the principle of inclusion-exclusion. The count of integers that are divisible by 7 is $\lfloor \frac{999}{7} \rfloor = 142$. The count of integers that are divisible by 11 is $\lfloor \frac{999}{11} \rfloor = 90$. The count of integers that are divisible by both 7 and 11 is $\lfloor \frac{999}{7 \times 11} \rfloor = 12$. Therefore, the solution is $142 + 90 - 12 = 220$.

2. How many positive integers less than 1000 are divisible by neither 7 or 11?

   Solution: We need to take out the count of integers that are divisible by either 7 or 11 from 999. That is $999 - 220 = 779$.

3. How many positive integers less than 1000 are divisible by exactly one of 7 and 11?

   Solution: We need to subtract the count of interges that are divisible by 7×11 twice from the sum of the counts of integers divisible by 7 and 11. It is $142 + 90 - 12 \times 2 = 208$.

4. How many positive integers less than 1000

have distinct digits?

Solution: We break the problem down to three cases: one-digit numbers, two-digit numbers and three-digit numbers. There are 9 one-digit numbers. There are $9 \times 9 = 81$ two-digit numbers. There are $9 \times 9 \times 8 = 648$ three-digit numbers. In total, it is $9 + 81 + 648 = 738$.

Note: We do not consider 0 for the first digit.

5. How many positive integers not exceeding 100 are divisible by either 4 or 6?

   Solution: Use the inclusion-exclusion principle. The solution is: $\lfloor \frac{100}{4} \rfloor + \lfloor \frac{100}{6} \rfloor - \lfloor \frac{100}{12} \rfloor = 33$.

   Why do we use 12 instead of 24?

6. How many strings of four decimal digits that do not contaion the same digit twice?

   Solution: Use the product rule: $10 \times 9 \times 8 \times 7$.

7. How many strings of four decimal digits that end with an even digit?

Solution: There are 10 ways for the first three digits and 5 ways for the last digit. So by using the product rule we get $10^3 \times 5$.

8. How many strings of four decimal digits that have exactly three digits that are 9s?

   Solution: There are 4 ways to choose the position that is to be different from 9, and 9 ways to choose the digit to go there. By using the product rule we get $4 \times 9 = 36$ such strings.

9. In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and groom are among these 10 people, if both bride and groom must be in the picture?

   Solution: We first place the bride in any of the 6 positions, and then place the groom in any of the 5 remaining positions. Then, from left to right in the remaining positions, wen choose the other four people to be in the picture. So the answer is $6 \times 5 \times 8 \times 7 \times 6 \times 5 = 50400$.

10. How many bit strings of length 10 contain five consecutive 0s?

Solution: We will base the count on where the string of five or more consecutive 0's starts. If it starts in the first bit, then the first five bits are all 0's, but there is free choice for the last five bits $2^5 = 32$. If it starts in the second bit, then the first bit has to be 1, the next five bits are 0's, the last four bits are free to choose $2^4 = 16$. Continue to reason in this way.

The solution is $32 + 5 \times 16 = 112$.

11. Let (x_i, y_i) be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

Solution: There are four possible pairs of parities: (odd,odd), (odd,even), (even,odd) and (even,even). Since we have five pairs of points, from the pigeonhole principle we know there are at least two of them will have the same pair of parities. The midpoint of the segment joining these two points will have integer coordinates.

12. Prove that a party where there are at least two people, there are two people who know the same number of other people there.

    Solution: Let $K(x)$ be the number of other people at the party that person x knows. The possible values for $K(x)$ are $0, 1, 2, \cdots, n-1$. However 0 and $n-1$ cannot happen at the same time. So $K(x)$ has $n-1$ possibilities and we have n people. The mapping is not one-to-one. we know at least two people who know the same number of other people there.