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11: Refinements

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We have already seen the refinement of Nash Equilibria to Subgame Perfect Equilibria. In this lecture, we see further refinements for multi-stage games of incomplete information.

1 Signaling Game

We consider an important instance of games of incomplete information. In these games, there are two players, a leader and a follower. The leader's action set is A_1, the follower's is A_2. The leader has a private type t taking values in T, and utility function u_1. The follower has a type that is common knowledge to everyone, and utility function u_2. The follower has an initial belief $p \in \Delta(T)$ about the distribution of the type $\mathbf{t}$, which is also common knowledge.

The leader acts first, and the follower observes this action before taking its own action. A strategy for the leader is a mapping $\sigma_1 : T \to \Delta(A_1)$ from its type $t \in T$ to a probability distribution

$$\sigma_1(z \mid t) \quad z \in A_1.$$

A strategy for the follower is a mapping $\sigma_2 : A_1 \to \Delta(A_2)$ from the leader's action $a_1 \in A_1$ to a probability distribution

$$\sigma_2(z \mid a_1) \quad z \in A_2.$$

Given a strategy profile (σ_1, σ_2), the expected payoff to the leader is straightforward:

$$\tilde{u}_1(\sigma_1, \sigma_2, t) = \mathbb{E}u_1(\mathbf{a}_1, \mathbf{a}_2, t) = \sum_{a_1 \in A_1} \sum_{a_2 \in A_2} \sigma_1(a_1 \mid t)\sigma_2(a_2 \mid a_1)u_1(a_1, a_2, t).$$

The expected payoff to the follower depends on both the prior belief p and the observation a_1. Before observing a_1, the ex-ante expected payoff corresponding to strategy profile (σ_1, σ_2) is

$$\hat{u}_2(\sigma_1, \sigma_2) = \sum_{t \in T} \sum_{a_1 \in A_1} \sum_{a_2 \in A_2} \sigma_1(a_1 \mid t)p(t)\sigma_2(a_2 \mid a_1)u_2(a_1, a_2, t).$$

However, observing a_1 affects the belief of the follower on the probability distribution of the leader's type $\mathbf{t}$. By Bayes' rule:

$$\begin{aligned}\mathbb{P}(\mathbf{a}_1 = a_1, \mathbf{t} = t) &= \mathbb{P}(\mathbf{a}_1 = a_1 \mid \mathbf{t} = t)\mathbb{P}(\mathbf{t} = t) \\ &= \mathbb{P}(\mathbf{t} = t \mid \mathbf{a}_1 = a_1)\mathbb{P}(\mathbf{a}_1 = a_1).\end{aligned}$$

Hence, if $\mathbb{P}(\mathbf{a}_1 = a_1) > 0$, then

$$\mathbb{P}(\mathbf{t} = t \mid \mathbf{a}_1 = a_1) = \frac{\mathbb{P}(\mathbf{a}_1 = a_1 \mid \mathbf{t} = t)\mathbb{P}(\mathbf{t} = t)}{\mathbb{P}(\mathbf{a}_1 = a_1)}.$$

Given strategy profile (σ_1, σ_2) and the observation a_1, the ex-post expected payoff to the follower is

$$\tilde{u}_2(\sigma_1, \sigma_2, a_1) = \sum_{t \in T} \sum_{a_2 \in A_2} \mu(t \mid \sigma_1, a_1)\sigma_2(a_2 \mid a_1)u_2(a_1, a_2, t),$$

$$\text{for all } t \in T, a_1 \in A_1 : \quad \mu(t \mid \sigma_1, a_1) = \begin{cases} \frac{\mathbb{P}(\mathbf{a}_1 = a_1 \mid \mathbf{t} = t)\mathbb{P}(\mathbf{t} = t)}{\mathbb{P}(\mathbf{a}_1 = a_1)} & \text{if } \mathbb{P}(\mathbf{a}_1 = a_1) > 0, \\ p(t) & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \frac{\sigma_1(a_1 \mid t)p(t)}{\sum_{t' \in T} \sigma_1(a_1 \mid t')p(t')} & \text{if } \sum_{t' \in T} \sigma_1(a_1 \mid t')p(t') > 0, \\ p(t) & \text{otherwise.} \end{cases}$$

We are ready to definite the following solution concept.

Definition 1.1 (PBE)**.** A Perfect Bayesian Equilibrium of a signaling game is a strategy profile (σ_1^*, σ_2^*) and a posterior belief μ such that for every $t \in T$, and $a_1 \in A_1$:

$$\sigma_1^*(\cdot \mid t) \in \arg\max_{z \in \Delta(A_1)} \quad \tilde{u}_1(z, \sigma_2^*, t),$$

$$\sigma_2^*(\cdot \mid t) \in \arg\max_{z \in \Delta(A_2)} \quad \tilde{u}_2(\sigma_1^*, z, a_1),$$

$$\mu = \begin{cases} \frac{\sigma_1^*(a_1 \mid t)p(t)}{\sum_{t' \in T} \sigma_1^*(a_1 \mid t')p(t')} & \text{if } \sum_{t' \in T} \sigma_1^*(a_1 \mid t')p(t') > 0, \\ p(t) & \text{otherwise.} \end{cases}$$

The following example illustrates one reason why people go to school.

Example 1.1 (Spence's Education Game)**.** The leader is a worker, the follower is an employer. The worker's type can be $t = 1, 2$; $\mathbf{t}$ is a random variable with distribution $\mathbb{P}(\mathbf{t} = 1) = q$. The worker's action is $a_1 \in \mathbb{R}_+$, the amount of investment in education. The employer's action is the wage $a_2 \in \mathbb{R}_+$ offered to the worker, based on the observation of a_1. Let the employer's belief for types 1,2 be denoted $1 - p, p$.

The employer's payoff is

$$u_2(a_1, a_2, t) = -(a_2 - t)^2.$$

The payoff of the worker is

$$u_1(a_1, a_2, t) = a_2 - a_1/t.$$

First, we show that a worker with higher ability prefers at least as much of education as if it has lower ability.

Proposition 1.1 (Monotonicity)**.** *Let σ_1^1 and σ_2^2 be equilibrium strategies for types 1 and 2 respectively. If $z^1 \in \text{support}(\sigma_1^1)$ and $z^2 \in \text{support}(\sigma_1^2)$, then $z^1 \leq z^2$.*

Proof. By definition of PBE, we have

$$a_2(z^1) - z^1/1 \geq a_2(z^2) - z^2/1,$$
$$a_2(z^2) - z^2/2 \geq a_2(z^1) - z^1/2.$$

Adding the two inequalities gives

$$-z^1/1 - z^2/2 \geq -z^2/1 - z^1/2$$
$$(z^2 - z^1)/2 \geq 0.$$

□

Let's look for equilibria for this game. First, consider a PBE where the leader (worker) chooses different actions if different types are observed (i.e., $a_1(1) \neq a_1(2)$). This is called a separating equilibrium. In this case, the follower can infer the type of the leader. The employer's expected payoff is

$$-\mathbb{E}(a_2 - \mathbf{t})^2,$$

and its optimal strategy is[1]

$$a_2(a_1) = \mathbb{E}[\mathbf{t} \mid a_1] = t.$$

Consider the following PBE strategy for the worker:

- if $t = 1$, then choose $a_1 = 0$,
- if $t = 2$, then choose $a_1 = a^*$ for some constant $a^* > 0$,

which is reasonable by the monotonicity proposition and since the expected payoff of the worker is

$$\mathbb{E}[\mathbf{t} \mid a_1] - a_1/t = t - a_1/t.$$

For this strategy to be optimal, we need:

$$(\text{for } t = 1) \quad \mathbb{E}[\mathbf{t} \mid 0] - 0/1 \geq \mathbb{E}[\mathbf{t} \mid a^*] - a^*/1$$
$$1 - 0 \geq 2 - a^*/1$$

and

$$(\text{for } t = 1) \quad \mathbb{E}[\mathbf{t} \mid a^*] - a^*/2 \geq \mathbb{E}[\mathbf{t} \mid 0] - 0/2$$
$$2 - a^*/2 \geq 1 - 0/2.$$

In other words,

$$1 \leq a^* \leq 2.$$

Every strategy for the worker with $a_1(1) = 0$ and $a_1(2) = a^*$ is an equilibrium strategy if $a^* \in [1, 2]$. For the employer, one possible posterior belief is

$$\mu(\mathbf{t} = 1 \mid a_1) = \begin{cases} 1 & \text{if } a_1 < a^* \\ 0 & \text{otherwise.} \end{cases}$$

[1] cf. `http://www.le.ac.uk/users/dsgp1/COURSES/TOPICS/meansqar.pdf`

2 Reading material

- Chapter 8 of Fudenberg and Tirole.