INSE6441 Jia Yuan Yu

10: Individual Rationality, Impossibility

Concordia April 10, 2017

Is it possible to have a cake and eat it too? We formalize this concept in the context of mechanism design.

Example 0.1 (Split the dollar). Each student i calls a number s_i at the same time. If the total $\sum_j s_j < 20$, then each student i receives the called number s_i. Otherwise, everyone gets 0.

1 Individual Rationality

Consider a mechanism that truthfully implements a social function. What happens when each player has an additional strategy, in addition to reporting his or her type: participate or not (reporting nothing, e.g., not bidding in an auction). Social choice functions that can be succesfully implemented must also ensure that players participate.

Consider implementable social functions. By the revelation principle, we can implement such social functions with $s^*(t) = t$. We can abuse the notation and write simply:

$$\begin{aligned} u_i(s_i^*(t_i), s_{-i}^*(t_{-i}), t_i, t_{-i}) &= u_i(t_i, t_{-i}), \\ v(s^*(t), t_i) &= v(t_i, t_{-i}), \\ p_i(s^*(t)) &= p_i(t_i, t_{-i}). \end{aligned}$$

Suppose that every player can obtain a payoff of 0 by not participating. After the outcome is determined, we use the notion of ex-post individual rationality:

$$u_i(t_i, t_{-i}) = v(t_i, t_{-i}) - p_i(t_i, t_{-i}) \geq 0, \quad \text{for all } t \in T, i \in \mathcal{I}.$$

If each player has observed its type t_i, but not yet the other players' types, we the notion of interim individual rationality:

$$\mathbb{E}_{\mathbf{t}_{-i}} u_i(t_i, \mathbf{t}_{-i}) = \mathbb{E}_{\mathbf{t}_{-i}} v(t_i, \mathbf{t}_{-i}) - \mathbb{E}_{\mathbf{t}_{-i}} p_i(t_i, \mathbf{t}_{-i}) \geq 0, \quad \text{for all } t_i \in T_i, i \in \mathcal{I}.$$

If each player has not yet observed its type t_i, we the notion of ex-ante individual rationality:

$$\mathbb{E}_{\mathbf{t}} u_i(\mathbf{t}_i, \mathbf{t}_{-i}) = \mathbb{E}_{\mathbf{t}} v(\mathbf{t}_i, \mathbf{t}_{-i}) - \mathbb{E}_{\mathbf{t}} p_i(\mathbf{t}_i, \mathbf{t}_{-i}) \geq 0, \quad \text{for all } i \in \mathcal{I}.$$

l

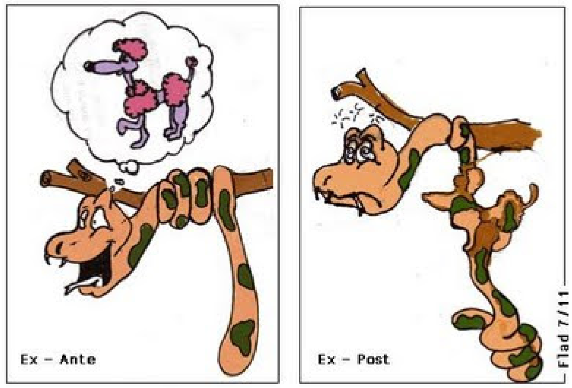

Figure 1: From `http://keywordsuggest.org/`

1.1 Budget Balance

Definition 1.1 (Budget balance)**.** Consider a mechanism where the revelation principle holds. Let $c : T \to \mathbb{R}$ denote the monetary cost of implementing an outcome specified by a strategy profile in this mechanism. This mechanism satisfies the budget balance condition if for every outcome profile $\hat{t} \in T$,

$$c(\hat{t}) \geq \sum_{i=1}^{I} p_i(\hat{t}).$$

Example 1.1 (ex-post IR for public good)**.** A public good costs $c \in (0, 1)$. There are two options: build or not. There are two players 1 and 2. Let $t_1 \in \{0, 1\}$ and $t_2 \in \{0, 1\}$ denote the valuations of the two players for the public good. We want to implement an efficient social choice function: build the public good if $t_1 = 1$ or $t_2 = 1$ (since $c < 1$); not building when $t_1 = t_2 = 0$. Ex-post Individual Rationality requires:

$$p_1(0, 0) \leq 0, \quad p_1(1, 0) \leq 1, \quad p_1(0, 1) \leq 0, \quad p_1(1, 1) \leq 1.$$

Incentive Compatibility (when a player can decide not to participate after finding out the outcome) requires (among other conditions) that for $(t_1, t_2) = (1, 1)$:

$$\begin{aligned} v(1, 1) - p_1(1, 1) &\geq v(0, 1) - p_1(0, 1) \\ \iff p_1(1, 1) &\leq p_1(0, 1). \end{aligned}$$

Combining IR and IC, we have

$$p_1(1, 1) \leq 0.$$

Similarly for player 2, we conclude (by symmetry) that

$$p_2(1, 1) \leq 0.$$

IR and IC imply that $p_1(1, 1) + p_2(1, 1) \leq 0$, which contradicts the budget-balance condition $p_1(1, 1) + p_2(1, 1) = c > 0$.

Example 1.2 (Interim IR for public good)**.** Consider the same setting as the previous example, but with the notion of interim IR. The valuations of the two players are now i.i.d. random variables $\mathbf{t}_1$ and $\mathbf{t}_2$ with $\mathbb{P}(\mathbf{t}_1 = 0) = \mathbb{P}(\mathbf{t}_1 = 1) = 1/2$. We want to implement an efficient social choice function: build the public good if $t_1 = 1$ or $t_2 = 1$ (since $c < 1$); not building when $t_1 = t_2 = 0$:

$$v_1(0,0) = 0, \quad v_1(0,1) = 1, \quad v_1(1,0) = 1, \quad v_1(1,1) = 1,$$

the associated payments are

$$\begin{aligned} p_1(0,0) &= 0, \quad p_2(0,0) = 0, \\ p_1(1,0) &= c, \quad p_2(1,0) = 0, \\ p_1(0,1) &= 0, \quad p_2(0,1) = c, \\ p_1(1,1) &= c/2, \quad p_2(1,1) = c/2. \end{aligned}$$

Interim Individual Rationality requires for $\mathbf{t}_1 = 0$:

$$\begin{aligned} \mathbb{P}(\mathbf{t}_2 = 0)p_1(0,0) + \mathbb{P}(\mathbf{t}_2 = 1)p_1(0,1) &\le \mathbb{P}(\mathbf{t}_2 = 0)v(0,0) + \mathbb{P}(\mathbf{t}_2 = 1)v(0,1) \\ \implies \quad p_1(0,0) + p_1(0,1) &\le 0. \end{aligned} \tag{1}$$

Incentive Compatibility requires that for $\mathbf{t}_1 = 1$:

$$\begin{aligned} &\mathbb{P}(\mathbf{t}_2 = 0)(v(1,0) - p_1(1,0)) + \mathbb{P}(\mathbf{t}_2 = 1)(v(1,1) - p_1(1,1)) \\ &\qquad \ge \mathbb{P}(\mathbf{t}_2 = 0)(v(0,0) - p_1(0,0)) + \mathbb{P}(\mathbf{t}_2 = 1)(v_1((0,1),1) - p_1(0,1)) \\ &\frac{1}{2}(1 - p_1(1,0)) + \frac{1}{2}(1 - p_1(1,1)) \ge \frac{1}{2}(0 - p_1(0,0)) + \frac{1}{2}(1 - p_1(0,1)) \\ &\implies \quad 2 - p_1(0,1) - p_1(1,1) \ge 1 - p_1(0,0) - p_1(0,1) \underbrace{\ge 1}_{\text{by}(1)} \end{aligned}$$

$$\implies \quad 1 \ge p_1(0,1) + p_1(1,1) \tag{2}$$

Similarly for player 2, we conclude (by symmetry) that

$$0 \ge p_2(0,0) + p_2(0,1) \tag{3}$$

$$1 \ge p_1(0,1) + p_1(1,1). \tag{4}$$

Combining (1)-(4), we obtain

$$2 \ge \sum_{i=1}^{2} p_i(0,0) + \sum_{i=1}^{2} p_i(0,1) + \sum_{i=1}^{2} p_i(1,0) + \sum_{i=1}^{2} p_i(1,1) = 3c.$$

This means that it is not possible to achieve budget balance if $c > 2/3$.

These examples demonstrate a phenomenon that is holds in general: we cannot have at the same time IR, IC, efficiency, and budget balance!

1

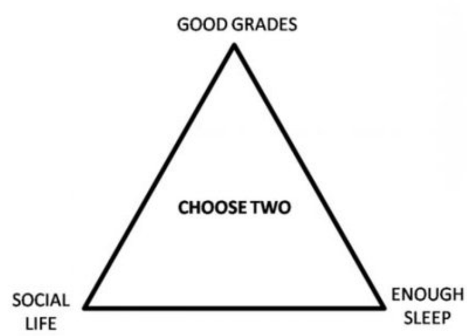

Figure 2: From `mdphdtobe.files.wordpress.com`

1.2 Bilateral trade, Double auction, Asymmetric information

Many important games have asymmetric information: salary negotiation, investing in startups, selling an used car.

Example 1.3 (Interim IR for bilateral trade[1]). Consider a bilateral trade situation with a seller and a buyer. Let $c \in [0,1]$. The type of the seller is a random variable $\mathbf{t}_1$ that takes one of two values $\{0, c\}$ with probability $1/2$ each. The type of the buyer is an independent random variable $\mathbf{t}_2$ that is uniformly distributed on $[0,1]$. We want to implement an efficient social choice function: trade if $t_2 \geq t_1$, no trade if $t_2 < t_1$. In this setting, the valuations are player specific:

$$v_2(t_1,t_2) = \begin{cases} t_2 & \text{if trade } (t_2 \geq t_1), \\ 0 & \text{if no trade.} \end{cases}$$

$$v_1(t_1,t_2) = \begin{cases} -t_1 & \text{if trade } (t_2 \geq t_1), \\ 0 & \text{if no trade.} \end{cases}$$

Interim Individual Rationality for the buyer requires for $t_2 \in [0,c)$:

$$\begin{aligned} \mathbb{E}v_2(\mathbf{t}_1,t_2) - \mathbb{E}p_2(\mathbf{t}_1,t_2) &\geq 0 \\ \mathbb{P}(\mathbf{t}_1 = 0)v_2(\underbrace{0,t_2}_{\text{trade}}) + \mathbb{P}(\mathbf{t}_1 = c)v_2(\underbrace{c,t_2}_{\text{notrade}}) - \mathbb{E}p_2(\mathbf{t}_1,t_2) &\geq 0 \\ \frac{1}{2}t_2 + \frac{1}{2}0 &\geq \mathbb{E}p_2(\mathbf{t}_1,t_2). \end{aligned} \tag{5}$$

[1]`https://econ.ucsb.edu/~tedb/Courses/UCSBpf/mechtheo.pdf`

Interim IR for the seller requires for $t_1 = c$:

$$\begin{aligned}
&\mathbb{E}v_1(c, \mathbf{t}_2) - \mathbb{E}p_1(c, \mathbf{t}_2) \geq 0 \\
&\mathbb{P}(\mathbf{t}_2 < c)\mathbb{E}[v_1(\underbrace{c, \mathbf{t}_2}_{\text{notrade}}) \mid \mathbf{t}_2 < c] + \mathbb{P}(\mathbf{t}_2 \geq c)\mathbb{E}[v_1(\underbrace{c, \mathbf{t}_2}_{\text{trade}}) \mid \mathbf{t}_2 \geq c] - \mathbb{E}p_1(c, \mathbf{t}_2) \geq 0 \\
&c \cdot 0 + (1-c)(-c) \geq \mathbb{E}p_1(c, \mathbf{t}_2). \qquad (6)
\end{aligned}$$

IC for the seller for $t_1 = 0$ requires:

$$\begin{aligned}
\mathbb{E}v_1(0, \mathbf{t}_2) - \mathbb{E}p_1(0, \mathbf{t}_2) &\geq \mathbb{E}v_1(c, \mathbf{t}_2) - \mathbb{E}p_1(c, \mathbf{t}_2), \\
\mathbb{E}v_1(0, \mathbf{t}_2) - \mathbb{E}v_1(c, \mathbf{t}_2) + \mathbb{E}p_1(c, \mathbf{t}_2) &\geq \mathbb{E}p_1(0, \mathbf{t}_2) \\
0 - 0 + \mathbb{E}p_1(c, \mathbf{t}_2) &\geq \mathbb{E}p_1(0, \mathbf{t}_2) \\
\mathbb{E}p_1(0, \mathbf{t}_2) &\leq -c(1-c), \qquad (7)
\end{aligned}$$

where the last inequality follows from (6). In turn, combining (6) and (7), we have

$$\mathbb{E}p_1(\mathbf{t}_1, \mathbf{t}_2) \leq -c(1-c). \qquad (8)$$

Recall that budget balance for bilateral trade requires that no payment is lost:

$$\mathbb{E}p_1(\mathbf{t}_1, \mathbf{t}_2) + \mathbb{E}p_2(\mathbf{t}_1, \mathbf{t}_2) = 0,$$

hence (8) implies that

$$\mathbb{E}p_2(\mathbf{t}_1, \mathbf{t}_2) \geq c(1-c). \qquad (9)$$

Consider a fixed $t_2 \in (0, c)$, Incentive Compatibility for the buyer with type t_2 requires that:

$$\begin{aligned}
\mathbb{E}v_2(\mathbf{t}_1, t_2) - \mathbb{E}p_2(\mathbf{t}_1, t_2) &\geq \mathbb{E}v_2(\mathbf{t}_1, z) - \mathbb{E}p_2(\mathbf{t}_1, z) \quad \text{for all } z \in [0, 1], \\
&\geq \mathbb{E}v_2(\mathbf{t}_1, 0) - \mathbb{E}p_2(\mathbf{t}_1, 0).
\end{aligned}$$

Observe that for $t_2 \in (0, c)$,

$$\mathbb{E}v_2(\mathbf{t}_1, t_2) = \mathbb{E}v_2(\mathbf{t}_1, 0),$$

since the outcome (t_1, t_2) is the same as the outcome $(t_1, 0)$ for all $t_2 \in (0, c)$. Hence, we obtain

$$\mathbb{E}p_2(\mathbf{t}_1, t_2) \leq \mathbb{E}p_2(\mathbf{t}_1, 0) \leq 0 \quad \text{for all } t_2 \in [0, c). \qquad (10)$$

Next, by definition, we have

$$\begin{aligned}
\mathbb{E}p_2(\mathbf{t}_1, \mathbf{t}_2) &= \mathbb{P}(\mathbf{t}_2 < c)\mathbb{E}[p_2(\mathbf{t}_1, \mathbf{t}_2) \mid \mathbf{t}_2 < c] + \mathbb{P}(\mathbf{t}_2 \geq c)\mathbb{E}[p_2(\mathbf{t}_1, \mathbf{t}_2) \mid \mathbf{t}_2 \geq c] \\
&\leq c \cdot 0 + (1-c)\mathbb{E}[p_2(\mathbf{t}_1, \mathbf{t}_2) \mid \mathbf{t}_2 > c],
\end{aligned}$$

where the last inequality uses (10). By (9), we have

$$\mathbb{E}[p_2(\mathbf{t}_1, \mathbf{t}_2) \mid \mathbf{t}_2 > c] \geq \frac{1}{1-c}\mathbb{E}p_2(\mathbf{t}_1, \mathbf{t}_2) \geq \frac{1}{1-c}c(1-c) = c,$$

so that

$$\mathbb{E}[v_2(\mathbf{t}_1, \mathbf{t}_2) \mid \mathbf{t}_2 > c] - \mathbb{E}[p_2(\mathbf{t}_1, \mathbf{t}_2) \mid \mathbf{t}_2 > c] \geq (1+c)/2 - c = 1/2 - c/2,$$

whereas

$$\mathbb{E}[v_2(\mathbf{t}_1, 0) \mid \mathbf{t}_2 > c] - \mathbb{E}[p_2(\mathbf{t}_1, 0) \mid \mathbf{t}_2 > c] \geq \mathbb{P}(\mathbf{t}_1 = 0)\mathbb{E}[\mathbf{t}_2 \mid \mathbf{t}_2 > c] + 0 \quad \geq (1+c)/4,$$

Observe that if $c > 1/3$, then $(1+c)/4 > 1/2 - c/2$, and the buyer's incentive compatibility condition no longer holds.

2 Myerson-Satterthwaite

We saw a negative result for preference aggregation (three choices for midterm date). Here's another negative result.

Theorem 2.1 (Myerson-Satterthwaite)**.** *Consider a Bayesian game where the player types are independent random variables whose probability distributions have strictly positive density functions, and the intersection of the support of these distributions is non-empty. There does not exist a Bayesian Nash equilibrium that is simultaneously IC, interim IR, budget balanced, and efficient.*

See Proposition 23.E.1 of MWG (for a proof for bilateral trade) or `https://pages.wustl.edu/files/pages/imce/nachbar/my-sat.pdf` for a proof in the general setting.

3 Reading material

- Chapter 23 of Mas-Colell, Whinston, Green.
- Chapter 7 of Fudenberg and Tirole.