Lecture 2: Jan. 8, 2007

Basic building blocks of logical circuits:

A logical device, e.g., a computer, a digital phone, an IPOD, etc. consists of the following blocks:

1) Memory: to hold information prior, during and after processing.

Take as an example the simple ~~task of~~ task of adding two numbers X and Y. You need a memory (a <u>register</u>) to hold X another to hold Y and possibly a third one to store X+Y. Of course you may store the result in one of the two registers initially holding X or Y if you do not need one of these after addition.

2) A processing unit: An adder, a multiplier, etc. These processing units are built from basic

elements called logic gates. We will talk about these gates later in this lecture and much more in future lectures.

3 - A Control unit: This unit specifies the sequence of operations.

4 - Input and output units: These are used to input data to the device or output the result of processing. The ones with which you are quite familiar are keyboard, scanners, monitors, printers.

Logic Gates

In previous lecture, we saw that we can represent any number with only two symbols, i.e., transform it to a binary format. In this format any processing task can be broken down into operation on binary digits (bits). There are three basic operations for bit manipulation. These are AND, OR and NOT.

NOT operates on a single bit, i.e., it has one input. The output of a NOT is 1 if its input is 0 and vice versa.

Truth table for NOT

X	X'
0	1
1	0

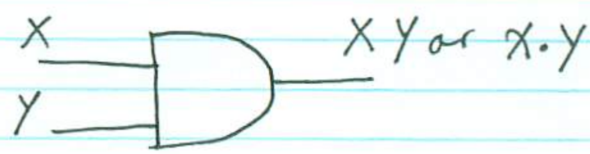
NOT is a unitary operation (or unitary gate) since it works on a single bit.

AND and OR are <u>binary</u> operations.

The output of an AND gate is 1 if both its inputs are 1 and <u>0</u> otherwise.

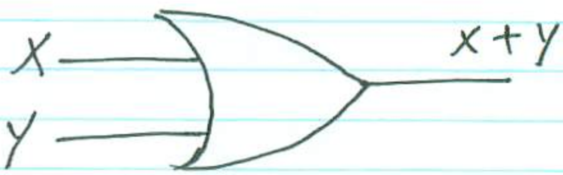
Truth Table for the AND gate:

X	Y	X•Y
0	0	0
0	1	0
1	0	0
1	1	1



The output of an OR gate is 0 if both its inputs are zero. Otherwise its output is 1. The truth table for OR gate is

x	y	x + y
0	0	0
0	1	1
1	0	1
1	1	1



De Morgan's Theorem

Note the similarity (duality) existing between the AND gate and the OR gate. If you exchange 0 and 1 in the truth table for 1 you get the other's truth table. That is if we negate the inputs two an OR gate (before feeding ~~it~~ them to it) and also invert the output we get the operation of an AND gate. That is:

$$(x' + y')' = xy$$

or $\boxed{(xy)' = x' + y'}$

Similarly,

$$(x'y')' = x+y$$

or

$$\boxed{(x+y)' = x'y'}$$

This is called De Morgan's Theorem.

This duality property makes the mathematics of binary systems (Boolean Algebra) different from the ordinary algebra.

Axiomatic Definition of Boolean Algebra

Boolean Algebra is an algebraic structure defined by a set of elements B and two operations + and • with the following properties:

1) a) The structure is closed under +, i.e.,

if $x \in B$ and $y \in B$ then $x+y \in B$

b) The structure is closed under •, i.e.,

if $x \in B$ and $y \in B$ then $x \cdot y \in B$

2) a) There is an identity element with respect to +, say, 0, i.e.

$x+0 = 0+x = x$ for all $x \in B$

b) There is an identity element, say 1, with respect to $\cdot$, i.e.

$$x \cdot 1 = 1 \cdot x = x$$

3) The structure is commutative with respect to both + and $\cdot$, i.e.,

$$x + y = y + x$$

and

$$x \cdot y = y \cdot x.$$

4) a) The operator $\cdot$ is distributive over +, i.e,

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

b) The operator + is distributive over $\cdot$, i.e.

$$x + y \cdot z = (x + y) \cdot (x + z)$$

please note that this is a major difference between the Boolean algebra and the regular algebra. That is in ordinary algebra addition is not distributive over multiplication.

5) For every element $x \in B$ there is an element $x' \in B$, called the complement of x such that a) $x + x' = 1$ and $x \cdot x' = 0$.

6) There exist at least two elements $x, y \in B$

such that $x \neq y$.

Two-valued Boolean Algebra.

Let $B = \{0, 1\}$, i.e., the set of elements of our algebra has only two elements.

Define the operations as

x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

from postulate 5 above we find that

if $x = 0$ then $x' = 1$

and if $x = 1$ then $x' = 0$

i.e.,

x	x'
0	1
1	0

Let's verify the validity of the Boolean algebra for our proposed structure:

1) Since $0+1=1+0=1+1=1$ and $0+0=0$

B is closed under $+$.

Since $0\cdot 0=0\cdot 1=1\cdot 0=0$ and $1\cdot 1=1$

then B is closed under $\cdot$.

2) a) $0+0=0$ and $0+1=1+0=1$

b) $1\cdot 1=1$ and $1\cdot 0=0\cdot 1=0$.

So, 0 is identity for $+$

and 1 is identity for $\cdot$

3) $0\cdot 1=1\cdot 0=0$, $1\cdot 1=1$, $0\cdot 0=0$

$0+1=1+0=1$, $1+1=1$, $0+0=0$

So Commutative laws with respect to $+$ and $\cdot$ hold.

4) a) distributivity of $\cdot$ over $+$:

x	y	z	$y+z$	$x\cdot(y+z)$	$x\cdot y$	$x\cdot z$	$x\cdot y+x\cdot z$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

b) Distributivity of + over $\cdot$:

x	y	z	$y \cdot z$	$x + y \cdot z$	$x + y$	$x + z$	$(x+y)\cdot(x+z)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

5)

a) $x + x' = 1$ since if $x = 0$, $x' = 1$

and $x + x' = 0 + 1 = 1$

if $x = 1$, then $x' = 0$ and $x + x' = 1 + 0 = 1$

b) $x \cdot x' = 0$ since for $x = 0$, $x' = 1$

and $0 \cdot 1 = 0$

for $x = 1$, $x' = 0 \Rightarrow x \cdot x' = 1 \cdot 0 = 0$

6) Since by design $1 \neq 0$.

Some Basic properties of Boolean Algebra

Theorem 1:

a) $x + x = x$

b) $x \cdot x = x$

proof

a) $x + x = (x + x) \cdot 1$ postulate 2

$= (x + x)(x + x')$ Postulate 5

$= x + xx'$ Postulate 4

$= x + 0$ Postulate 2

$= x$

b) $x \cdot x = x \cdot x + 0$ P2

$= x \cdot x + x \cdot x'$ P5

$= x \cdot (x + x')$ P4

$= x \cdot 1$ P5

$= x$

Theorem 2: a) $x + 1 = 1$

b) $x \cdot 0 = 0$

Proof: a) $x + 1 = 1 \cdot (x + 1) = (x + x') \cdot (x + 1)$

$= x + x' \cdot 1 = x + x' = 1.$

2)

b) $x \cdot 0 = 0$

The proof is by duality principle.

The duality principle states that any property remains unchanged if ~~$\cdot$~~ $\cdot$ and $+$ are interchanged and any 0 is changed to 1 and vice versa.

So

$$x + 1 = 1 \Rightarrow x \cdot 0 = 0$$

Theorem 3: $(x')' = x$

Proof: We have (postulate 5) that

$$x + x' = 1 \text{ and } x \cdot x' = 0$$

using Commutativity (postulate 3) we get

$$x' + x = 1 \text{ and } x' \cdot x = 0$$

i.e., x serves as the complement to x', so,

$$(x')' = x.$$

Theorem 4:

a) $x + xy = x$

b) $x(x+y) = x$

Boolean Functions:

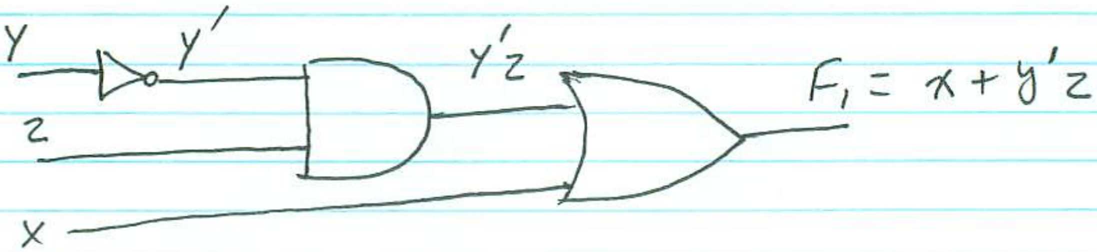
A Boolean function is an expression containing binary variables, Constants 0 and 1 and logic operations. For example:

$$F_1 = x + y'z$$

is a function of 3 variables x, y and z.

A Boolean function can be implemented using basic gates: AND, OR and NOT.

For example, F_1 can be implemented as:



A Boolean function can also be represented by a truth table. A truth table has 2^n rows, where n is the number of variables in the function. The truth Table for F_1 is:

X	Y	Z	F_1
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

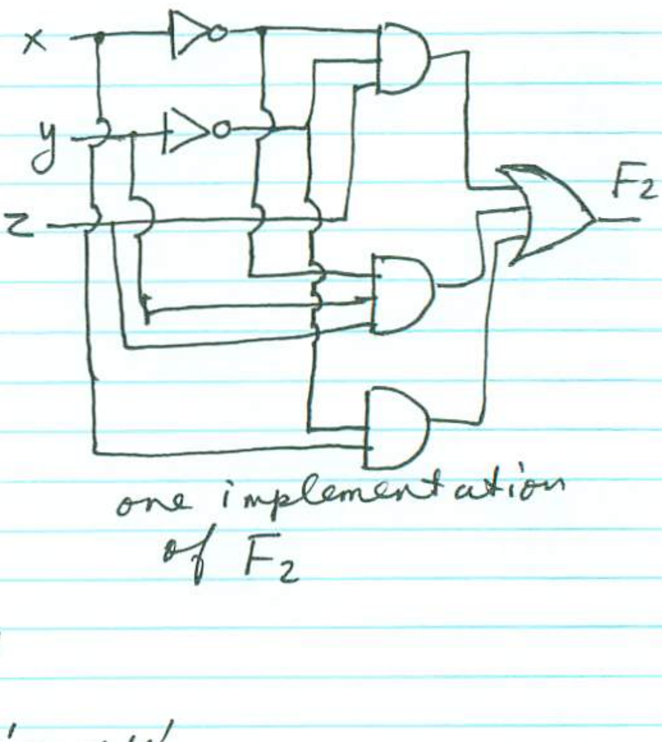
Each row contains one of the 2^n possibilities of the variables in the first n positions and a 1 or 0 in the last place showing the value of the function for the values of the variables. Each function may be represented by one truth table. However, it may have different implementations.

Take as an example:

$$F_2 = x'y'z + x'yz + xy'$$

It has the following truth table:

x	y	z	F_2
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

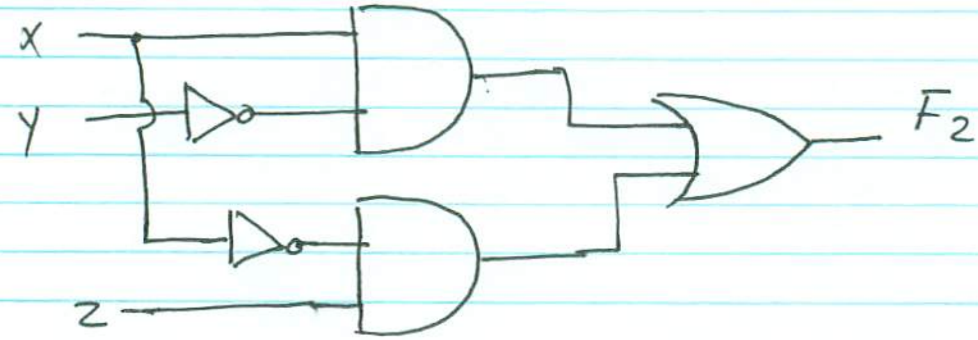


one implementation of F_2

We can write F_2 as

$$F_2 = x'y'z + x'yz + xy'$$
$$= x'z(y'+y) + xy' = x'z + xy'$$

So, we can have the following implementation which is less complex:



Complement of a Function

The complement of a function F is another function F' derived by changing each 0 to 1 and each 1 to 0.

We use DeMorgan's theorem to show that

$$(A+B+C+\cdots+F)' = A'B'C'\cdots F'$$

and

$$(ABCD\cdots F)' = A'+B'+C'+\cdots+F'$$

Using these generalized De Morgan formulas, we can find the complement of any Boolean function.

Example: Find the complement of

$$F = xy' + x'y$$

$$F' = (x'+y)(x+y') = x'x + x'y' + xy + yy'$$

or

$$F' = xy + x'y'$$

since $x'x = yy' = 0$

Example: Find the complement of

$$F = x(y'z' + yz)$$

$$F' = [x(y'z' + yz)]' = x' + (y'z' + yz)'$$

$$= x' + (y+z)(y'+z')$$

$$= x' + yy' + yz' + zy' + zz'$$

$$= x' + yz' + zy'$$

Example: Find the complement of

$$F = (x'+y+z')(x+y')(x+z)$$

$$F' = [(x'+y+z')(x+y')(x+z)]'$$

$$= (x'+y+z')' + (x+y')' + (x+z)'$$

$$= xy'z + x'y + x'z'$$