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| 5: Subgame Perfection | |
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**Example 0.1** (No Nash equilibrium)**.** Auction for 3 CAD, where the winner is the highest bid in the open interval $(0, 1)$.

**Example 0.2** (Relance with 52 cards)**.** One card poker with two players. One action each player P1: bet or fold, P2: call or fold. Among cards of the same number, assume that $\spadesuit > \heartsuit > \clubsuit > \diamondsuit$. Initial ante of 1 CAD each.

**Example 0.3** (Finite Bargaining)**.** Finite horizon: two players take turn proposing how to divide 1 CAD, starting with P1. There's a discounting factor $\delta \in (0, 1)$ at each iteration. Nash equilibrium by backward induction.

**Example 0.4** (Infinite Bargaining)**.** Consider the infinite version. For a fixed $x$, consider the profile:

- for every information set, P1 proposes $(x, 1 - x)$, and rejects $(z, 1 - z)$ if $z < x$.
- for every information set, P2 proposes $(x, 1 - x)$, and rejects $(z, 1 - z)$ if $1 - z < 1 - x$.

This is a Nash equilibrium. However, suppose that P1 proposes at the first time step:

$$(k, 1 - k) \quad \text{such that } \delta x < k < x,$$

the above strategy for P1 is no longer a best response.

Consider the following alternative strategy profile:

- for every information set, P1 proposes $(x, 1 - x)$, and accepts $(z, 1 - z)$ if $z \geq y$,
- for every information set, P2 proposes $(y, 1 - y)$, and accepts $(z, 1 - z)$ if $1 - z \geq 1 - x$,

and suppose that $x$ and $y$ satisfy the following equations:

$$y = \delta x,$$
$$1 - x = \delta(1 - y).$$

P1 is then indifferent between accepting $y$ and rejecting. Similarly for P2. By solving the above equations, we get a Nash equilibrium with $x = 1/(1 + \delta)$. Observe that since $x > 1 - x$ in this case, the first player has an advantage.

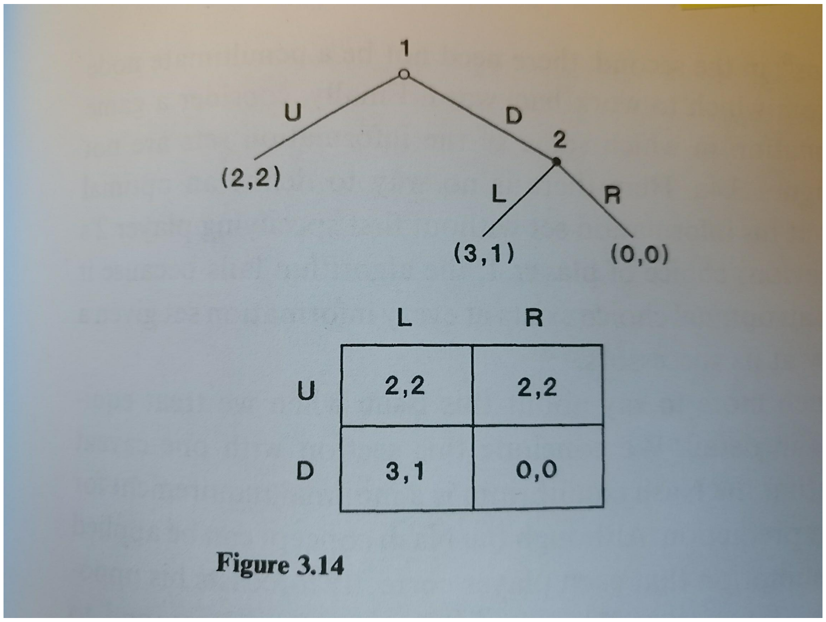


Figure 1: Dynamic game with two equilibria $(U, R)$ and $(D, L)$. However, $(U, R)$ is not credible: even if player 2 declares that he will only play R, player 1 can play D first and force player 2 to reconsider. What if U and D lead to the same information set for player 2?

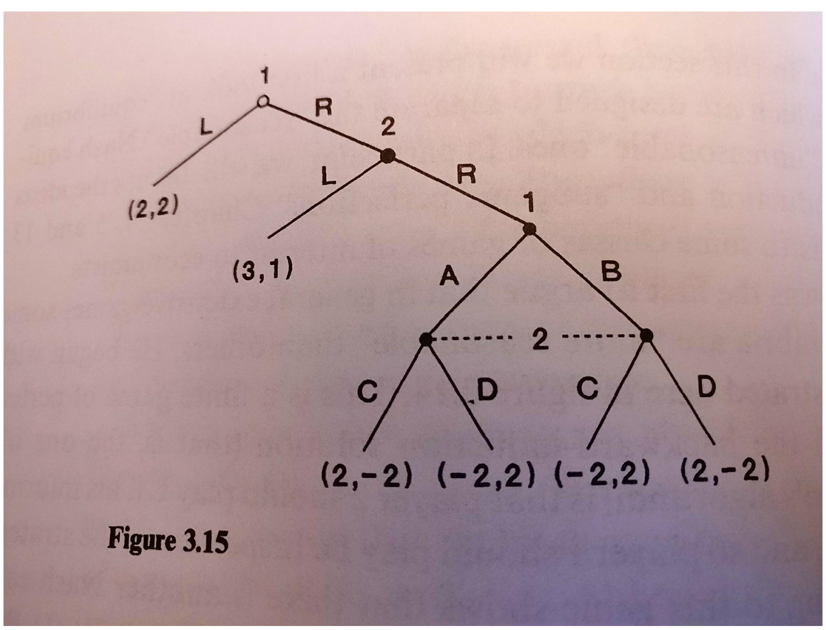


Figure 2: Subgame example.

# 1 Subgame-perfect equilibrium

Although Nash equilibria are well-defined to dynamic games, we can refine the notion of Nash equilibria to a special class of Nash equilibria that better describe reality (cf. Figure 1). We introduce the notion of subgame-perfect equilibrium.

**Definition 1.1** (Subgame)**.** A proper subgame of a game tree $T$ is a subtree of $T$ (with a single root node) such that two nodes of the subtree are in the same information set if and only if they are in the same information set in $T$.

**Definition 1.2** (SPE)**.** A behavior strategy profile $\sigma$ is a subgame-perfect equilibrium if for every proper subgame $G$, the restriction of $\sigma$ to $G$ is a Nash equilibrium of $G$.

*Remark* 1. A SPE is a Nash equilibrium. In finite games of perfect information, SPE can be found by backward induction.

**Example 1.1** (Backward induction with multiple players)**.** Consider a finite game with perfect information. We can draw the game tree and specify a pure Nash equilibrium by backward induction.

Next, let's analyze an extensive-form game with Nature moves.

**Example 1.2** (Borel's Relance[1])**.** Nature sends private signals $X$ and $Y$ uniformly drawn from $[0, 1]$ to P1 and P2, respectively. For simplicity, assume that $X$ and $Y$ are independent. Each players puts up an ante of 1 CAD at the start. This is a zero-sum game. P1 decides to bet $\beta$ or fold (payoff of -1 to P1, +1 to P2). Then, P2 decides to fold (payoff of +1 to P1, -1 to P2) or call (payoff of $1 + \beta$ to P1 if $X > Y$, $-(1 + \beta)$ if $X < Y$, 0 if $X = Y$; opposite for P2). The extensive-form game tree proceeds as follows: Nature sends signal to P1, then P1 moves, then Nature sends signal to P2, then P2 moves.

First, observe that, only strategies of the following form are not weakly dominated: for fixed constants $a, b$,

$$s_1(X) = \begin{cases} \text{fold} & \text{if } X < a, \\ \text{bet } \beta & \text{if } X \geq a. \end{cases}$$

$$s_2(Y) = \begin{cases} \text{fold} & \text{if } Y < b, \\ \text{call } \beta & \text{if } Y \geq b. \end{cases}$$

We will only consider these strategies.

Next, observe that $b \geq a$ because choosing $b < a$ is weakly dominated by choosing $b \geq a$ for P2. We cannot use backward induction because of the information sets containing multiple nodes.

[1] See "Best Response to Tight and Loose Opponents in the Borel and von Neumann Poker Models" by Casey Warmbrand and "On the Borel and von Neumann Poker Models" by Chris Ferguson and Thomas S. Ferguson.

If P2 uses the above strategy with parameter $b$, P1 must be indifferent[2] between betting and folding at the information set $X = a$; the corresponding expected payoffs must be the same:

$$
\begin{aligned}
& +1\mathbb{P}(Y < b) - (1+\beta)\mathbb{P}(Y > b) = -1 & (1)\\
\implies \quad & b - (1+\beta)(1-b) = -1 & (2)\\
\implies \quad & b + b - 1 - \beta + \beta b = -1 & (3)\\
\implies \quad & b = \beta/(\beta+2). & (4)
\end{aligned}
$$

Similarly, P2 must be indifferent at information set $Y = b$ between calling and folding:

$$(1+\beta)\mathbb{P}(\text{P2 wins} \mid \text{P1 bets}, Y = b) - (1+\beta)\mathbb{P}(\text{P2 loses} \mid \text{P1 bets}, Y = b) = -1.$$

Since

$$
\begin{aligned}
\mathbb{P}(\text{P2 wins} \mid \text{P1 bets}, Y = b) &= \mathbb{P}(X < b \mid \text{P1 bets})\\
&= \mathbb{P}(X \in (a, b) \mid X > a) = \frac{b-a}{1-a},\\
\mathbb{P}(\text{P2 loses} \mid \text{P1 bets}, Y = b) &= \frac{1-b}{1-a},
\end{aligned}
$$

we obtain

$$(1+\beta)\frac{b-a}{1-a} - (1+\beta)\frac{1-b}{1-a} = -1. \qquad (5)$$

By solving (1) and (5) for $a$ and $b$, we find:

$$a = \beta^2/(2+\beta)^2.$$

For the example the Relance game with 52 cards, and $\beta = 2$ (pot-sized bet), we have $b = 1/2$ and $a = 1/4$. The approximate optimal strategy is for P1 to fold about 13 cards (deuce (♠♡♣♢), three (♠♡♣♢), four (♠♡♣♢), five (♢)), and for P2 to fold about half the cards (deuce, three, four, five, six, seven, eight (♡♣♢)).

# 2 Reading material

- Chapters 3, 4.4, 6 of Fudenberg and Tirole.

[2] The best-response function is upper semicontinuous.