INSE6441	Jia Yuan Yu
3: Existence of equilibria	
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Last time, we introduced mixed strategies and equilibria. Today, we show an important result on the existence of equilibria, and how to compute them for very special cases.

Example 0.1 (Dollar auction)**.** Bidders must pay theirs bids win or lose, iterate between bidders.

1 Existence of Nash equilibria

Recall that each strategy σ_i belongs to a simplex Σ_i. Let $\Sigma = \Sigma_1 \times \ldots \times \Sigma_I$. Recall that the best-response function of each player i is a set-valued function, which we will denote

$$r_i : \Sigma \to 2^{\Sigma_i},$$

where 2^{Σ} denotes the power set of Σ (e.g., the set of all possible subsets of Σ). The domain of each r_i can be restricted to Σ_{-i}, but this complicates notation.

Define the best-response profile:

$$r = (r_1, \ldots, r_I) : \Sigma \to 2^{\Sigma}.$$

A fixed point of r is a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ such that

$$\sigma \in r(\sigma),$$

which is a Nash equilibrium by definition. Under what conditions does r have a fixed point?

Theorem 1.1 (Kakutani)**.** *The following are sufficient conditions for r to have a fixed point:*

1. *Σ is compact[1], convex, and nonempty subset of a finite-dimensional Euclidean space,*
2. *$r(\sigma)$ is nonempty for all σ,*
3. *$r(\sigma)$ is convex for all σ.*
4. *r has a closed graph: for every sequence $\{\sigma^n\}$ and $\{\hat{\sigma}^n\}$ such that $\hat{\sigma}^n \in r(\sigma^n)$ for all n, $\sigma^n \to \sigma$, and $\hat{\sigma}^n \to \hat{\sigma}$, we have $\hat{\sigma} \in r(\sigma)$.*

[1] In $\mathbb{R}^d$, compact means closed and bounded.

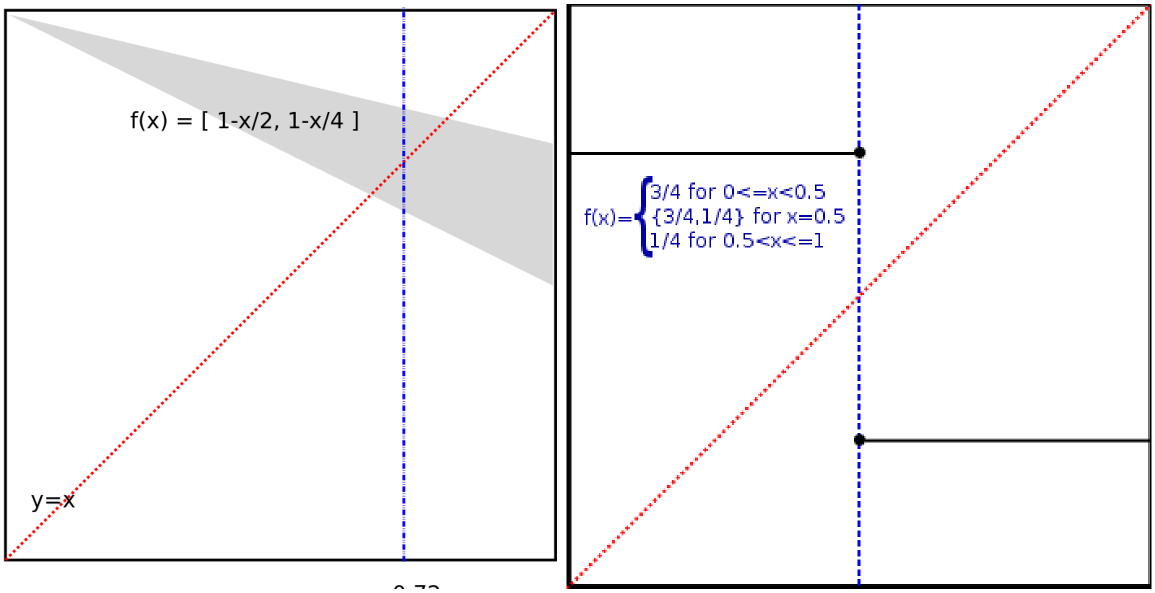

Figure 1: Functions with fixed points and without fixed points(from Wikipedia).

In our case, let's check the four conditions:

1. $\Sigma = \Sigma_1 \times \ldots \times \Sigma_I$, where each Σ_i is a simplex, which is nonempty, compact, and convex.

2. Consider each best-response function r_i: for a fixed σ_{-i}, the payoff function is continuous (linear) in σ_i. By Weierstrass' Theorem, it attains a maximum, hence, $r_i(\sigma)$ is nonempty.

3. Consider an arbitrary i. If $\sigma_i' \in r_i(\sigma)$ and $\sigma_i'' \in r_i(\sigma)$, then by definition:

$$\tilde{u}_i(\lambda\sigma_i' + (1-\lambda)\sigma_i'', \sigma_{-i}) = \lambda\tilde{u}_i(\sigma_i', \sigma_{-i}) + (1-\lambda)\tilde{u}_i(\sigma_i'', \sigma_{-i}),$$

   hence, $\lambda\sigma_i' + (1-\lambda)\sigma_i''$ is also a best response.

4. Consider an arbitrary i. Since $\hat{\sigma}^n \in r(\sigma^n)$ for all n, we have

$$\tilde{u}_i(\hat{\sigma}_i^n, \sigma_{-i}^n) \geq \tilde{u}_i(z, \sigma_{-i}^n) \quad \text{for all } z.$$

   Since the expected payoff $\tilde{u}_i$ is continuous in Σ, we have

$$\tilde{u}_i(\hat{\sigma}_i, \sigma_{-i}) \geq \tilde{u}_i(z, \sigma_{-i}) \quad \text{for all } z,$$

   so that $\hat{\sigma}_i \in r_i(\sigma)$ by definition. Since this holds for an arbitrary i, we have also $\hat{\sigma} \in r(\sigma)$.

Remark 1 (Infinite games). For games where the strategy spaces $S_1, \ldots, S_I$ are nonempty, compact, and convex, if every utility function u_i is continuous with respect to the strategy profile, and concave with respect to the action x_i, then there exists a (pure) Nash equilibrium.

2 Computing equilibria

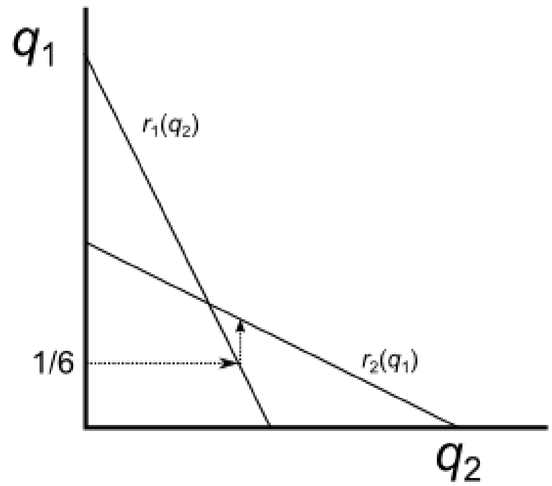

Figure 2: Optimal quantity $q_1^{\text{opt}} = r_1$ as a function of q_2 and vice versa. From `http://mindyourdecisions.com/`.

Tatonnement in Cournot competition (asynchronous, synchronous, average of opponent's past plays, etc.). Most of the time, tatonnement does not work (Example).

In general, computing equilibria for games is hard. In the case of finite zero-sum two-player games, we can use linear programming to compute Nash equilibria.

3 Reading material

- Chapter 1 of Fudenberg and Tirole.
- Chapter 12 of Microeconomics Theory (Mas-Colell, Whinston, Green).
- Chapter 2 of Myerson.