Department of Electrical & Computer Engineering

COEN 212:
DIGITAL SYSTEMS DESIGN
Lecture 2: Boolean Algebra

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Lecture 2:
Objectives of this lecture

- Definition of Boolean Algebra
- Properties of Boolean Algebra
- Boolean representation of Digital Circuits

Lecture 2:
Reading for this lecture

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- Chapter 2 (sections 2.1 to 2.5)

of Digital Design by M. Morris R. Mano and Michael D. Ciletti, 6th Edition, Pearson, 2018.

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Lecture 2: Axiom's of Boolean Algebra

Boolean Algebra is an algebraic structure defined by a set of elements B and two operations + and . with the following properties:

1) The structure is closed under + and ., i.e., if $x \epsilon B$ and $y \epsilon B$ then $x + y \epsilon B$ and $x.y \epsilon B$.

2) Identity element for +, $x + 0 = 0 + x = x$ for all $x \epsilon B$.

Identity element for ., say 1, $x.1 = 1.x = x$ for all $x \epsilon B$.

3) Commutativity for both + and ., i.e.,

$$x + y = y + x \text{ and } x.y = y.x.$$

4) Distributivity: $x.(y + z) = x.y + x.z$ and $x + y.z = (x + y).(x + z)$

5) For every element $x \epsilon B$ there is an element $x' \epsilon B$ such that:

a) $x + x' = 1$ and b) $x.x' = 0$

6) B has at least two distinct elements $x, y \epsilon B$, $x \neq y$.

Lecture 2:

Two-valued Boolean Algebra:

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- Let' $\mathrm{B} = \{0,1\}$
- Define operations:

x	y	$x.y$
0	0	0
0	1	0
1	0	0
1	1	1

- and

x	y	$x+y$
0	0	0
0	1	1
1	0	1
1	1	1

- In the next two slides, we verify the validity of the Boolean algebra for our proposed structure.

Lecture 2:

Two-valued Boolean Algebra:

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- Axiom 1: Since 0+1=1+0=1+1=1 and 0+0=0, B is closed under +

Since 0.0=1.0=0.1=10 and 1.1+0=1, B is closed under .

- Axiom 2: a) 0+0=0 and 0+1=1+0=1 and b) 1.1=1 and 1.0=0.1=0.

So, 0 is the identity for + and 1 is the identity for .

- Axiom 3: 0.1=1.0=0, 1.1=1, 0.0=0 and 0+1=1+0=1,1+1=1, 0+0=0
- Axiom 4 a:

x y z	$y+z$	$x.(y+z)$	$x.y$	$x.z$	$x.y+x.z$
0 0 0	0	0	0	0	0
0 0 1	1	0	0	0	0
0 1 0	1	0	0	0	0
0 1 1	1	0	0	0	0
1 0 0	0	0	0	0	0
1 0 1	1	1	0	1	1
1 1 0	1	1	1	0	1
1 1 1	1	1	1	1	1

So: $x.(y+z) = x.y + x.z$

Lecture 2:
Two-valued Boolean Algebra:

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Axiom 4 b:

x y z	$y.z$	$x+y.z$	$x+y$	$x+z$	$(x+y).(x+z)$
0 0 0	0	0	0	0	0
0 0 1	0	0	0	1	0
0 1 0	0	0	1	0	0
0 1 1	1	1	1	1	1
1 0 0	0	1	1	1	1
1 0 1	0	1	1	1	1
1 1 0	0	1	1	1	1
1 1 1	1	1	1	1	1

So: $x+y.z=(x+y).(x+z)$

- Axiom 5: if $x=0$, then $x+1=1$ and $x.1=0$

Similarly if $x=1$, then $x+0=1$ and $x.0=0$.

- So 1 is complement of 0 and 0 is complement of 1.

x	x'
0	1
1	0

- Axiom 6: by definition B has two distinct elements.

Lecture 2:

Properties of Boolean Algebra: Theorem 1

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Theorem 1: a) $x + x = x$ and b) $x.x = x$

To prove a:

$x + x = (x + x).1$	Axiom 2
$=(x + x).(x + x')$	Axiom 5
$= x + x.x'$	Axiom 4
$= x + 0$	Axiom 2
$= x$	

To prove b:

$x.x = x.x + 0$	Axiom 2
$=x.x + x.x'$	Axiom 5
$= x.(x + x')$	Axiom 4
$= x.1$	Axiom 2
$= x$	

Lecture 2:
Properties of Boolean Algebra: Theorem 2

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Theorem 2: a) $x + 1 = 1$ and b) $x.0 = 0$

To prove a:

$$x + 1 = 1.(x + 1) = (x + x').(x + 1) = x + x'.1$$
$$= x + x' = 1$$

To prove b:

$$x.0 = 0 + x.0 = x.x' + x.0 = x.(x' + 0)$$
$$= x.x' = 0$$

Proof of b using De Morgan Duality:

Start with $x' + 1 = 1$ and change any entity to its inverse (complement) and every + to . and vice versa to get $x.0 = 0$.

Lecture 2:
Properties of Boolean Algebra: Theorems 3 and 4

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Theorem 3: $(x')' = x$

Simply put: Inverting twice is like doing nothing.

Proof:

Axiom 5 states that: $x + x' = 1$ and $x.x' = 0$.

Using Commutativity (Axiom 3)

$x' + x = 1$ and $x'.x = 0$.

So : x is the inverse of x'.

Theorem 4: a)$x + xy = x$, b)$x.(x + y) = x$

Proof: a) $x + xy = x(1 + y) = x.1 = x$.

b)$x.(x + y) = x.x + x.y = x + x.y = x$.

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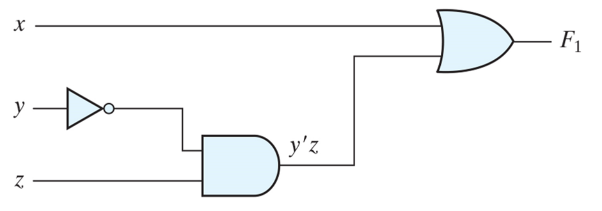
- A Boolean function is an expression containing binary variables, constants and logical operations.
- Example: $F_1 = x + y'z$
- Implementation of F_1:



- Truth table of F_1:

x	y	z	F_1
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

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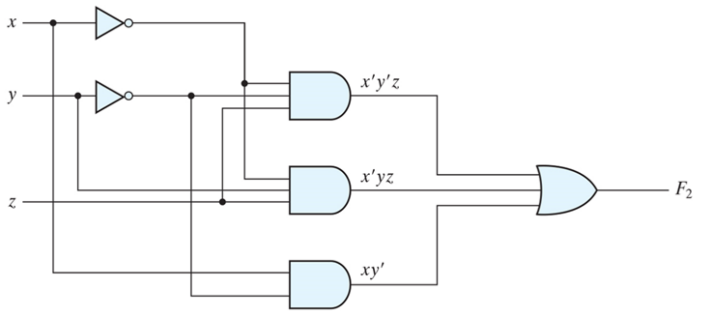
- Another Example: $F_2 = x'y'z + x'yz + xy'$
- Direct implementation of F_2:



- Truth table for F_2:

x	y	z	F_2
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Lecture 2:
Boolean Function minimization

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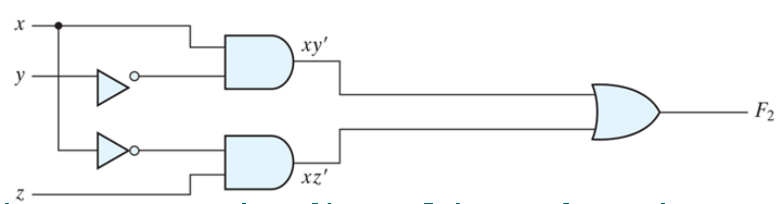
- We can write F_2 as

$$F_2 = x'y'z + x'yz + xy' = x'z(y' + y) + xy' = x'z + xy'$$

- This leads to another implementation with less gates:



- Comparison of the complexity of two implementation:

	AND	OR	NOT
Direct Implementation	5	2	2
Minimized Complexity	2	1	2

- Having less gates means, lower cost, lower delay, lower power consumption, more reliability.
- Soon, we will learn techniques for minimizing the gate count.

Lecture 2:
Complement of Boolean functions

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- Complement of a function F denoted as F' is a function that produces a 1 when F produces a 0 and produces a zero when F's output is one.
- We use De Morgan's theorem to show that

$$(A + B + C + \cdots + F)' = A'B'C' \ldots F'$$

and $$(ABC \ldots F)' = A' + B' + C' + \cdots + F'.$$

- Using these generalized De Morgan formulas, we can find the complement of any Boolean function.
- **Example:** find the complement of $F = xy' + x'y$.

$$F' = (x' + y)(x + y') = x'x + x'y' + xy + yy'$$ or

since $x'x = yy' = 0$, we get:

$$F' = xy + x'y'.$$

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- **Example 2:** find the complement of $F = x(y'z' + yz)$.

$$F' = [x(y'z' + yz)]' = x' + (y'z' + yz)' = x' + (y + z)(y' + z')$$
$$= x' + yy' + yz' + zy' + zz' = x' + yz' + zy'.$$

Example 3: find the complement of $F = (x' + y + z')(x + y')(x + z)$.

$$F' = [(x' + y + z')(x + y')(x + z)]'$$
$$= (x' + y + z')' + (x + y')' + (x + z)'$$
$$= xy'z + x'y + x'z'$$

Lecture 2:
Knowledge Check

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- **Question 1:** Implement the function for the following truth table:

x	y	z
0	0	1
0	1	1
1	0	1
1	1	0

- **Question 2:** Complement of $F = x'y + y'x$ is:

a) $F' = (x'+y')(x+y)$ b) $F' = xy + x'y'$ c) $F' = x'y + xy$ b) None

Question 3: Derive the truth table of:

$$F = x'y'z + xy'z' + x'yz' + xyz$$