1. Find a recurrence relation for the number of strictly increasing sequences of positive integers that have 1 as their first term and n as their last term where n is a positive integer.

   Solution: Let s_n be the number of such sequences. A string ending in n must consist of a string ending in something less than n, followed by an n as the last term. Therefore
   $s_n = s_{n-1} + s_{n-2} + \cdots + s_1$.
   The initial condition is $s_1 = 1$.

2. Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0s.

   Solution: The string can start with a 1 and be followed by a string containing a pair of consecutive 0s.

   1——00——

   The string can also start with 01 and be followed by a string containing a pair of consecutive 0s.

   01——00——

The string can also start with 00 and be followed by any strings of length $n - 2$.

00———————-

Therefore, we have $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$. The initial conditions are: $a_0 = a_1 = 0$.

3. Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s.

   Solution: The string can start with a 1 and be followed by a string containing three consecutive 0s.

   1———000———

   The string can start with 01 and be followed by a string containing three consecutive 0s.

   01——000———

   The string can start with 001 and be followed by a string containing three consecutive 0s.

   001——-000———

   The string can also start with 000 and be followed by any strings of length $n - 3$.

   000———————-

Therefore, we have $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$.

The initial conditions are: $a_0 = a_1 = a_2 = 0$.

4. Find a recurrence relation for the number of bit strings of length n that do not contain three consecutive 0s.

   Solution: Let a_n be the number of such strings. When the last bit is 1, the previous $n-1$ bits do not contain three consecutive 0s. (a_{n-1}). When the string ends with 10, the previous $n-2$ bits do not contain three consecutive 0s. (a_{n-2}). When the string ends with 100, the previous $n-3$ bits do not contain three consecutive 0s. (a_{n-3}). Therefore, we have $a_n = a_{n-1} + a_{n-2} + a_{n-3}$.

   The initial conditions are: $a_0 = 1$, $a_1 = 2$, $a_2 = 4$.

5. Find a recurrence relation for the number of bit strings of length n that contain the string 01.

   Solution: Let a_n be the number of bit strings of length n that contain 01. If we want to construct such a string, we could start with a 1 and follow it with a bit string of length $n - 1$ that contains 01, and there are a_{n-1} of these. Alternatively, for any k from 1 to $n - 1$, we could start with k 0's, follow this by a 1, and then follow this by any $n - k - 1$ bits. For such k there are 2^{n-k-1} such strings, since the final bits are free. So $a_n = a_{n-1} + 2^0 + 2^1 + \cdots + 2^{n-2} = a_{n-1} + 2^{n-1} - 1$.

The initial conditions are: $a_0 = a_1 = 0$.

6. Find a recurrence relation for the number of ternary strings that do not contain two consecutive 0s.

   Solution: Let a_n be the number of such strings. When the last bit is 1 or 2, the previous $n - 1$ bits do not contain two consecutive 0s. (a_{n-1}). When the string end with 10 or 20, the previous $n - 2$ bits do not contain two consecutive 0s. (a_{n-2}). So, $a_n = 2a_{n-1} + 2a_{n-2}$.

   The initial conditions are: $a_0 = 1$, $a_1 = 3$.

7. Solve the linear nonhomogeneous recurrence relation: $a_n = 2a_{n-1} + 3n2^n$, for $a_0 = 1$ and $n \geq 1$.

   Solution: $a_n^{(h)} = \alpha 2^n$. The root of the characteristic equation is 2.

   $a_n^{(p)} = (cn + b)n2^n$.

   Substitute $a_n^{(p)} = (cn + b)n2^n$ into the original equation and collect terms containing n on one side and other terms on the other side to obtain: $n(3 - 2c) = b - c$. So $b = c = \frac{3}{2}$. The general solution is: $a_n = \alpha 2^n + 3(n + 1)n2^{n-1}$. Now use the initial condition: $a_0 = 1$ to obtain $\alpha = 1$. Therefore $a_n = [2 + 3n(n + 1)]2^{n-1}$.

8. Solve the following recurrence relation for n is a power of 2, i.e. $n = 2^k$:

   $f(n) = 8f(n/2) + n^2$.

   for $n \geq 1$ and $f(1) = 1$.

   Solution:

$$f(n) = 8f(n/2) + n^2$$

$$f(n) = 8^2 f(n/2^2) + 2n^2 + n^2$$

$$f(n) = 8^3 f(n/2^3) + 2^2 n^2 + 2n^2 + n^2$$

$$\dots$$

$$f(n) = 8^k f(1) + n^2 \sum_{i=0}^{k-1} 2^i$$

$$f(n) = 8^{log_2 n} + n^2(2^{log_2 n} - 1)$$

$$f(n) = n^{log_2 8} + n^2(n - 1)$$

$$f(n) = n^3 + n^2(n - 1).$$