1. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 21$ where $x_i, i = 1, 2, 3, 4, 5$, is a nonnegative integer such that

(a) $x_1 \geq 1$?

Solution: Let $x_1 = x_1' + 1$, then the constrain on x_1' becomes a nonnegative integer and the equation becomes $x_1' + x_2 + x_3 + x_4 + x_5 = 20$. From Theorem 2 we get $C(5 + 20 - 1, 20) = 10626$.

(b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?

Solution: Let $x_i = x_i' + 2$, then the equation becomes $x_1' + x_2' + x_3' + x_4' + x_5' = 11$. From Theorem 2 we get $C(5 + 11 - 1, 11) = 1365$.

(c) $0 \leq x_1 \leq 10$?

Solution: The number of solutions for $x_1 \geq 11$ is $C(5 + 10 - 1, 10)$. The number of solutions for $x_1 \geq 0$ is $C(5 + 21 - 1, 21)$. So we get: $C(5+21-1, 21)-C(5+10-1, 10) = 11649$.

(d) $0 \leq x_1 \leq 3$, $1 \leq x_2 < 4$ and $x_3 \geq 15$?

Solution: Let $x_2 = x_2' + 1$ and $x_3 =$

$x_3' + 15$, then the problem becomes
$x_1 + x_2' + x_3' + x_4 + x_5 = 5$
where $0 \leq x_1 \leq 3$, $0 \leq x_2' \leq 2$ and $x_3' \geq 0$.

A: if $x_1 \geq 4$, then $C(5+1-1, 1) = 5$ solutions.

B: if $x_2' \geq 3$, then $C(5+2-1, 2) = 15$ solutions.

$A \cap B$: if $x_1 \geq 4$ and $x_2' \geq 3$, then 0 solutions.

$A \cup B$: if $x_1 \geq 4$ or $x_2' \geq 3$, then $5 + 15 = 20$ solutions.

By De Morgan's laws for set we get the number of solutions for this question is $C(5+5-1, 5) - 20 = 106$.

2. How many solutions are there to the inequality $x_1 + x_2 + x_3 \leq 11$ where $x_i, i = 1, 2, 3$, is a nonnegative integer ?

   Solution: Let's introduce an auxiliary variable x_4, then the problem becomes to find the number of solution of the equation $x_1 + x_2 + x_3 + x_4 = 11$, where x_i is a nonnegative integer. From Theorem 2 we get $C(4+11-1,11) = C(14,3) = 364$.

3. How many positive integers less than 1000000 have the sum of their digits equal to 19 ?

   Solution: We can have at most 6 digits and the problem becomes

   $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 19$, where $0 \leq x_i \leq 9$.

   A_i: if $x_i \geq 10$, then the solution is $C(6+9-1,9) = 2002$.

   The number of solution without restriction is $C(6+19-1,19)$. By De Morgan's laws for set we get the number of solutions for this question is $C(6+19-1,19) - 2002 \times 6 = 30492$.

4. How many different strings can be made from the letters in $ORONO$, using some or all of the letters ?

   Solution: *Length 5*: From Theorem 3 we get $\frac{5!}{3!1!1!} = 20$.

   Length 4: Omit one letter 'O', then $\frac{4!}{2!1!1!} =$

12. Omit one letter 'R', then $\frac{4!}{3!1!} = 4$. Omit one letter 'N', then $\frac{4!}{3!1!} = 4$.

Length 3: Omit two letters 'OO', then $\frac{3!}{1!1!1!} = 6$. Omit two letters 'OR', then $\frac{3!}{2!1!} = 3$. Omit two letters 'ON', then $\frac{3!}{2!1!} = 3$. Omit two letters 'RN', then $\frac{3!}{3!} = 1$.

Length 2: String 'OO', 'RN', 'NR', 'OR', 'RO', 'ON' and 'NO', so there are 6 strings of length 2.

Length 1: String 'O', 'R' and 'N', so there are 3 strings of length 1.

Sum all of them together, we get 63 strings.

5. How many ways are there to distribute five indistinguishable objects into three indistinguishable boxes ?

   Solution: We can have the following 5 ways to do it:

   $5, 0, 0$

   $4, 1, 0$

   $3, 2, 0$

$3, 1, 1$

$2, 2, 1$

6. How many different terms are there in the expansion of $(x_1+x_2+\cdots+x_m)^n$ after all terms with identical sets of exponents are added ?

   Solution: Let n_i be the exponent of x_i in the term $x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$. Apparently we have $n_1 + n_2 + \cdots n_m = n$ and n_i is a non-negative integer. From Theorem 2 we get $C(m + n - 1, n)$ different terms.