1. The English alphabet contains 21 consonants and 5 vowels. How many strings of six lowercase letters of the English alphabet contain exactly 1 vowel?

   Solution: The vowel can be placed to 6 positions and we have 5 choices to choose the vowel. The rest letters have 21 choices each (note: it can repeat). So by using the product rule we have: $6 \times 5 \times (21 \times 21 \times 21 \times 21 \times 21) = 122523030$.

2. How many strings of six lowercase letters of the English alphabet contain exactly 2 vowels?

   Solution: We have $C(6, 2)$ ways to select the two vowels' positions. Each vowel can be chosen in 5 ways. The rest letters have 21 choices each (note: it can repeat). So by using the product rule we have: $5 \times 5 \times C(6, 2) \times (21 \times 21 \times 21 \times 21) = 72930375$.

3. How many strings of six lowercase letters of the English alphabet contain at least 1 vowel?

   Solution: When you encounter *at least*, you should use *subtraction.* Without restriction, we have 26^6 strings. Without vowels, we have 21^6 strings. By using subtraction, we get $26^6 - 21^6 = 223149655$.

4. How many strings of six lowercase letters of the English alphabet contain at least 2 vowels?

   Solution: When you encounter *at least*, you should use *subtraction.* You should subtract the number of strings that conains exactly one vowel from the number of strings that conatins at least one vowel. So the solution is: $223149655 - 122523030 = 100626625$.

5. How many 4-permutations of positive integers not exceeding 100 contain three consecutive integers in the correct order where these consecutive integers can be separated by other integers?

   Solution: Examples of three consecutive integers are $5, 6, 7$ etc. When you choose the starting position of the three consecutive integers from locations 1 to 98, you can insert the 4th integer into the three consecutive integers in 4 ways. This integer has $97 = 100 - 3$ choices. By using the product rule you have: $98 \times (4 \times 97) = 38024$. However, we have overcounted 97 4-permutations of the form $(n, n+1, n+2, n+3)$. So we get $38024 - 97 = 37927$.

6. How many 4-permutations of positive integers not exceeding 100 contain three consecutive integers in the correct order where these consecutive integers must be in consecutive locations?

   Solution: When you choose the starting position of the three consecutive integers from locations 1 to 98, you can put the 4th integer to the three consecutive integers in 2 ways. This integer has $97 = 100 - 3$ choices. By using the product rule you have: $98 \times (2 \times 97) = 19012$. However, we have overcounted 97 4-permutations of the form $(n, n+1, n+2, n+3)$. So we get $19012 - 97 = 18915$.

7. Give a combinatorial proof that $\Sigma_{k=1}^{n} kC(n,k)^2 = nC(2n-1, n-1)$.

   Solution: Suppose we want to select a committe, with n members from a group of n mathematics professors and n computer science professors, such that the chairman is a mathematics professor.

   LHS: For any positive integer k, we have $C(n,k)$ ways to choose k mathematics professors and $C(n, n-k)$ ways to choose $n-k$ computer science professors. We have k ways to select the chairman. By product rule we get $kC(n,k)C(n,n-k) = kC(n,k)^2$. Using the sum rule on different k we get the left hand side of the equation.

   RHS: We can select the chairman from the n mathematics professors first. Then select $n-1$ memebers from all the rest people. By the product rule we get $RHS = nC(2n-1, n-1)$.

   This complete the proof.

8. Give a combinatorial proof that $\Sigma_{k=1}^{n} kC(n,k) =$

$n2^{n-1}$.

Solution: We can choose the leader first in n different ways. We can then choose the rest of the committe in 2^{n-1} ways. Hence there are $n2^{n-1}$ ways to choose the committee and its leader. Meanwhile, the number of ways to select a committee with k people is $C(n, k)$. Once we have chosen a committe with k people, there are k ways to choose its leader. Hence there are $\Sigma_{k=1}^{n} kC(n, k)$ ways to choose the committee and its leader. Hence $\Sigma_{k=1}^{n} kC(n, k) = n2^{n-1}$.