Lecture 1: Jan. 3, 2007

Representation of numbers in different radix (basis).

The set of integers and real numbers contain an infinite number of elements.

However, we are able to represent any real number using a finite number of symbols (digits).

We are all familiar with decimal numbers.

We use digits 1, 2, ---, 9 as well as 0 to show any number. For example 35987 is represented as

$$30000 + 5000 + 900 + 80 + 7$$

$$= 3 \times 10^4 + 5 \times 10^3 + 9 \times 10^2 + 8 \times 10^1 + 7 \times 10^0$$

and 275.69 is represented as

$$200 + 70 + 5 + \frac{6}{10} + \frac{9}{100}$$

$$= 2 \times 10^2 + 7 \times 10^1 + 5 \times 10^0 + 6 \times 10^{-1} + 9 \times 10^{-2}$$

It is said that people have used base 10 because they have 10 fingers. The term <u>digit</u> means <u>finger</u>.

Now assume that like Simpsons we had 4 fingers on each hand. Then, most likely, we would have used base 8. That means that after counting 1, 2, 3, ..., 7 we had to move 1 digit up and represent 8 as 10.

In such a system (called octal), a number $a_3 a_2 a_1 a_0 . a_{-1} a_{-2} a_{-3}$ is represented as

$$8^3 a_3 + 8^2 a_2 + 8^1 a_1 + 8^0 a_0 + 8^{-1} a_{-1} + 8^{-2} a_{-2} + \cdots$$

For example 253 (in decimal) will be represented as $(375)_8$ since

$$3 * 8^2 + 7 \times 8^1 + 5 \times 8^0 = 253$$

to translate from decimal (base 10) to octal (base 8) we do the following

253		
31	5	dividing 253 by 8
3	7	dividing 31 by 8
0	3	dividing 3 by 8

read from bottom to top $\rightarrow (375)_8$

binary numbers: A representation usually used in digital systems is binary. It uses an alphabet with only two symbols, 0 and 1. A number can be represented in binary as

$$a_n * 2^n + a_{n-1} * 2^{n-1} + \cdots + a_1 * 2^1 + a_0 * 2^0 + a_{-1} * 2^{-1} + \cdots$$

For example $(75)_{10}$, i.e., 75 in our familiar decimal system will be represented as

75	
37	1
18	1
9	0
4	1
2	0
1	0
0	1

$(1001011)_2$

Binary to octal conversion, we can divide binary digits of a binary numbers into groups of 3 and represent each with their octal value, e.g.,

$$(1,001,011)_2 = (113)_8$$

$$\begin{array}{ccc} 1 & 001 & 011 \\ \downarrow & \downarrow & \downarrow \\ 1 & 1 & 3 \end{array}$$

We can check this by transforming $(75)_{10}$ directly into octal:

$$\begin{array}{r|l} 75 & \\ 9 & 3 \\ 1 & 1 \\ 0 & 1 \end{array} \quad \uparrow (113)_8$$

Hexa-decimal numbers:

This is a representation in base (radix) 16. Here, we use 1, 2, ..., 9, A, B, C, D, E and F. Where A mean 10, B is 11 ... and F is 15.

To change a binary number into hexa-decimal, we separate its bits into groups of 4 and replace each by its hex value.

Example:

$$\underbrace{10}_{2}\ \underbrace{1100}_{C}\ \underbrace{0110}_{6}\ \underbrace{1011}_{B}\ .\ \underbrace{1111}_{F}\ \underbrace{0010}_{2}$$

So, we get

$$(2C6B.F2)_{16}$$

Note: Start from the "decimal" (binary) point and choose groups of 4. If you end up with less than 4 bits in a group add zeros to make it four (without changing the value).

Complements: - Radix Complement
- Diminished radix Complement

If we have a number X in base r and we subtract it from r^n to get $r^n - X$. We call this r's complement of X.

For example 10' Complement of 3250 is

$$10^4 - 3250 = 6750$$

Note that, it can be formed by leaving the least significant zeros intact, subtract least significant digit from 10 and..

the rest from 9. Note that the above is equivalent to 999...96750, because having any number of zeros in before the first digit does not ~~make~~ change anything, i.e., 03250 is the same as 003250 and 3250.

Two's Complement of X (binary) is

$$2^n - X$$

For example for $X = 1101100$ we get

0010100

and 2's complement of 0110111 is

1001001

$(r-1)$'s complement of X is

$$r^n - 1 - X$$

note that $r^n - 1$ is an n digital ~~word~~ number with all digits equal to $r-1$.

For example 9's complement of 546700 is $999999 - 546700 = 453299$

and 9's complement of 012398 is

999999 − 012398 = 987601

For binary case 1's complement of a number is formed by subtracting each bit from 1.

1's complement of 1011000 is 0100111

1's complement of 0101101 is 1010010

Subtraction in radix complement

To form $M - N$

1) Add M to r's complement of N to get

$$M + r^n - N = r^n + M - N$$

2) if $M > N$ then there will be a carry which can be discarded.

3) if $M < N$, there would not be a carry and is equal to $r^n - (N - M)$ which is r's complement of $(N - M)$.

So take r's complement of the result and add a minus sign.

Example: Find X − Y and Y − X

if X = 1010100 and Y = 1000011

to find X − y

$$
\begin{array}{rl}
X = & 1010100 \\
 & 0111101 \leftarrow \text{2's complement of } Y \\
\hline
 & 1\,0010001
\end{array}
$$

Carry →

discard the carry to get $\boxed{0010001}$

for Y − X:

$$
\begin{array}{rl}
Y = & 1000011 \\
 & 0101010 \quad \text{2's complement of } X \\
\hline
 & 1101111
\end{array}
$$

There is no carry. So, find 2's complement

0010001

and add a minus sign to get −0010001

In r−1's complement, e.g., in 1's complement since the numbers are one less than r's (e.g., 2's) complement, we need to subtract one from the result when there is a carry. When there is no carry the r−1's complement takes care of the one.

Example:

Subtract X from y (and Y from X)

in 1's complement when:

X = 1100101 and Y = 1011010

for X − y

$$
\begin{array}{r}
X = 1100101 \\
0100101 \quad \text{1's complement of Y} \\
\hline
1000101 0
\end{array}
$$

Carry

Add the carry to the result to get:

0001011

for Y − X

$$
\begin{array}{r}
Y = 1011010 \\
0011010 \quad \text{1's complement of X} \\
\hline
1110100
\end{array}
$$

no carry ⇒ find one's complement of 1110100

as 0001011 and add a minus sign:

$$\boxed{-0001011}$$

Signed binary numbers

In previous examples, we indicated a negative number using a minus (−) sign the same way we do with negative real or integer numbers. However, in digital (binary) systems using an extra symbol for sign is not convenient. That is why usually the following convention is used: The most significant bit of a negative number is a 1 and the most significant bit of a positive number is 0.

Of course, we need to specify the word length in advance so that there is no confusion about the place of the sign bit.

For example 11100 can be either −12 or +28 depending whether we deal with unsigned 5-bit numbers of 5-bit signed numbers (1 bit sign and 4 bits magnitude)

There are three ways to represent signed binary numbers:

1- Signed magnitude:

1 sign bit + magnitude

e.g.,

11001 is -9

01001 is +9

2- 1's Complement representation

leave positive numbers as they are

take 1's Complement of negative ones.

Since taking complement changes the ~~sign~~ polarity of the most significan bit the ~~sign of the~~ most significant bit of negative numbers will be 1.

e.g. +9 01001

-9 10110

3- 2's complement representation

the same as above except take 2's Complement.

e.g. -9 is 10111