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9: Revenue Equivalence	
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1 Revenue equivalence

Let's compare the payments for the first- and second-price auction with player types drawn from the same distribution.

For first-price auction with two players, and valuations V_1, V_2 i.i.d. according to the uniform distribution F on $[0, 1]$, the BNE bids are

$$\frac{1}{2}V_1, \frac{1}{2}V_2.$$

If we let $V_{(2)}$ denote the highest valuation[1], the payment by the winner is

$$\max\left\{\frac{1}{2}V_1, \frac{1}{2}V_2\right\} = \frac{1}{2}V_{(2)},$$

which is also the revenue of the auctioneer. The probability distribution of $V_{(2)}$ is

$$\begin{aligned}\mathbb{P}(V_{(2)} \leq z) &= \mathbb{P}(V_1 \leq z, V_2 \leq z)\\ &= \mathbb{P}(V_1 \leq z)\mathbb{P}(V_2 \leq z) = F^2(z).\end{aligned}$$

The probability density of $V_{(2)}$ is

$$2F(z)F'(z) = 2z, \quad z \in [0, 1].$$

The expected revenue is

$$\mathbb{E}\frac{1}{2}V_{(2)} = \frac{1}{2}\int_0^1 2z \cdot z\mathrm{d}z = \left[\frac{z^3}{3}\right]_0^1 = 1/3.$$

For second-price auction with two players, and valuations V_1, V_2 i.i.d. according to the uniform distribution F on $[0, 1]$, the BNE bids are

$$V_1, V_2.$$

If we let $V_{(2)}$ denote the highest valuation and $V_{(1)}$ the second highest, the payment by the winner is

$$V_{(1)} = \min\{V_1, V_2\},$$

[1] This is an order statistics.

which is the revenue of the auctioneer. The probability distribution of $V_{(2)}$ is

$$\begin{aligned}\mathbb{P}(V_{(1)} \leq z) &= 1 - \mathbb{P}(V_{(1)} > z) \\ &= 1 - \mathbb{P}(V_1 > z, V_2 > z) \\ &= 1 - \mathbb{P}(V_1 > z)\mathbb{P}(V_2 > z) = 1 - (1 - F(z))^2.\end{aligned}$$

The probability density of $V_{(1)}$ is

$$2(1 - F(z))F'(z) = 2(1 - z) \quad z \in [0, 1].$$

The expected revenue is

$$\mathbb{E}V_{(1)} = \int_0^1 2(1 - z) \cdot z \mathrm{d}z = 2\left[\frac{z^2}{2} - \frac{z^3}{3}\right]_0^1 = 1/3.$$

Is it a coincidence that first- and second-price auction yield the same revenue for the auctioneer?

Theorem 1.1 (Revenue Equivalence). *Consider two symmetric auctions with the same I buyers, whose valuations are i.i.d. random variables, and where winner of the auction is the same and the expected payment from a buyer with a valuation of 0 is also 0. Then, the two auctions generate the same expected revenue at their respective Bayesian Nash Equilibria.*

Proof. By the revelation principle, we consider incentive compatible implementations of the two auction mechanisms. Consider one of the two auctions. Suppose that buyer i is the winner. Let F denote the probability distribution of the valuation random variables $V_1, \ldots, V_I$. Let $G(z) \triangleq F^{I-1}(z)$ denote the probability that z is the highest valuation—or the probability distribution of $V_{(I-1)}$. The Nash equilibrium profile is $(v_1, \ldots, v_I)$, the expected payoff of buyer i reporting a valuation z is

$$v_i \cdot G(z) - \mathbb{E}p_i(V_1, \ldots, z, \ldots, V_I) = v_i \cdot G(z) - \phi_i(z)$$

Since v_i is a best response, we must have

$$v_i G'(v_i) - \phi_i'(v_i) = 0,$$

in other words,

$$v_i G'(v_i) = \phi_i'(v_i).$$

By the fundamental theorem of calculus[2], we obtain:

$$\phi_i(v_i) - \phi_i(0) = \int_0^{v_i} zG'(z)\mathrm{d}z.$$

[2] $\phi(b) - \phi(a) = \int_a^b \phi'(z)\mathrm{d}z$.

By assumption, we have $\phi_i(0) = 0$. Observe that

$$\begin{aligned}\mathbb{E}[V_{(I-1)} \mid V_{(I-1)} \leq v_i] &= \frac{\int_0^{v_i} z 1_{[z \leq v_i]} \mathrm{d}\mathbb{P}(V_{(I-1)} \leq z)}{\mathbb{P}(V_{(I-1)} \leq v_i)} \\ &= \frac{\int_0^{v_i} z \mathrm{d}\mathbb{P}(V_{(I-1)} \leq z)}{\mathbb{P}(V_{(I-1)} \leq v_i)} \\ &= \frac{\int_0^{v_i} z G'(z) \mathrm{d}z}{\mathbb{P}(V_{(I-1)} \leq v_i)} = \frac{\int_0^{v_i} z G'(z) \mathrm{d}z}{G(v_i)}.\end{aligned}$$

Finally, we obtain

$$\phi_i(v_i) = G(v_i)\mathbb{E}[V_{(I-1)} \mid V_{(I-1)} \leq v_i],$$

which does not depend on the auction chosen. □

2 Reading material

- Chapter 7 of Fudenberg and Tirole.
- Chapter 6 of Myerson.
- Chapter 23 of Mas-Colell, Whinston, Green.