

## 5: Linear optimization

All the problems considered so far require only one decision (e.g., one order size) at a time. Let's look at decision problems involving multiple quantities at the same time.

These notes are based on the books on Linear Programming by Ferguson and by Vanderbei.

## 1 Resource allocation problem

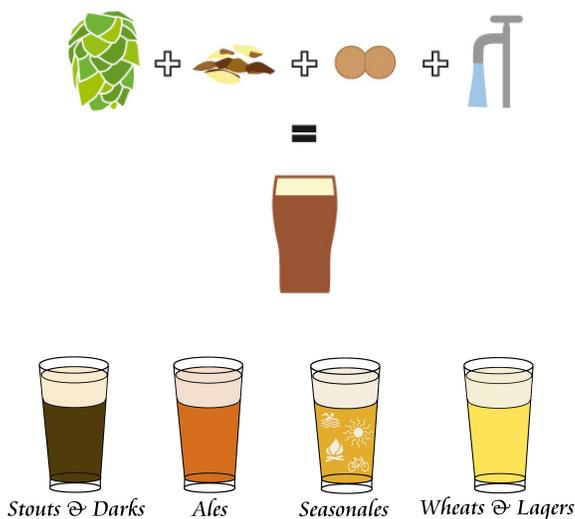


Figure 1: Types of resources and finished goods. From <http://www.danielle-adam.com/> and <http://homebrewmanual.com/>.

Consider a factory producing a number  $n$  of different goods (say goods  $1, \dots, n$ ). These goods use  $m$  different raw resources (say resources  $1, \dots, m$ ). Suppose that the decision maker observes that the amounts of raw resources are  $b_1, \dots, b_m$ . Each unit of resource  $i$  costs  $\rho_i$ .

Producing each unit of good  $j$  requires  $a_{1j}$  units of resource 1,  $a_{2j}$  units of resource 2,  $\dots$ , and  $a_{mj}$  units of resource  $m$ . Finally, each unit of good  $j$  can be sold for  $\sigma_j$  dollars. Consequently, the profit for each unit of good  $j$  produced is

$$c_j \triangleq \sigma_j - \sum_{i=1}^m \rho_i a_{ij}, \quad \text{for } j = 1, \dots, n.$$

As the operator of this factory, the decision maker, you must decide how many units of each good to produce in order to maximize total profit.

The standard approach to solving such decision problems involving many decision variables is through optimization. The simplest version of optimization problems involve linear functions, as is the case for the above resource allocation problem. The solution approach to such problem is called linear programming.

## 2 Standard form of linear program

To use linear programming, we first describe our problem with a number of linear functions. The inputs are the values  $a, b, c$ , the decisions or outputs are the  $n$  quantities  $x_1, \dots, x_n$  of each good to produce.

The first linear function describes the objective of the decision maker:

$$\sum_{j=1}^n c_j x_j,$$

which is the total profit. Next, we have  $m$  linear functions describing the amounts of resources required:

$$\sum_{j=1}^n a_{ij} x_j, \quad \text{for } i = 1, \dots, m.$$

The standard form of presenting the problem to solve is

$$\begin{aligned} & \max_{x_1, \dots, x_n} \sum_{j=1}^n c_j x_j \\ & \text{subject to } \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \text{for } i = 1, \dots, m, \\ & \quad \quad \quad x_j \geq 0, \quad \text{for } i = 1, \dots, n. \end{aligned}$$

We call  $x_1, \dots, x_n$  decision variables,  $\sum_{j=1}^n c_j x_j$  the objective function, and  $\sum_{j=1}^n a_{ij} x_j \leq b_i$  constraints. Observe that we can include equality constraints by introducing two inequality constraints.

*Remark 1* (Comparison with other problems). In contrast to problems modeled as Markov decision problems, which required sequential decision  $(a_1, \dots, a_T)$ , linear program require multiple decisions  $(x_1, \dots, x_N)$  at the same time. How does it compare with EOQ and Newsboy?

## 3 Optimal Transportation problem

Optimal transportation problems abstract many problems: match supply and demand, match married couples, building highways, etc. Their study dates back to Gaspard Monge's "Mémoire sur les déblais et les remblais."

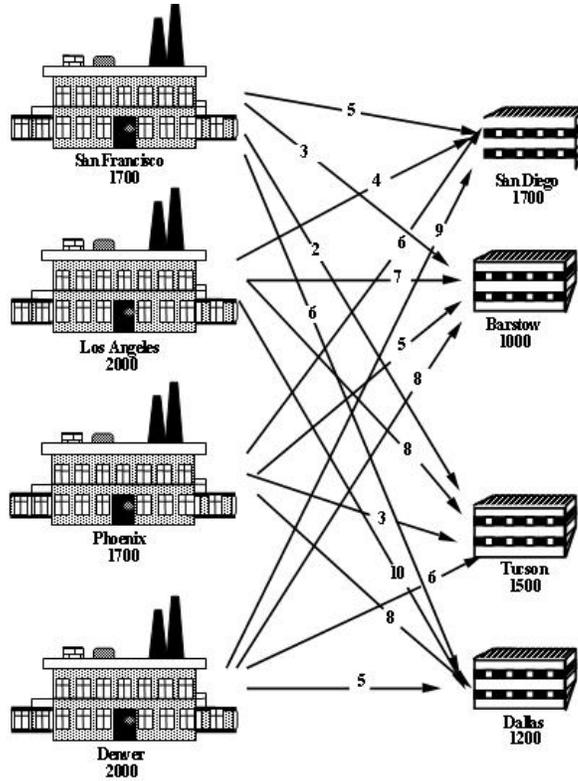


Figure 2: From <http://twiki.esc.auckland.ac.nz/>.

Consider  $I$  warehouses and  $J$  stores. Each warehouse  $i$  has an inventory  $s_i$  of a good (beer). Each store  $j$  has ordered a quantity  $r_j$  of this good. Let  $b_{ij}$  denote the cost of transporting each unit of the good from warehouse  $i$  to store  $j$ . How do we meet the orders from the stores with the minimum transportation cost?

Let  $x_{ij}$  denotes the quantity of good shipped from warehouse  $i$  to store  $j$ . The decision variables are  $\{x_{ij}\}$ .

The objective function is

$$\sum_{i=1}^I \sum_{j=1}^J b_{ij} x_{ij}.$$

The constraints come from inventory levels and order sizes. The inventory constraints are

$$\sum_{j=1}^J x_{ij} \leq s_i, \quad \text{for } i = 1, \dots, I.$$

The order constraints are

$$\sum_{i=1}^I x_{ij} \leq r_j, \quad \text{for } j = 1, \dots, J.$$

There are also non-negative constraints  $x_{ij} \geq 0$  for all  $i$  and  $j$ .

### 3.1 Solving LPs

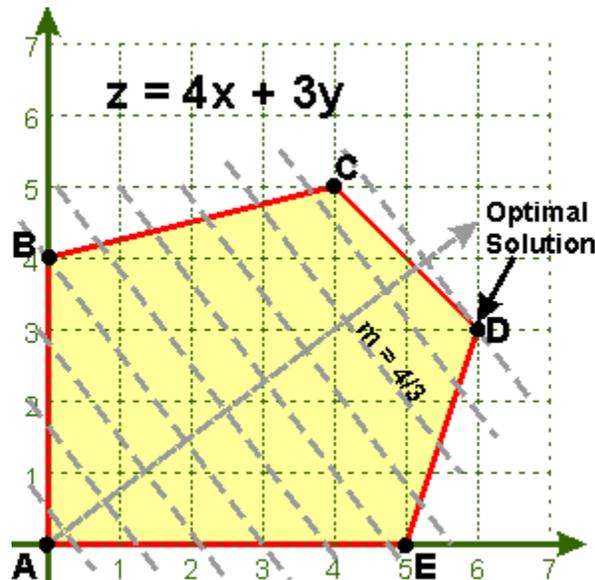


Figure 3: From <https://people.richland.edu/james/lecture/m116/systems/linear.html>

In previous lectures, we solved optimization problems with a single or two decision variables using either calculus or visual inspection of a plotted function. These approaches become more difficult when there are more variables, as in the case of linear programs. Solving linear programs is beyond the scope of the course. We will solve one toy problem in class and rely on software packages for bigger problems.

How many solutions can there be?

- None, when the feasible set is empty, or when unbounded feasible set;
- One unique (vertex of the feasible set);
- Infinitely many (an edge of the feasible set).

We can also solve it by hand (cf. <http://mat.gsia.cmu.edu/classes/QUANT/NOTES/chap7.pdf> for example). There are numerous solvers (software packages) for linear programs (e.g., <http://www.phpsimplex.com/simplex/simplex.htm?l=en>). These solvers use methods such as the simplex algorithm, interior-point algorithms, etc.

### 3.2 How long does it take to find the optimal solution?

Simplex-based algorithms can take exponential time, interior-point methods take polynomial time (of the order of  $O(n^4)$ ). In general, integer linear programs (with integer-valued decisions) take exponential time in the worst case (there are no known efficient algorithm right now).

An example of integer linear program is the Uncapacitated Facility Location Problem. Inputs:

- Set of customers  $I$
- Set of facilities  $J$
- Fixed cost  $f_j$  for opening each facility  $j \in J$
- Service cost  $c_{ij}$  for each pair  $i \in I$  and  $j \in J$  (e.g., the service cost can be proportional to distance)

Outputs (decisions):

- Whether facility  $j$  is open:  $x_j$ .
- Whether facility  $j$  serves customer  $i$ :  $y_{ij}$ .

The objective is to minimize total cost, subject to constraints that every customer is served and that only opened facilities can offer service.

$$\begin{aligned} \min_{\{x,y\}} \quad & \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} c_{ij} y_{ij} \\ \text{subject to} \quad & \sum_{j \in J} y_{ij} = 1, \quad \text{for } i \in I, \\ & y_{ij} \leq x_j, \quad \text{for } i \in I \text{ and } j \in J, \\ & y_{ij} \in \{0, 1\}, \quad \text{for } i \in I \text{ and } j \in J, \\ & x_j \in \{0, 1\}, \quad \text{for } j \in J. \end{aligned}$$

An approximate solution can be obtained by replacing the integer constraints with non-negative constraints and rounding the solution (cf. Section 3.5 of <http://www.or.uni-bonn.de/~vygen/files/fl.pdf>).

### 3.3 Other applications of LP

- Markov decision problems can be formulated as linear programs (cf. Puterman Chapter 6.9 for discounted infinite-horizon MDP).
- Risk management.
- Scheduling shifts for airline crews, nurses, etc.

## 4 Reading material

- Linear Programming (Vanderbei): Chapter 2: solving linear programming problems by the simplex method.
- Linear Programming (Ferguson): Chapter 1: examples.