6.5 Malus's law Find the angle between the transmission axes of two polarizers for the transmitted light through both to be 50%, and 25%.

Solution
Malus's law for the transmitted intensity is

\[ I(\theta) = I(0) \cos^2 \theta \]

so that \( \cos \theta = \left[ \frac{I(\theta)}{I(0)} \right]^{1/2} = (0.5)^{1/2} \)

solving \( \theta = 45^\circ \)

Similarly, \( \cos \theta = \left[ \frac{I(\theta)}{I(0)} \right]^{1/2} = (0.25)^{1/2} \)

solving \( \theta = 60^\circ \)

6.8 Wave plates Calculate and compare the thickness of quarter-wave plates made from calcite, quartz and LiNbO₃ crystals all operating at a wavelength of \( \lambda \approx 590 \) nm. What is your conclusion? Assuming little relative change in the indices, what are the thicknesses at double the wavelength?

Solution
Quarter-wavelength retardation is a phase difference of \( \pi/2 \) so that

\[ \phi = \frac{2\pi}{\lambda} |n_e - n_o| L = \frac{\pi}{2} \]

giving,

\[ L = \frac{\frac{1}{4} \lambda}{|n_e - n_o|} \]

Using \( n_o \) and \( n_e \) values from Table 6.1 in the textbook,

Quartz \[ L = \frac{\frac{1}{4} \lambda}{|n_e - n_o|} = \frac{\frac{1}{4} (590 \times 10^{-9} \text{ m})}{|1.5533 - 1.5442|} = 16.2 \text{ \mu m.} \]

Calcite \[ L = \frac{\frac{1}{4} \lambda}{|n_e - n_o|} = \frac{\frac{1}{4} (590 \times 10^{-9} \text{ m})}{|1.486 - 1.658|} = 0.86 \text{ \mu m.} \]

LiNbO₃ \[ L = \frac{\frac{1}{4} \lambda}{|n_e - n_o|} = \frac{\frac{1}{4} (590 \times 10^{-9} \text{ m})}{|2.29 - 2.26|} = 1.6 \text{ \mu m.} \]

At double the wavelength, simply double the above calculated thicknesses; and the thickness for calcite and LiNbO₃ become a little more manageable.
6.11 Glan-Foucault prism Error! Reference source not found. shows the cross section of a Glan-Foucault prism which is made of two right angle calcite prisms with a prism angle of $38.5^\circ$. Both have their optic axes parallel to each other and to the block faces as in the figure. Explain the operation of the prisms and show that the o-wave does indeed experience total internal reflection.

![Diagram of Glan-Foucault prism](image_url)

Error! Reference source not found. The Glan-Foucault prism provides linearly polarized light

**Solution**

As shown in Error! Reference source not found., the light in the left prism travel as o- and e-waves (with field $E_o$ and $E_e$, with refractive indices $n_o = 1.658$ and $n_e = 1.486$). The critical angles for TIR for the o- and e-waves at the calcite/air interface are

\[
\theta_o(\text{o-wave}) = \arcsin(1/n_o) = 37.1^\circ.
\]

and

\[
\theta_e(\text{e-wave}) = \arcsin(1/n_e) = 42.3^\circ.
\]

If the angle of incidence is $\theta$ at the calcite/air interface then from Figure 6.45,

\[
90^\circ + (90^\circ - \theta) + 38.5^\circ = 180^\circ
\]

or

\[
\theta = 38.5^\circ > \theta_o(\text{o-wave})
\]

but

\[
\theta < \theta_e(\text{e-wave})
\]

Thus, the o-wave suffers TIR while the e-wave does not. Hence the beam that emerges is the e-wave, with a field $E_e$ along the optic axis.

![Diagram of Glan-Foucault prism](image_url)

**Figure 6.45** The Glan-Foucault prism provides linearly polarized light
6.12 Optical activity

(a) Consider an optically active medium. The experimenter $A$ (Alan) sends a vertically polarized light into the this medium as in Error! Reference source not found.. The light that emerges from the back of the crystal is received by an experimenter $B$ (Barbara). $B$ observes that the optical field $\mathbf{E}$ has been rotated to $\mathbf{E}'$ counterclockwise. She reflects the wave back into the medium so that $A$ can receive it. Describe the observations of $A$ and $B$. What is your conclusion?

(b) Error! Reference source not found. shows a simplified version of the Fresnel prism that converts an incoming unpolarized light into two divergent beams that have opposite circular polarizations. Explain the principle of operation.

![Diagram](image)

Figure 6.17 An optically active material such as quartz rotates the plane of polarization of the incident wave. The optical field $\mathbf{E}$ rotated to $\mathbf{E}'$. If we reflect the wave back into the material, $\mathbf{E}'$ rotates back to $\mathbf{E}$.

![Diagram](image)

Figure 6.45 The Fresnel prism for separating unpolarized light into two divergent beams with opposite circular polarizations ($R = right, L = left$; divergence is exaggerated)
Solution

(a)

Figure 6Q12-1 What is observed is optical activity in the medium

The medium rotates \( E \) to \( E' \). A, sender, sees his initial \( E \) rotated clockwise through medium to become \( E' \). B, receiver, sees \( E \) rotated counterclockwise. \( B \) reflects the wave and becomes the sender. \( A \) is now the receiver and sees \( E' \) rotated counterclockwise to \( E \). \( B \) is now the sender and sees \( E' \) rotated clockwise. If \( B \) magically runs around and receives it, she will again see a counterclockwise rotation. See Figure 7Q12-1.

For a stationary observer (looking along the same direction), the sense of rotation changes with the direction of light. For an observer who always receives the light, the sense of rotation is either always clockwise or counterclockwise. Hence, the convention in assigning dextrorotatory and levorotatory forms of optical activity in terms of a receiver. This is called reciprocal rotation.

(b) Light that enters the right-handed quartz prism (R-prism) can be thought of as two different circularly polarized waves, right circular polarization (R-polarization) and left circular polarization (L-polarization). These travel with different velocities and experience different refractive indices \( n_R \) and \( n_L \), where \( n_R < n_L \) so that R-polarization propagates faster. In the left-handed quartz prism (L-prism), however, the indices are reversed \( n_R > n_L \) as L-polarization moves faster in L-quartz. Thus, R-polarization wave at the interface experience an increase in \( n_R \) and L-polarized wave experiences a decrease in \( n_L \). Therefore, they have different angles of refraction and hence enter the L-prism diverging from each other.

6.16 Transverse Pockels cell with LiNbO₃ Suppose that instead of the configuration in Figure 6.24, the field is applied along the z-axis of the crystal, the light propagates along the y-axis. The z-axis is the polarization of the ordinary wave and x-axis that of the extraordinary wave. Light propagates through as \( o \)- and \( e \)-waves. Given that \( E_o = V/d \), where \( d \) the crystal length along \( z \), the indices are

\[
n_o' = n_o + \frac{1}{2} n_{o3}^3 E_o \quad \text{and} \quad n_e' = n_e + \frac{1}{2} n_{e3}^3 E_o
\]

Show that the phase difference between the \( o \)- and \( e \)-waves emerging from the crystal is

\[
\Delta \phi = \phi_o - \phi_e = \frac{2\pi L}{\lambda} (n_e - n_o) + \frac{2\pi L}{\lambda} \left( \frac{1}{2} (n_o n_{o3}^3 - n_e n_{e3}^3) \right) \frac{V}{d}
\]

where \( L \) is the crystal length along the y-axis.
Explain the first and second terms. How would you use two such Pockels cells to cancel the first terms in the total phase shift for the two cells.

If the light beam entering the crystal is linearly polarized in the $z$-direction, show that

$$\Delta \phi = \frac{2\pi n_e L}{\lambda} + \frac{2\pi L}{\lambda} \left( \frac{n_e^3 r_{33}}{2} \right) \frac{V}{d}$$

Consider a nearly monochromatic light beam of the free-space wavelength $\lambda = 633$ nm and polarization along $z$-axis. Calculate the voltage $V$ needed to change the output phase $\Delta \phi$ by $\pi$ given a LiNbO$_3$ crystal with $d/L = 0.01$ (see table 6.2).

**Solution**

Consider the phase change between the two electric field components,

$$\Delta \phi = \phi_e - \phi_o = \frac{2\pi L}{\lambda} (n_e - n_o) + \frac{2\pi L}{\lambda} \left( \frac{n_e^3 r_{33}}{2} - n_o^3 r_{33} \right) \frac{V}{d}$$

The first term is the *natural* birefringence of the crystal, just as in the calcite crystal, and occurs all the time, even without an applied field. The second term is the Pockels effect, that is, the applied field induces a change in the refractive indices. Figure 6Q16-1 shows how two Pockels cells may be used to cancel the first terms in the combined system.

![Figure 6Q16-1](image)

Figure 6Q16-1 Two transverse Pockels cell phase modulators together cancel the natural birefringence in each crystal.

If the light beam is linearly polarized with its field parallel to $z$, we only need to consider the extraordinary ray, thus we can set $\phi_o = 0$.

$$\Delta \phi = \phi_e = \frac{2\pi L n_e}{\lambda} + \frac{2\pi L}{\lambda} \left( \frac{n_e^3 r_{33}}{2} \right) \frac{V}{d}$$

The first term does not depend on the voltage. The voltage $V$ that changes the output phase by $\pi$ is

$$\frac{2\pi L}{\lambda} \left( \frac{n_e^3 r_{33}}{2} \right) \frac{V}{d} = \pi$$

From table 6.2, $n_e = 2.000$, $r_{33} = 30.9 \times 10^{-12}$ m V$^{-1}$, and taking $d/L = 0.01$, $\lambda = 633$ nm,

or

$$V = \frac{d}{L} \left( \frac{\lambda}{n_e^3 r_{33}} \right) = (0.01) \left( \frac{633 \times 10^{-9} \text{ m}}{(2.000)^3 (30.9 \times 10^{-12} \text{ m \ V}^{-1})} \right)$$

$$V = 19.2 \text{ V}$$
6.25 **Faraday effect** Consider using a ZnTe crystal to rotate the optical field of the 633 nm polarized laser beam from a HeNe laser. If the crystal length is 1 cm, and the magnetic field is 0.8 T, what is the rotation of the optical field?

**Solution**

Use \[ \theta = VBL = [(188 \text{ rad m}^{-1}) (0.8 \text{ T}) (0.01 \text{ m})] = 1.50 \text{ rad or } 86^\circ \]