1   Beer game

Take-away messages from the beer game:

- Inventory management is hard: optimal inventory is a nontrivial value, neither too high nor too low.

- Making decision takes forever, especially with many decision-makers with different opinions.

- Relatively small changes in demand create big changes in orders. Costs can easily lead to bankruptcy.

- Orders are "quick," shipping delays complicate things.

In real-life, we have more complex scenarios:

- Multiple items,

- Fixed cost per order, discounts for order size, limits on order size,

- Uncertainty in demand, uncertainty in shipping delays, worker productivity,

- Pricing, advertising on retail side,

- Dependence of demand on inventory.

We will consider the problem of optimal decision-making for order sizes. First, we do it for the deterministic case, then later in the course, we consider the non-deterministic one (dealing with uncertainty).

2   Deterministic inventory management

In the beer game, if cost is the objective, then a nontrivial inventory level (neither zero, nor infinite) is required due to holding costs and backorder costs. The inventory level is a state of the system that evolves from one step to another in a Markovian fashion: depending on the previous inventory level and previous orders. It can be controled by the decision maker through orders.
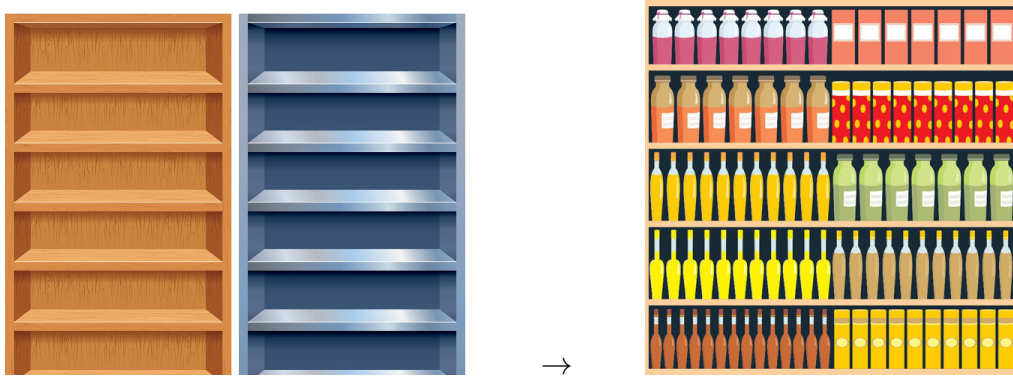
Figure 1: In the EOQ model, the optimal policy is to order a fixed amount of goods when the inventory drops to a fixed threshold. From `http://blog.omnipress.com/` and `http://freedesignfile.com/`.

2.1 EOQ model without backorder

The Economic order quantity model is a well-known model for inventory management (cf. Harris 1913). The goal is to size orders (decisions) so as to minimize the average cost over time. It has a number of assumptions:

- The demand is constant and given (known): λ units of inventory per day,

- The demand and inventory levels are real-numbers that evolve in continuous time,

- The cost of an order of size $x \in [0, \infty)$ is $K + cx$ dollars, where K and c are given constants,

- There is a given constant cost for holding inventory of h dollars per unit of inventory per day,

- There is zero lead time: all orders arrive instantaneously (however, there is a cost K per order),

- Backorders are not allowed: inventory must be non-negative at all time.

Observe that:

- It only makes sense to order when inventory is 0 (due to zero lead time).

Let Q denote the order size at time 0, which is the decision that we have to make. Suppose that the objective of our decision is to minimize the average cost per day (from time 0 to the next instant when inventory depletes). Suppose that the inventory level is Q at time 0, due to constant demand, the inventory level is 0 at time Q/λ. The

total cost from time 0 to time Q/λ is

$$K + cQ + \int_0^{Q/\lambda} h \underbrace{(Q - \lambda t)}_{\text{inventory level}} \mathrm{d}t = K + cQ + hQ^2/\lambda - h\lambda \int_0^{Q/\lambda} t\mathrm{d}t$$

$$= K + cQ + hQ^2/\lambda - h\lambda[t^2/2]_0^{Q/\lambda}$$

$$= K + cQ + hQ^2/\lambda - \frac{1}{2}hQ^2/\lambda.$$

By dividing the total cost by the time period Q/λ, we obtain the average cost:

$$g(Q) = \underbrace{\frac{K}{Q/\lambda}}_{\text{order overhead}} + \underbrace{c\lambda}_{\text{order cost}} + \underbrace{hQ/2}_{\text{holding}}.$$

Observe that Q/λ is the number of days required to deplete an inventory of Q at rate λ. The optimal value of Q can be found using calculus:

$$\frac{dg(Q)}{dQ} = -K\lambda/Q^2 + h/2,$$

$$\frac{d^2g(Q)}{d^2Q} = 2K\lambda/Q^3.$$

The second derivative is positive for all allowed values of Q, hence, we have a convex function g. Setting the first derivative to 0, the optimal (i.e., minimizing the average cost) order size is $Q^* = \sqrt{2K\lambda/h}$. We can also find Q^* by visual inspection.
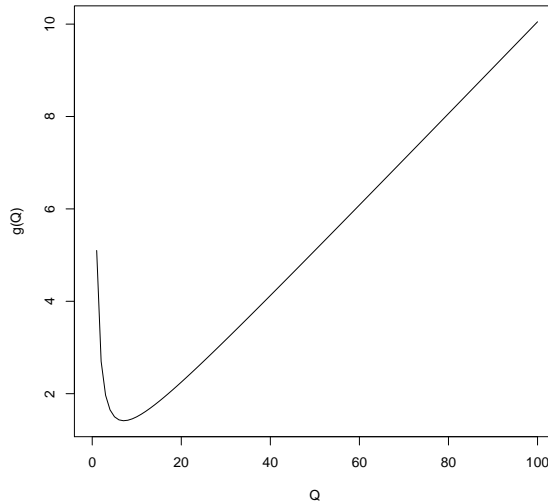

Figure 2: Cost per day function.

Remark 1. Quantity discounts: replace $c\lambda$ by $c(Q)\lambda$, where $c(Q)$ is the quantity dependent price per unit.
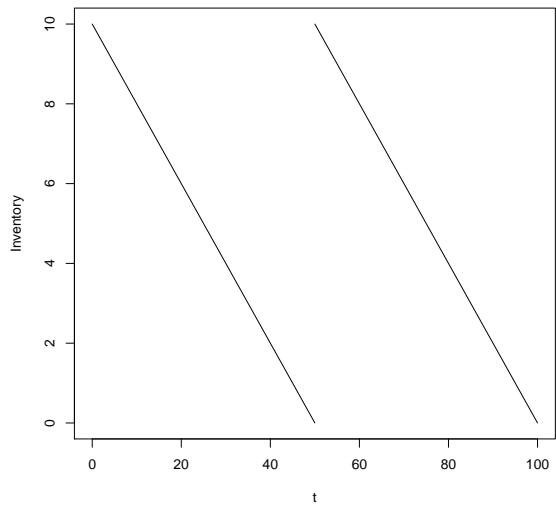
3

Figure 3: Optimal inventory level over time, with $Q^* = 10$.
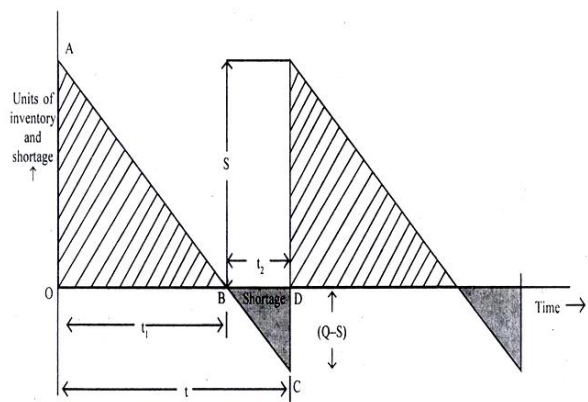
2.2 EOQ with backorders

Figure 4: EOQ with backorder.

Previously, we had a single decision: the quantity Q of each order. Now, we allow backorders, incurring a cost of p dollars per unit per day. With backorders, things are a little more complex. We need a second decision: the amount of backorder W before we reorder.

For presentation purpose, let us introduce variable x such that $Qx = W$, so that at time 0, the inventory level is $(1-x)Q$, and that we place an order for a quantity Q at time Q/λ, which corresponds to when the inventory has dropped to $-xQ$. Holding cost is incurred from time 0 until the inventory level drops to zero, which occurs at

4

time $(1-x)Q/\lambda$; hence, the total holding cost from time 0 to Q/λ is

$$\int_0^{(1-x)Q/\lambda} h\underbrace{\left((1-x)Q - \lambda t\right)}_{\text{inventory level}} dt = h\lambda \int_0^{(1-x)Q/\lambda} t dt$$

$$= h\lambda \frac{(1-x)^2 Q^2/\lambda^2}{2}.$$

Similarly, the total backorder cost from time 0 to Q/λ is

$$p\lambda \int_0^{xQ/\lambda} t dt = p\lambda \frac{x^2 Q^2/\lambda^2}{2}.$$

Consequently, the average cost per day is:

$$g(Q, x) = \frac{1}{2} \frac{Q(1-x)Q(1-x)/\lambda}{Q/\lambda} h + \frac{1}{2} \frac{QxQx/\lambda}{Q/\lambda} p + K\lambda/Q + c\lambda.$$

Take partial derivatives with respect to x and Q:

$$\frac{\partial}{\partial x} g(Q, x) = -hQ(1-x) + pQx,$$

$$\frac{\partial}{\partial Q} g(Q, x) = -h(1-x)^2/2 + px^2/2 - K\lambda/Q^2.$$

We find that the following optimal solution:

$$x^* = \frac{h}{h+p},$$

$$Q^* = \sqrt{\frac{2K\lambda(h+p)}{hp}},$$

$$W^* = \frac{h}{h+p}\sqrt{\frac{2K\lambda(h+p)}{hp}}.$$

This means that the optimal decision is to order Q^* units everytime that the inventory drops down to level $-W^*$ (i.e., a backorder of W^*).

Backorder can model a number of things:

- lost opportunity,

- lost of goodwill (brand image),

- penalty, excess fees for express delivery.

2.3 What more can be incorporated into EOQ?

- Lead time (easy: homework),

- Perishables, obsolescence,

- Uncertain demand. To model uncertainty, the most common tool is probability theory, which will be reviewed next lecture.

2.4 Newsboy problem

Figure 5: From `http://akvo.org/wp-content/uploads/2013/10/Extra.newsboy.jpg`.

Consider a perishable product (newspaper) with uncertain demand. How much inventory (copies of newspaper) to order at the start of each day (a single decision)? The decision is again nontrivial: newspapers cost money, and excess newspapers become worthless at the end of the day. This problem dates back to 1951 (Arrow, Harris, Marshak). We will use random variables to model the uncertainty.

Let h denote the cost per unit of excess inventory (at end of day). Let p be the backorder cost: the cost per unit of inventory shortfall (at end of day). Let s denote the decision: order size at start of the day. Let random variable D denote the random demand during the day. We assume for simplicity that the demand D is real-valued, as opposed to integer-valued, and that the probability distribution F of D has a density function, which we denote by f.

The vanilla version of the Newboy problem takes as objective the expected value of the cost at the end of the day. The (random) cost is the random variable

$$g(s, D) = h \max(s - D, 0) + p \max(D - s, 0).$$

The expected cost $\mathbb{E}g(s, D)$ can be computed analytically or plotted on computer for different values of s. More on this next week.

3 Reading material

Books:

- FOSCT: Chapter 3 (EOQ), Chapter 4.4.2 (Newsboy).