Finite-SNR Diversity-Multiplexing Tradeoff for Rayleigh MIMO Channels

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***Abstract*—In this paper, exact diversity-multiplexing tradeoff (DMT) at finite signal to noise ratio (SNR) for multiple-input multiple-output (MIMO) systems with dual antennas at the transmitter ($2 \times N_r$) and/or at the receiver ($N_t \times 2$) is derived over Rayleigh fading channels. We derive the outage probability of the mutual information between transmitted and received signals versus SNR. It is shown that at realistic SNRs, achievable diversity gains are significantly lower than asymptotic values. While space-time codes (STCs) for MIMO systems are conventionally designed to achieve the asymptotic DMT frontier, this finite-SNR DMT could provide a new insight to design STCs for practical MIMO systems optimized at operational SNRs.**

***Index Terms*—MIMO systems, diversity multiplexing tradeoff.**

I. Introduction

MULTIPLE-input multiple-output (MIMO) systems have been adopted in most of the recently developed wireless communication systems, e.g., WiMAX, Wi-Fi, LTE, etc. Indeed, their potential benefits are data rate increase and better reliability. Spatial multiplexing techniques [1] have been incorporated for data rate increase, while space-time codes (STCs) [2] have been designed to improve the channel reliability through spatial diversity. Zheng and Tse in [3] have shown that both gains can be simultaneously delivered with a fundamental tradeoff between them. The diversity-multiplexing tradeoff (DMT) defines the optimal tradeoff between achievable diversity and multiplexing gains of any transmission over $N_t \times N_r$ MIMO channels.

The DMT formulated in [3], for uncorrelated Rayleigh MIMO channels, is an asymptotic framework as the signal to noise ratio (SNR) tends to infinity. In [2], asymptotic analysis was also used to obtain the well-known rank-determinant criteria for space-time block codes (STBCs) design. STBCs for MIMO systems are commonly designed according to these rank-determinant criteria e.g., [4]. A nonzero lower bound on the coding gain i.e., a nonzero minimum determinant is sufficient to reach the frontier of the asymptotic DMT [5]. Recently, several papers have noted that STBCs designed to perform better at high (asymptotic) SNRs are not effective at low to medium SNRs [6] (and references therein) where practical communication systems operate. Moreover, we have shown in [7] that STBCs design parameters are SNR-dependent and that adaptive STBCs are more effective for practical communication systems.

A novel framework has been proposed in [8–10] to characterize the tradeoff between diversity and multiplexing gains at finite-SNRs. This finite-SNR DMT could provide new insights to design MIMO systems optimized at operational SNRs. In [9, 10], Narasimhan has pointed out that exact form of the outage probability and therefore the finite-SNR DMT for any $N_t \times N_r$ MIMO channels are not tractable. Therefore, estimates of the finite-SNR DMT are given in [9–12] for uncorrelated, correlated Rayleigh and Rician fading $N_t \times N_r$ MIMO channels where $N_r \geq N_t$. In some cases, the exact expression of the outage probability and the finite-SNR DMT can be derived. To the best of our knowledge, exact expression of the finite-SNR DMT is derived only for 2×2 MIMO systems in [13] with uncorrelated Rayleigh fading and for both multiple-input single-output (MISO) and single-input multiple-output (SIMO) systems with uncorrelated [14] and correlated [12] Rayleigh fading. Inspired by the method in [13], we derive in this paper the exact finite-SNR DMT for any uncorrelated Rayleigh fading MIMO channels with dual antennas i.e., $N_t \times 2$ and $2 \times N_r$ MIMO systems.

II. System Model and Definitions

We consider a MIMO system with N_t transmit antennas, N_r receive antennas operating over a flat Rayleigh fading channel. A perfect channel state information (CSI) is assumed at the receiver, but not at the transmitter. A quasi-static fading is assumed where the channel remains constant over one space-time coding block of length T and changes independently across blocks. The MIMO channel is defined by the matrix $\mathbf{H}_{[N_r \times N_t]}$ where its entries are assumed to be independent and identically distributed (i.i.d.) circularly symmetric complex Gaussian random variables with probability density function (pdf) $\sim CN(0, 1)$.

The channel input-output relation is given by:

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{N} \quad (1)$$

where $\mathbf{X}_{[N_t \times T]}$ is the transmitted space-time block code, $\mathbf{Y}_{[N_r \times T]}$ is the matrix of received N_t noisy signals over T

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channel uses and $\mathbf{N}_{[N_r \times T]}$ is the matrix of additive complex white Gaussian noise with i.i.d. entries and pdf $\sim CN(0,\sigma^2)$.

In [3], conventional asymptotic definitions of multiplexing and diversity gains for a MIMO channel are given by:

$$r_{\text{asymptotic}} = \lim_{\rho\to\infty} \frac{R}{\log \rho} \tag{2}$$

$$d_{\text{asymptotic}} = -\lim_{\rho\to\infty} \frac{\log P_{\text{out}}}{\log \rho} \tag{3}$$

where $r_{\text{asymptotic}}$ and $d_{\text{asymptotic}}$ represent the asymptotic multiplexing and diversity gains respectively, ρ is the average SNR per receive antenna, R is the MIMO system data rate and P_{out} is the outage probability. As no CSI is available at the transmitter, an equal power across transmit antennas is adopted and P_{out} is defined by:

$$P_{\text{out}} = \Pr\left[I \le R\right], \tag{4}$$

where I is the mutual information between received and transmitted signals over the MIMO channel.

The asymptotic diversity-multiplexing or data rate-reliability tradeoff represents the upper bound achievable by any transmission over a $N_t \times N_r$ MIMO system. The asymptotic DMT has been established in [3] under the condition that $T \ge N_t + N_r - 1$, otherwise only upper and lower bounds on DMT exist. These bounds match at high multiplexing gains and they do not when r is small. Later, [5] has noted that $T \ge N_t$ is a sufficient condition. The asymptotic DMT is given by the piece-wise linear function connecting the points $(k, d^*(k))$ where $d^*(k)$ is given by:

$$d^*\left(k\right) = \left(N_t - k\right)\left(N_r - k\right); k = 1, \cdots, \min\left(N_t, N_r\right) \tag{5}$$

At SNR ρ, the multiplexing gain r is defined as the ratio of the system data rate R to the capacity of an additive white Gaussian noise (AWGN) channel with array gain G [10]:

$$r = \frac{R}{\log\left(1 + G\rho\right)} \tag{6}$$

In order to have a fair comparison of diversity and outage performance across different N_t and N_r antennas at low to medium SNRs, the array gain is chosen such that $G = N_r$ [9]. However, in a real scenario, the receiver can always extract a processing gain $G = N_r$ as full CSI is available at the receiver.

Considering a capacity-approaching channel code in the communication chain, the probability of error is well approximated by the channel outage at finite-SNR [8–10]. Therefore, the diversity gain $d\left(r,\rho\right)$ of a system with a fixed multiplexing gain r at SNR ρ is defined by the negative slope of the log-log curve of the outage probability versus SNR, leading to:

$$d\left(r,\rho\right) = -\frac{\partial \log P_{\text{out}}\left(r,\rho\right)}{\partial \log \rho} = -\frac{\rho}{P_{\text{out}}\left(r,\rho\right)} \frac{\partial P_{\text{out}}\left(r,\rho\right)}{\partial \rho} \tag{7}$$

This definition is important for system design as the diversity gain at a particular operating SNR can be used to estimate the additional SNR required to reach a specified outage probability, for a given data rate represented by the multiplexing gain.

In the sequel, motivated by the work in [13], we derive the exact finite-SNR DMT for systems with dual transmit and/or receive antennas over uncorrelated Rayleigh fading channels.

III. Computation of finite-SNR DMT

A. Mutual information pdf

In this section, we derive an analytical expression for the pdf of the mutual information between received and transmitted signals for $N_t \times 2$ and $2 \times N_r$ MIMO systems. By assuming that $\mathbf{X}$ is a zero-mean white complex Gaussian random variable, the MIMO mutual information I conditioned on the channel realization $\mathbf{H}$ is given by [15]:

$$\begin{aligned} I &= \log\det\left(\mathbf{I}_{N_r} + \tfrac{\rho}{N_t}\mathbf{H}\mathbf{H}^H\right) \\ &= \log\det\left(\mathbf{I}_{N_t} + \tfrac{\rho}{N_t}\mathbf{H}^H\mathbf{H}\right) \end{aligned} \tag{8}$$

where the superscript ^{H} stands for conjugate transpose.

It is known that when $\mathbf{H}$ is a complex normally distributed matrix, matrices $\mathbf{H}\mathbf{H}^H$ and $\mathbf{H}^H\mathbf{H}$ are central complex Wishart distributed [16]. We define $m \triangleq \min\left(N_t, N_r\right)$ and $n \triangleq \max\left(N_t, N_r\right)$. The joint pdf of the m nonzero ordered eigenvalues $\lambda_1, ..., \lambda_m$ of the complex Wishart matrix is given in [16]. For $m = 2$, the joint pdf of the two nonzero ordered eigenvalues λ_1 and λ_2 ($\lambda_1 \ge \lambda_2$) is given by:

$$f(\lambda_1, \lambda_2) = \frac{\left(\lambda_1\lambda_2\right)^{n-2}}{\Gamma\left(n\right)\Gamma\left(n-1\right)}\left(\lambda_1 - \lambda_2\right)^2 e^{-\left(\lambda_1 + \lambda_2\right)} \tag{9}$$

where $\Gamma(x)$ is the gamma function defined by $\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$. Therefore, the mutual information is expressed by:

$$I = \log\left(1 + \lambda_1\frac{\rho}{N_t}\right) + \log\left(1 + \lambda_2\frac{\rho}{N_t}\right). \tag{10}$$

Let us define new variables x and y as:

$$x \triangleq \log\left(1 + \lambda_1\frac{\rho}{N_t}\right) \text{ and } y \triangleq \log\left(1 + \lambda_2\frac{\rho}{N_t}\right). \tag{11}$$

Using the rules in [17],

$$\begin{aligned} f(x,y) = {}& \left(\tfrac{N_t}{\rho}\right)^{2n} \tfrac{1}{\Gamma(n)\Gamma(n-1)} e^{x+y}\left(e^x - e^y\right)^2 \\ & \times \left(e^x - 1\right)^{n-2}\left(e^y - 1\right)^{n-2} e^{-\frac{N_t}{\rho}\left(e^x + e^y - 2\right)} \end{aligned} \tag{12}$$

where $x \ge y \ge 0$.

From the definition in (11) the mutual information between received and transmitted signals is $I = x + y$. The joint pdf of I and y is then given by:

$$\begin{aligned} f(I,y,\rho) = {}& \left(\tfrac{N_t}{\rho}\right)^{2n} \tfrac{1}{\Gamma(n)\Gamma(n-1)} e^{I}\left(e^{I-y} - e^y\right)^2 \\ & \times \left(e^y - 1\right)^{n-2}\left(e^{I-y} - 1\right)^{n-2} e^{-\frac{N_t}{\rho}\left(e^{I-y} + e^y - 2\right)} \end{aligned} \tag{13}$$

where $I/2 \ge y \ge 0$.

From (13), and after integration, one can show that:

$$\begin{aligned} f(I,\rho) = {}& \left(\tfrac{N_t}{\rho}\right)^{2n} \tfrac{1}{\Gamma(n)\Gamma(n-1)} e^{I} \\ & \times \int_0^{I/2} g(y, I, \rho)\mathrm{d}y \end{aligned} \tag{14}$$

where $g(y, I, \rho)$ is expressed by:

$$\begin{aligned} g(y,I,\rho) = {}& \left(e^{I-y} - e^y\right)^2\left(e^{I-y} - 1\right)^{n-2} \\ & \times \left(e^y - 1\right)^{n-2} e^{-\frac{N_t}{\rho}\left(e^{I-y} + e^y - 2\right)}. \end{aligned} \tag{15}$$

For each value of I and ρ, the expression $\int_0^{I/2} g(y, I, \rho)\mathrm{d}y$ can be calculated by numerical integration.

B. Outage probability

From (14), and thanks to numerical integration, the outage probability $P_{\text{out}}(r,\rho)$ is computed for each ρ and r by:

$$\begin{aligned} P_{\text{out}}(r,\rho) &= \Pr\left[I \le r\log(1+G\rho)\right] \\ &= \int_0^R f(I,\rho)\mathrm{d}I \\ &= \left(\frac{N_t}{\rho}\right)^{2n} \frac{1}{\Gamma(n)\Gamma(n-1)} \\ \times \int_0^{r\log(1+G\rho)} e^I & \left(\int_0^{I/2} g(y,I,\rho)\mathrm{d}y\right)\mathrm{d}I \end{aligned} \tag{16}$$

This outage probability describes the performance of MIMO systems and its slope defines the finite-SNR DMT. It is to be noted that $P_{\text{out}}(r,\rho)$ decreases with ρ^{-2n} and at high SNRs, the slope of outage probability curves defines the maximum achievable diversity equal to $2n$.

C. Analytical finite-SNR DMT

The finite-SNR DMT is computed using (7). Using the well-known Leibniz integral rule [17] in (16), $\frac{\partial P_{\text{out}}(r,\rho)}{\partial\rho}$ is given by:

$$\frac{\partial P_{\text{out}}(r,\rho)}{\partial\rho} = A_1(r,\rho) + A_2(r,\rho) + A_3(r,\rho) \tag{17}$$

where

$$\begin{aligned} A_1(r,\rho) &= -2n\left(N_t\right)^{2n} \frac{1}{\rho^{2n+1}} \frac{1}{\Gamma(n)\Gamma(n-1)} \\ \times \int_0^{r\log(1+G\rho)} e^I & \left(\int_0^{I/2} g(y,I,\rho)\mathrm{d}y\right)\mathrm{d}I \\ &= -\frac{2n}{\rho} P_{\text{out}}(r,\rho), \end{aligned} \tag{18}$$

$$\begin{aligned} A_2(r,\rho) = \left(\frac{N_t}{\rho}\right)^{2n} \frac{1}{\Gamma(n)\Gamma(n-1)} \frac{rG}{1+G\rho} e^{r\log(1+G\rho)} \\ \times\left(\int_0^{r\log(1+G\rho)/2} g(y, r\log(1+G\rho), \rho)\mathrm{d}y\right), \end{aligned} \tag{19}$$

and

$$\begin{aligned} A_3(r,\rho) = \left(\frac{N_t}{\rho}\right)^{2n} \frac{1}{\Gamma(n)\Gamma(n-1)} \\ \times \int_0^{r\log(1+G\rho)} e^I \left(\int_0^{I/2} \frac{\partial g(y,I,\rho)}{\partial\rho}\mathrm{d}y\right)\mathrm{d}I \end{aligned} \tag{20}$$

where

$$\begin{aligned} \frac{\partial g(y,I,\rho)}{\partial\rho} = \left(e^{I-y}-e^y\right)^2 \left(e^{I-y}-1\right)^{n-2} \left(e^y-1\right)^{n-2} \\ \times e^{-\frac{N_t}{\rho}\left(e^{I-y}+e^y-2\right)} \frac{N_t}{\rho^2}\left(e^{I-y}+e^y-2\right). \end{aligned} \tag{21}$$

The above expressions can also be calculated using numerical integration. After deriving $P_{\text{out}}(r,\rho)$ and $\frac{\partial P_{\text{out}}(r,\rho)}{\partial\rho}$, the finite-SNR DMT can be easily computed using (7).

IV. Numerical results

In order to examine the derived finite-SNR DMT, in Fig. 1 we plot the derived exact finite-SNR DMT curves for $n=2$ at $\text{SNR}=5, 10$ dB and the asymptotic DMT [3]. We noted that the obtained DMT curves are identical to the ones obtained in [9, 11] by Monte Carlo simulations, which validates the derived exact DMT. Furthermore, the finite-SNR DMT derived in [9, 11] overestimate the achievable DMT. In order to discuss the achievability of our exact finite-SNR DMT, we have plotted the exact finite-SNR DMT for Alamouti code derived in [8]. It is noted that the Alamouti code always benefits from all the available diversity of the system (when the multiplexing gain r approaches zero), not only at asymptotic SNRs, but also at realistic SNRs.

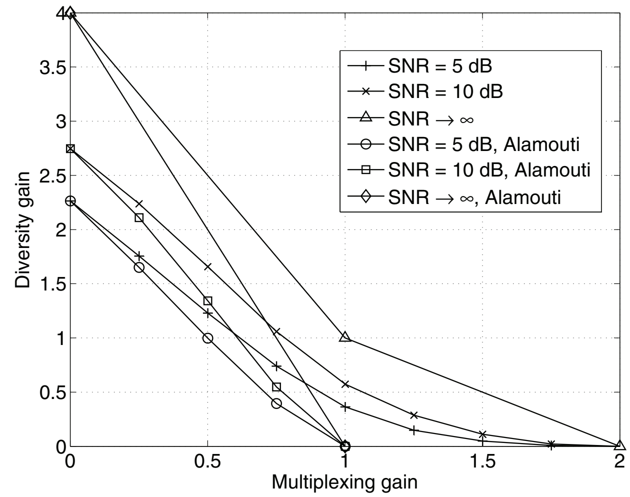

Fig. 1. Asymptotic and exact finite-SNR DMT curves for 2×2 MIMO channel and Alamouti code at various SNRs.

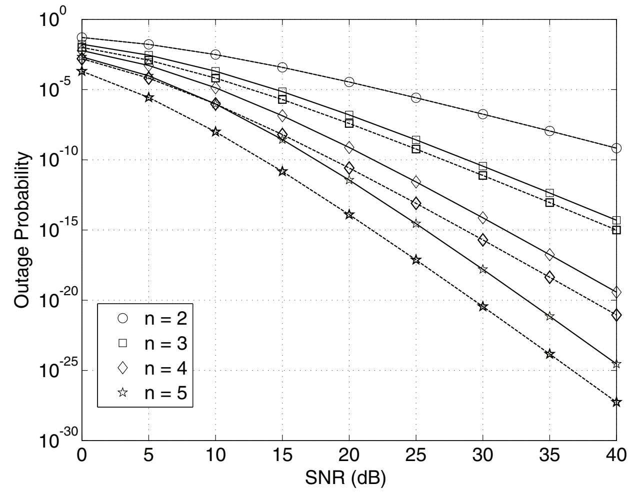

Fig. 2. Exact outage probability at a multiplexing gain $r=0.5$ for $n\times 2$ MIMO channel with solid lines and $2\times n$ MIMO channel with dashed lines, with $n=2,3,4$ and 5.

Fig. 2 plots the outage probability as a function of SNR at a multiplexing gain $r=0.5$ for $n\times 2$ MIMO channel with solid lines and $2\times n$ MIMO channel with dashed lines, with $n=2,3,4$ and 5. For $n=2$, the obtained curve is identical to the one obtained in [9, 11] by Monte Carlo simulations. Also the simulated outage probability values obtained by Monte Carlo simulations are relatively high e.g., up to 10^{-6} in [9, 11]. Our exact expression permits the assessment of very low values for the outage probability. In addition, Fig. 2 shows the superiority of $2\times N_r$ systems relative to $N_t\times 2$ systems. At high SNRs, both systems have the same slope and therefore the same diversity gain.

Fig. 3 and Fig. 4 show the asymptotic and exact finite-SNR DMT for $n\times 2$ MIMO system with solid lines and exact finite-SNR DMT for $2\times n$ MIMO system with dashed lines, for $n=3$ and $n=5$ respectively. Both figures show that the achievable diversity gains at finite-SNR are significantly lower than the asymptotic values. Furthermore, the results show that receive antennas provide higher diversity gain than transmit

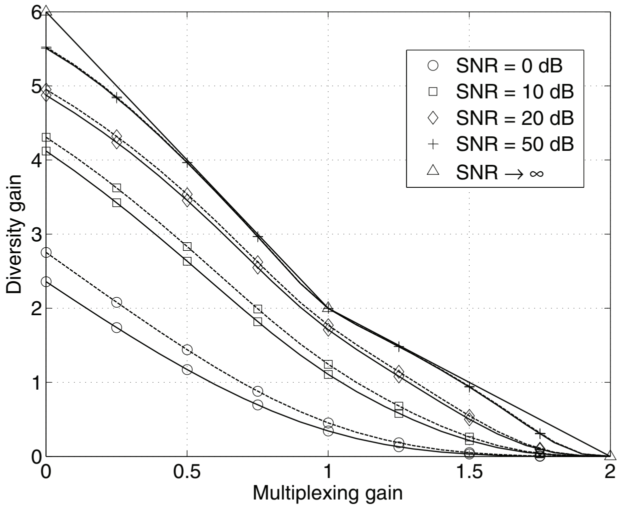

Fig. 3. Asymptotic and exact finite-SNR DMT for 3×2 MIMO channel with solid lines and 2×3 MIMO channel with dashed lines at various SNRs.

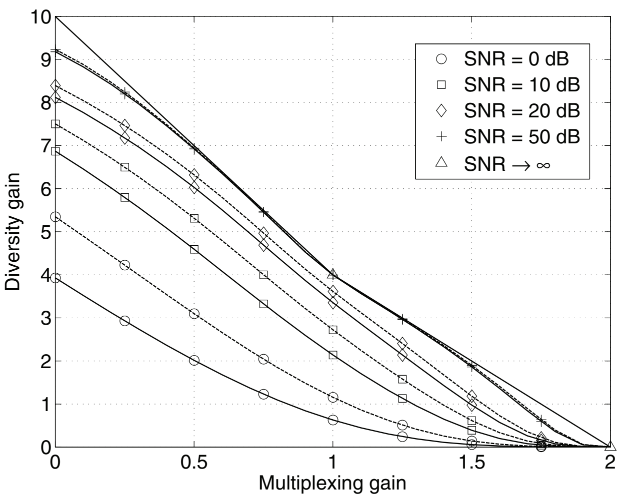

Fig. 4. Asymptotic and exact finite-SNR DMT for 5×2 MIMO channel with solid lines and 2×5 MIMO channel with dashed lines at various SNRs.

antennas especially at low to moderate SNRs. We recall that no CSI is available at the transmitter and an equal power across transmit antennas is adopted. Under this assumption, the outage probability is defined in (4) and the finite-SNR DMT in (7). Therefore, the superiority of $2 \times N_r$ with respect to $N_t \times 2$ can be explained by the fact that receive diversity is always obtained regardless of the transmission technique. On the other hand, a judicious transmission technique is required to benefit from transmit diversity, and an equal power across transmit antennas is not necessarily the best strategy at low to moderate SNRs. It is to be noted that the gap between $N_t \times 2$ and $2 \times N_r$ systems increases with the increase of n. Also, this gap decreases with the SNR increase where both systems converge to the asymptotic DMT [3] e.g. $\rho = 50$ dB where this convergence is not uniform. Unfortunately, an analytical proof of the convergence of the derived finite-SNR DMT to asymptotic DMT by letting $\rho \to \infty$ is not easily tractable.

V. Conclusions

In this paper, we have derived the exact finite-SNR diversity-multiplexing tradeoff for $N_t \times 2$ and $2 \times N_r$ Rayleigh MIMO channels. The finite-SNR DMT characterizes the system at operational SNRs where available diversity gains are computed. It has been shown that the Alamouti code benefits from all the available diversity at the operational SNR. Furthermore, we have shown that receive antennas provide higher diversity gains with respect to transmit antennas especially for low to moderate SNRs. Analysis on finite-SNR DMT shows that the achievable diversity gains are significantly lower than asymptotic values for SNR $\to \infty$. STCs for MIMO systems are conventionally designed to achieve the asymptotic DMT frontier and therefore are not efficient at realistic SNRs. This finite-SNR DMT could provide a new insight to design STCs for practical MIMO systems optimized at operational SNRs.

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