

1. Find the number of positive integers not exceeding 1000 that are either the square or the cube of an integer.

**Solution:** From the principle of inclusion-exclusion we get  $\lfloor \sqrt{1000} \rfloor + \lfloor \sqrt[3]{1000} \rfloor - \lfloor \sqrt[6]{1000} \rfloor = 31 + 10 - 3 = 38$ .

2. How many bit strings of length eight do not contain six consecutive zeros ?

**Solution:** The number of bit strings of length eight that contain six consecutive zeros is:  $2^2 + 2^1 + 2^1 = 8$ . So we get:  $2^8 - 8 = 248$ .

3. How many permutations of the 26 letters of English alphabet do not contain any of the strings *fish*, *rat*, or *bird* ?

**Solution:** There are  $26!$  strings in total. To count the strings that contain *fish*, we glue these four letters together as one and permute it and the 22 other letters, so there are  $23!$  such strings. Similarly there are  $24!$  strings that contain *rat* and  $23!$  strings that contain *bird*. Furthermore, there are

21! strings that contain both *fish* and *rat* (glue each set of letters together), but there are no strings that contain both *bird* and another of these strings. So the answer is:  $26! - 23! - 24! - 23! + 21!$ .

4. Prove the principle of inclusion-exclusion using mathematical induction.

**Solution:** The base case is  $n = 2$ , for which we already know the formula is true. Assume that the formula is true for  $n$  sets. Look at a situation with  $n + 1$  sets, and temporarily consider  $A_n \cup A_{n+1}$  as one set. Then by the induction hypothesis we have

$$|A_1 \cup \dots \cup A_{n+1}| = \sum_{i < n} |A_i| + |A_n \cup A_{n+1}| - \sum_{i < j < n} |A_i \cap A_j| - \sum_{i < n} |A_i \cap (A_n \cup A_{n+1})| + \dots + (-1)^n |A_1 \cap \dots \cap A_{n-1} \cap (A_n \cup A_{n+1})|.$$

Next we apply the distributive law to each term on the right involving  $A_n \cup A_{n+1}$ , giving us

$$\begin{aligned} & \sum |(A_{i_1} \cap \dots \cap A_{i_m}) \cap (A_n \cup A_{n+1})| = \\ & \sum |(A_{i_1} \cap \dots \cap A_{i_m} \cap A_n) \cup (A_{i_1} \cap \dots \cap A_{i_m} \cap A_{n+1})| \end{aligned}$$

Now we apply the base case to rewrite each

of these terms as

$$\begin{aligned} & \Sigma |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n| + \\ & \Sigma |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_{n+1}| - \\ & \Sigma |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n \cap A_{n+1}| \text{ which gives us} \\ & \text{precisely the summation we want.} \end{aligned}$$

5. Find the probability that when a coin is flipped five times tails come up exactly three times, the first and last flips come up tails, or the second and fourth flips come up heads.

**Solution:** Let  $E_1$ ,  $E_2$ , and  $E_3$  be these three events, in the order given. Then  $p(E_1) = C(5, 3)/2^5 = 10/32$ ;  $p(E_2) = 2^3/2^5 = 8/32$ ; and  $p(E_3) = 2^3/2^5 = 8/32$ . Furthermore  $p(E_1 \cap E_2) = C(3, 1)/2^5 = 3/32$ ;  $p(E_1 \cap E_3) = 1/32$ ; and  $p(E_2 \cap E_3) = 2/32$ . Finally  $p(E_1 \cap E_2 \cap E_3) = 1/32$ . Therefore the probability that at least one of these events occurs is  $(10 + 8 + 8 - 3 - 1 - 2 + 1)/32 = 21/32$ .

6. How many solutions does the equation  $x_1 + x_2 + x_3 = 13$  have where  $x_1$ ,  $x_2$ , and  $x_3$  are nonnegative integers less than 6 ?

**Solution:**  $A_i : x_i \geq 6$  the number of

solutions is  $C(3+7-1, 7) = 36$  for  $i = 1, 2, 3$ .

The number of solutions for  $A_i \cap A_j$  is  $C(3+1-1, 1) = 3$  for  $i, j = 1, 2, 3$  and  $i < j$ .

The number of solutions for  $A_1 \cap A_2 \cap A_3$  is 0.

So the solution is  $C(3+13-1, 13) - (C(3, 1) \times 36 - C(3, 2) \times 3 + 0) = 6$ .

7. An integer is called squarefree if it is not divisible by the square of a positive integer greater than 1. Find the number of square-free positive integers less than 100.

**Solution:** Square-free numbers are those not divisible by the square of a prime. We count them as follows:

$$99 - (\lfloor 99/2^2 \rfloor + \lfloor 99/3^2 \rfloor + \lfloor 99/5^2 \rfloor + \lfloor 99/7^2 \rfloor - \lfloor 99/(2^2 3^2) \rfloor) = 61.$$

8. Use a combinatorial argument to show that the sequence  $\{D_n\}$ , where  $D_n$  denotes the number of derangements of  $n$  objects, satisfies the recurrence relation  $D_n = (n-1)(D_{n-1} + D_{n-2})$  for  $n \geq 2$ .

**Solution:** In a derangement of the num-

bers from 1 to  $n$ , the number 1 cannot go first, so let  $k$  be different from 1 and be the number that goes first. There are  $n - 1$  choices for  $k$ . Now there are two ways to get a derangement with  $k$  first. One way is to have 1 in the  $k$ th position. If we do this, then there are exactly  $D_{n-2}$  ways to derange the rest of the numbers. On the other hand, if 1 does not go to the  $k$ th position, then think of the number 1 as being temporarily relabeled  $k$ . A derangement is completed in this case by finding a derangement of the numbers 2 through  $n$  in positions 2 through  $n$ , so there are  $D_{n-1}$  of them. Combining all this, by the product and the sum rules, we obtain the desired recurrence relation.

The initial conditions are  $D_1 = 0$ ,  $D_2 = 1$ .

9. Solve the recurrence relation  $a_n = a_{n-1}^3 a_{n-2}^2$  if  $a_0 = 2$  and  $a_1 = 2$ .

**Solution:** Let  $b_n = \log a_n$ . Then the recurrence relation becomes  $b_n = 3b_{n-1} + 2b_{n-2}$ , with initial conditions  $b_0 = b_1 = 1$ .

The characteristic equation is  $r^2 - 3r - 2 = 0$ , which gives roots  $(3 + \sqrt{17})/2$  and  $(3 - \sqrt{17})/2$ . Plugging the initial conditions into the general solution and doing some messy algebra gives

$$b_n = \frac{17 - \sqrt{17}}{34} \left( \frac{3 + \sqrt{17}}{2} \right)^n + \frac{17 + \sqrt{17}}{34} \left( \frac{3 - \sqrt{17}}{2} \right)^n$$

The solution to the original problem is then  $a_n = 2^{b_n}$ .

10. How many ways are there to assign six different jobs to three different employees if the hardest job is assigned to the most experienced employee and the easiest job is assigned to the least experienced employee?

**Solution:** After the assignments of the hardest and easiest job has been made, there are 4 different jobs to assign to 3 different employees. No restrictions are stated, so we assume that there are none. Therefore we are just looking for the number of functions from a set with 4 elements to a set with 3 elements, and there are  $3^4 = 81$  such functions.