ELEC 342

Discrete - Time Signals and Systems

Lecture 1

In this lecture:

- Course outline: topics to be Covered, marking scheme, etc.
- Definition of Signals and Systems
- Classification of Signals.
- Classification of Systems
- Dirac Delta Function
- Impulse Response of Linear - Time - Iinvariant (LTI) Discrete - Time Signals.

Next Lecture:

- Convolution Sum.

Definition

A signal is a time-function representing the values of a measurable event over time.

Examples are: audio and video signals,
price of a commodity,
popularity of a statesman.

Classification of Signals:

1) • A signal may be continuous-time or Discrete-time.

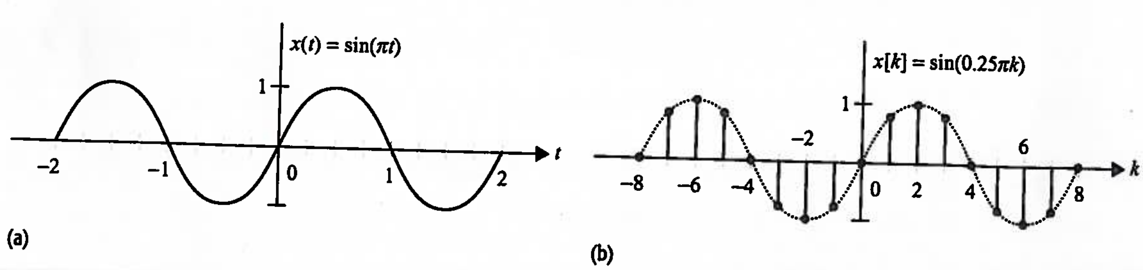
A Continuous-time signal has value for all times, e.g.,

$$x(t) = \sin(\pi t) \quad \text{all } t \in \mathbb{R}$$

A Discrete-Time Signal has value only at times $t_0, t_1, \dots, t_k, \dots$

usually these time instants are uniformally spaced, i.e., $t_k = kT$, e.g.,

$$x[kT] = \sin(0.25kT), \quad k = 0, \pm 1, \pm 2, \dots$$



(a) (b)

Note that a Discrete-Time (DT) Signal does not necessarily have to be generated by sampling a Continuous-Time (CT) Signal. A Signal can be discrete-time by itself.

2) Analog versus Digital:

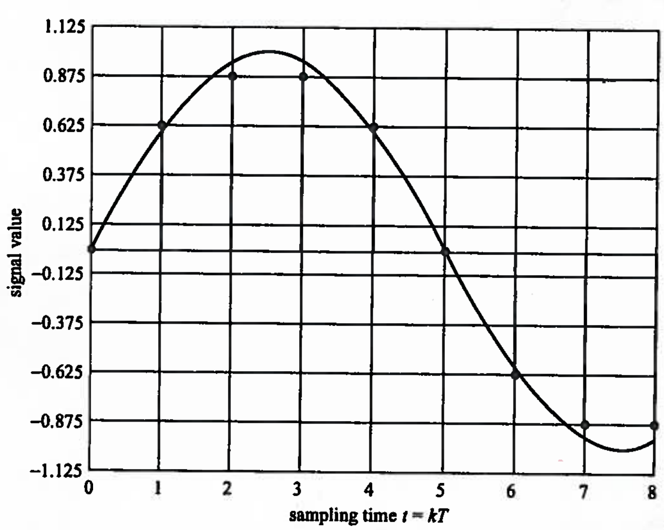
The property of being Discrete or Continuous time is related to sampling of time axis.

Being Digital or Analog relates to the value of the Signal.

If the amplitude of a signal is represented by a real number it is called analog.

If it is represented by an iteger, it is called Digital.

Fig. 1.5. Analog signal with its digital approximation. The waveform for the analog signal is shown with a line plot; the quantized digital approximation is shown with a stem plot.



3) A Signal may be Periodic or aperiodic

Definition: A DT Signal is Periodic if

$$x[k] = x[k+K_0]$$

K_0 is called the period of the signal.

For CT Signal

$$x(t) = x(t+T_0)$$

is the condition for being periodic.

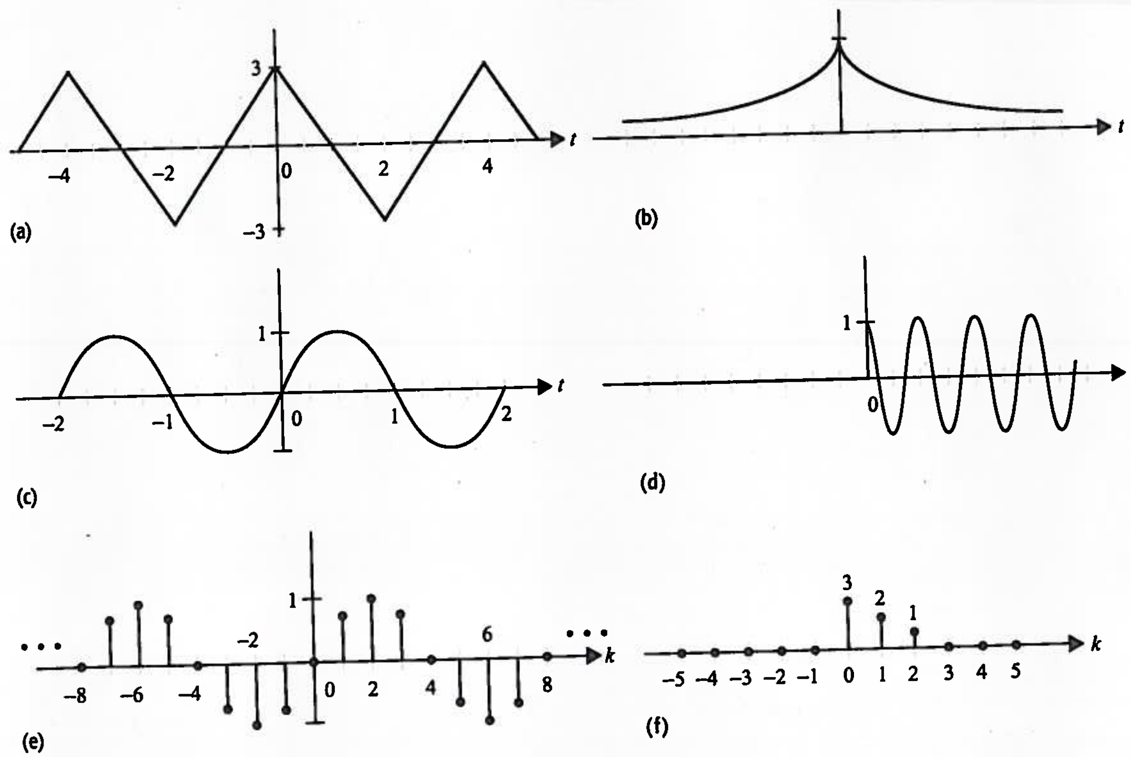


Fig. 1.6. Examples of periodic ((a), (c), and (e)) and aperiodic ((b), (d), and (f)) signals. The line plots (a) and (c) represent CT periodic signals with fundamental periods T_0 of 4 and 2, while the stem plot (e) represents a DT periodic signal with fundamental period $K_0 = 8$.

Proposition 1.1 *An arbitrary DT sinusoidal sequence $x[k] = A\sin(\Omega_0 k + \theta)$ is periodic iff $\Omega_0/2\pi$ is a rational number.*

The term *rational number* used in Proposition 1.1 is defined as a fraction of two integers. Given that the DT sinusoidal sequence $x[k] = A\sin(\Omega_0 k + \theta)$ is periodic, its fundamental period is evaluated from the relationship

$$\frac{\Omega_0}{2\pi} = \frac{m}{K_0} \tag{1.7}$$

as

$$K_0 = \frac{2\pi}{\Omega_0} m. \tag{1.8}$$

Proposition 1.1 can be extended to include DT complex exponential signals. Collectively, we state the following.

(1) The fundamental period of a sinusoidal signal that satisfies Proposition 1.1 is calculated from Eq. (1.8) with m set to the smallest integer that results in an integer value for K_0.
(2) A complex exponential $x[k] = A\exp[j(\Omega_0 k + \theta)]$ must also satisfy Proposition 1.1 to be periodic. The fundamental period of a complex exponential is also given by Eq. (1.8).

Example 1.4
Determine if the sinusoidal DT sequences (i)–(iv) are periodic:

(i) $f[k] = \sin(\pi k/12 + \pi/4)$;
(ii) $g[k] = \cos(3\pi k/10 + \theta)$;
(iii) $h[k] = \cos(0.5k + \phi)$;
(iv) $p[k] = e^{j(7\pi k/8+\theta)}$.

Solution
(i) The value of Ω_0 in $f[k]$ is $\pi/12$. Since $\Omega_0/2\pi = 1/24$ is a rational number, the DT sequence $f[k]$ is periodic. Using Eq. (1.8), the fundamental period of $f[k]$ is given by

$$K_0 = \frac{2\pi}{\Omega_0} m = 24m.$$

Setting $m = 1$ yields the fundamental period $K_0 = 24$.
To demonstrate that $f[k]$ is indeed a periodic signal, consider the following:

$$f[k + K_0] = \sin(\pi[k + K_0]/12 + \pi/4).$$

Substituting $K_0 = 24$ in the above equation, we obtain

$$\begin{aligned} f[k + K_0] &= \sin(\pi[k + K_0]/12 + \pi/4) = \sin(\pi k + 2\pi + \pi/4) \\ &= \sin(\pi k/12 + \pi/4) = f[k]. \end{aligned}$$

(ii) The value of Ω_0 in $g[k]$ is $3\pi/10$. Since $\Omega_0/2\pi = 3/20$ is a rational number, the DT sequence $g[k]$ is periodic. Using Eq. (1.8), the fundamental period of $g[k]$ is given by

$$K_0 = \frac{2\pi}{\Omega_0} m = \frac{20m}{3}.$$

Setting $m = 3$ yields the fundamental period $K_0 = 20$.
(iii) The value of Ω_0 in $h[k]$ is 0.5. Since $\Omega_0/2\pi = 1/4\pi$ is not a rational number, the DT sequence $h[k]$ is not periodic.
(iv) The value of Ω_0 in $p[k]$ is $7\pi/8$. Since $\Omega_0/2\pi = 7/16$ is a rational number, the DT sequence $p[k]$ is periodic. Using Eq. (1.8), the fundamental period of $p[k]$ is given by

$$K_0 = \frac{2\pi}{\Omega_0} m = \frac{16m}{7}.$$

Setting $m = 7$ yields the fundamental period $K_0 = 16$

Note that while $x(t) = \sin(\omega_0 t + \theta)$ is always periodic with fundamental frequency $\frac{2\pi}{\omega_0}$ the DT signal is only periodic if $\frac{\Omega_0}{2\pi}$ is a rational number (a fraction).

Example 1.3 shows that CT sinusoidal signals of the form $x(t) = \sin(\omega_0 t + \theta)$ are always periodic with fundamental period $2\pi/\omega_0$ irrespective of the value of ω_0. However, Example 1.4 shows that the DT sinusoidal sequences are not always periodic. The DT sequences are periodic only when $\Omega_0/2\pi$ is a rational number. This leads to the following interesting observation.

Consider the periodic signal $x(t) = \sin(\omega_0 t + \theta)$. Sample the signal with a sampling interval T. The DT sequence is represented as $x[k] = \sin(\omega_0 kT + \theta)$. The DT signal will be periodic if $\Omega_0/2\pi = \omega_0 T/2\pi$ is a rational number. In other words, if you sample a CT periodic signal, the DT signal need not always be periodic. The signal will be periodic only if you choose a sampling interval T such that the term $\omega_0 T/2\pi$ is a rational number.

4) Energy and Power Signals

The total energy of a signal is:

$$E_x = \sum_{k=-\infty}^{+\infty} |x[k]|^2$$

The power is defined as

$$P_x = \lim_{K \to \infty} \frac{1}{2K+1} \sum_{k=-K}^{K} |x[k]|^2$$

For periodic signals, we can simply find the power by considering one period:

$$P_x = \frac{1}{K_0} \sum_{k=k_1}^{k_1+K_0-1} |x[k]|^2$$

A signal is an energy signal if

$$0 < E_x < \infty$$

that is, if its energy is a non-zero finite value.
A signal is a power signal if its power is non-zero and finite:

$$0 < P_x < \infty$$

Example 1.7

Consider the following DT sequence:

$$f[k] = \begin{cases} e^{-0.5k} & k \geq 0 \\ 0 & k < 0. \end{cases}$$

Determine if the signal is a power or an energy signal.

Solution

The total energy of the DT sequence is calculated as follows:

$$E_f = \sum_{k=-\infty}^{\infty} |f[k]|^2 = \sum_{k=0}^{\infty} |e^{-0.5k}|^2 = \sum_{k=0}^{\infty} (e^{-1})^k = \frac{1}{1-e^{-1}} \approx 1.582.$$

Because E_f is finite, the DT sequence $f[k]$ is an energy signal.

In computing E_f, we make use of the geometric progression (GP) series to calculate the summation. The formulas for the GP series are considered in Appendix A.3.

Example 1.8

Determine if the DT sequence $g[k] = 3\cos(\pi k/10)$ is a power or an energy signal.

Solution

The DT sequence $g[k] = 3\cos(\pi k/10)$ is a periodic signal with a fundamental period of 20. All periodic signals are power signals. Hence, the DT sequence $g[k]$ is a power signal.

Using Eq. (1.15), the average power of $g[k]$ is given by

$$\begin{aligned} P_g &= \frac{1}{20}\sum_{k=0}^{19} 9\cos^2\left(\frac{\pi k}{10}\right) = \frac{9}{20}\sum_{k=0}^{19}\frac{1}{2}\left[1+\cos\left(\frac{2\pi k}{10}\right)\right] \\ &= \underbrace{\frac{9}{40}\sum_{k=0}^{19} 1}_{\text{term I}} + \underbrace{\frac{9}{40}\sum_{k=0}^{19}\cos\left(\frac{2\pi k}{10}\right)}_{\text{term II}}. \end{aligned}$$

Clearly, the summation represented by term I equals $9(20)/40 = 4.5$. To compute the summation in term II, we express the cosine as follows:

$$\text{term II} = \frac{9}{40}\sum_{k=0}^{19}\frac{1}{2}\left[e^{j\pi k/5} + e^{-j\pi k/5}\right] = \frac{9}{80}\sum_{k=0}^{19}(e^{j\pi/5})^k + \frac{9}{80}\sum_{k=0}^{19}(e^{-j\pi/5})^k.$$

Using the formulas for the GP series yields

$$\sum_{k=0}^{19}(e^{j\pi/5})^k = \frac{1-(e^{j\pi/5})^{20}}{1-(e^{j\pi/5})} = \frac{1-e^{j\pi 4}}{1-(e^{j\pi/5})} = \frac{1-1}{1-(e^{j\pi/5})} = 0$$

and

$$\sum_{k=0}^{19}(e^{-j\pi/5})^k = \frac{1-(e^{-j\pi/5})^{20}}{1-(e^{j\pi/5})} = \frac{1-e^{-j\pi 4}}{1-(e^{j\pi/5})} = \frac{1-1}{1-(e^{j\pi/5})} = 0.$$

Term II, therefore, equals zero. The average power of $g[k]$ is therefore given by

$$P_g = 4.5 + 0 = 4.5.$$

In general, a periodic DT sinusoidal signal of the form $x[k] - A\cos(\omega_0 k + \theta)$ has an average power $P_x = A^2/2$.

5) Deterministic versus Random Signals

Deterministic and random signals

If the value of a signal can be predicted for all time (t or k) in advance without any error, it is referred to as a *deterministic signal*. Conversely, signals whose values cannot be predicted with complete accuracy for all time are known as *random signals*.

Deterministic signals can generally be expressed in a mathematical, or graphical, form. Some examples of deterministic signals are as follows.

(1) CT sinusoidal signal: $x_1(t) = 5\sin(20\pi t + 6)$;
(2) CT exponentially decaying sinusoidal signal: $x_2(t) = 2e^{-t}\sin(7t)$;
(3) CT finite duration complex exponential signal: $x_3(t) = \begin{cases} e^{j4\pi t} & |t| < 5 \\ 0 & \text{elsewhere;} \end{cases}$
(4) DT real-valued exponential sequence: $x_4[k] = 4e^{-2k}$;
(5) DT exponentially decaying sinusoidal sequence: $x_5[k] = 3e^{-2k} \times \sin\left(\frac{16\pi k}{5}\right)$.

Unlike deterministic signals, random signals cannot be modeled precisely. Random signals are generally characterized by statistical measures such as means, standard deviations, and mean squared values. In electrical engineering, most meaningful information-bearing signals are random signals. In a digital communication system, for example, data are generally transmitted using a sequence of zeros and ones. The binary signal is corrupted with interference from other channels and additive noise from the transmission media, resulting in a received signal that is random in nature. Another example of a random signal in electrical engineering is the thermal noise generated by a resistor. The intensity of the thermal noise depends on the movement of billions of electrons and cannot be predicted accurately.

The study of random signals is beyond the scope of this book. We therefore restrict our discussion to deterministic signals. However, most principles and techniques that we develop are generalizable to random signals. The readers are advised to consult more advanced books for analysis of random signals.

Odd and even signals

A CT signal $x_e(t)$ is said to be an even signal if

$$x_e(t) = x_e(-t). \tag{1.16}$$

Conversely, a CT signal $x_o(t)$ is said to be an odd signal if

$$x_o(t) = -x_o(-t). \tag{1.17}$$

A DT signal $x_e[k]$ is said to be an even signal if

$$x_e[k] = x_e[-k]. \tag{1.18}$$

Conversely, a DT signal $x_o[k]$ is said to be an odd signal if

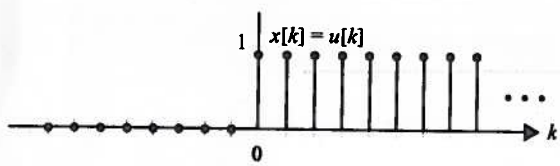
$$x_o[k] = -x_o[-k]. \tag{1.19}$$

The even signal property, Eq. (1.16) for CT signals or Eq. (1.18) for DT signals, implies that an even signal is symmetric about the *vertical* axis ($t = 0$). Likewise, the odd signal property, Eq. (1.17) for CT signals or Eq. (1.19) for DT signals, implies that an odd signal is antisymmetric about the *vertical* axis ($t = 0$). The symmetry characteristics of even and odd signals are illustrated in Fig. 1.10.

Some Elementary Signals

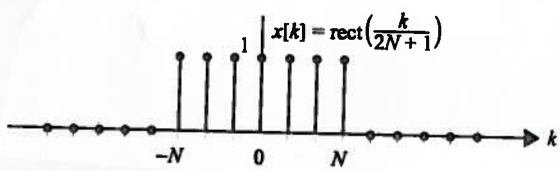
1) Unit step function:

$$u[k] = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$



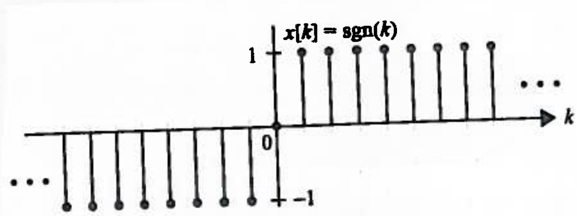
2) Rectangular Pulse function:

$$\text{rect}\left[\frac{k}{2N+1}\right] = \begin{cases} 1 & |k| \leq N \\ 0 & |k| > N \end{cases}$$



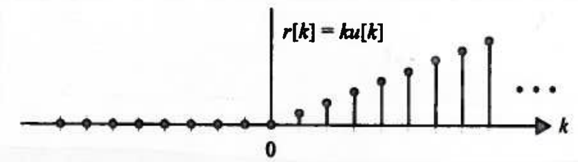
3) Signum function:

$$\text{Sgn}[k] = \begin{cases} 1 & k > 0 \\ 0 & k = 0 \\ -1 & k < 0 \end{cases}$$



4) Ramp function:

$$r[k] = k\,u[k] = \begin{cases} k & k \geq 0 \\ 0 & k < 0 \end{cases}$$



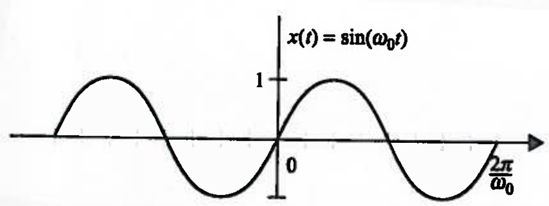
5) Sinusoidal function (signal):

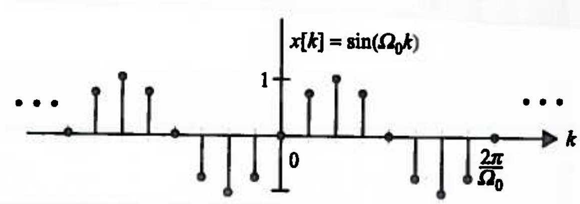
for CT:

$$x(t) = \sin(\omega_0 t + \theta) = \sin(2\pi f_0 t + \theta)$$

for DT:

$$x[k] = \sin(\omega_0 k + \theta) = \sin[2\pi f_0 k + \theta]$$





6) Sinc function:

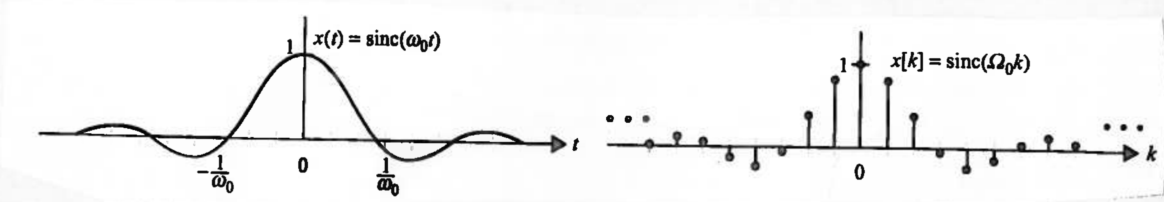
$$\mathrm{Sinc}(\omega_0 t) = \frac{\sin(\pi \omega_0 t)}{\pi \omega_0 t}$$

or in some text books as

$$\mathrm{Sinc}(\omega_0 t) = \frac{\sin(\omega_0 t)}{\omega_0 t}$$

in DT:

$$\mathrm{Sinc}[\Omega_0 k] = \frac{\sin[\pi \Omega_0 k]}{\pi \Omega_0 k}$$



7) DT exponential function

The DT complex exponential function with radian frequency Ω_0 is defined as follows:

$$x[k] = e^{(\sigma+j\Omega_0)k} = e^{\sigma t}(\cos \Omega_0 k + j \sin \Omega_0 k.) \tag{1.39}$$

As an example of the DT complex exponential function, we consider $x[k] = \exp(j0.2\pi - 0.05k)$, which is plotted in Fig. 1.16, where plot (a) shows the real component and plot (b) shows the imaginary part of the complex signal.

Case 1 Imaginary component is zero ($\Omega_0 = 0$). The signal takes the following form:

$$x[k] = e^{\sigma k}$$

when the imaginary component Ω_0 of the DT complex frequency is zero. Similar to CT exponential functions, the DT exponential functions can be classified as rising, decaying, and constant-valued exponentials depending upon the value of σ.

Case 2 Real component is zero ($\sigma = 0$). The DT exponential function takes the following form:

$$x[k] = e^{j\omega_0 k} = \cos \omega_0 k + j \sin \omega_0 k.$$

Recall that a complex-valued exponential is periodic iff $\Omega_0/2\pi$ is a rational number. An alternative representation of the DT complex exponential function

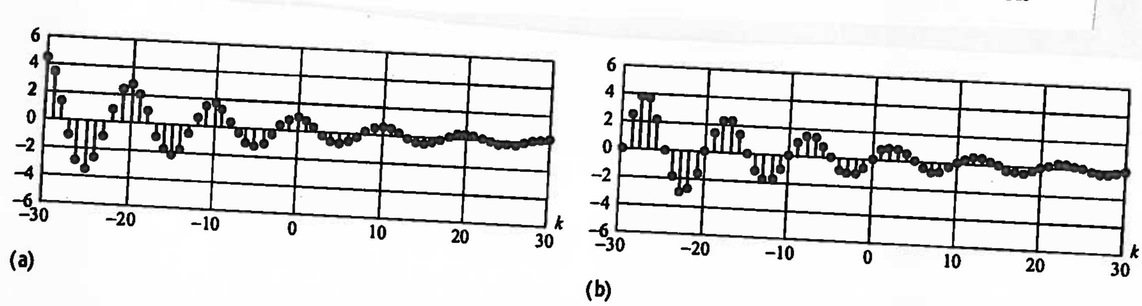


Fig. 1.16. DT complex exponential function $x[k] = \exp(j0.2\pi k - 0.05k)$. (a) Real component; (b) imaginary component.

is obtained by expanding

$$x[k] = \left(e^{(\sigma+j\Omega_0)}\right)^k = \gamma^k, \tag{1.40}$$

where $\gamma = (\sigma + j\Omega_0)$ is a complex number. Equation (1.40) is more compact than Eq. (1.39).

DT unit impulse function

The DT impulse function, also referred to as the Kronecker delta function or the DT unit sample function, is defined as follows:

$$\delta[k] = u[k] - u[k-1] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0. \end{cases} \tag{1.50}$$

Unlike the CT unit impulse function, the DT impulse function has no ambiguity in its definition; it is well defined for all values of k. The waveform for a DT unit impulse function is shown in Fig. 1.19.

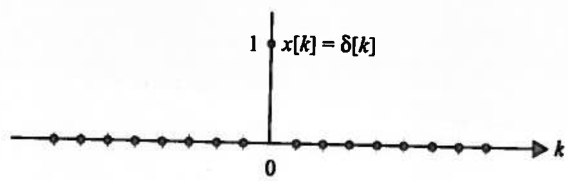


Fig. 1.19. DT unit impulse function.

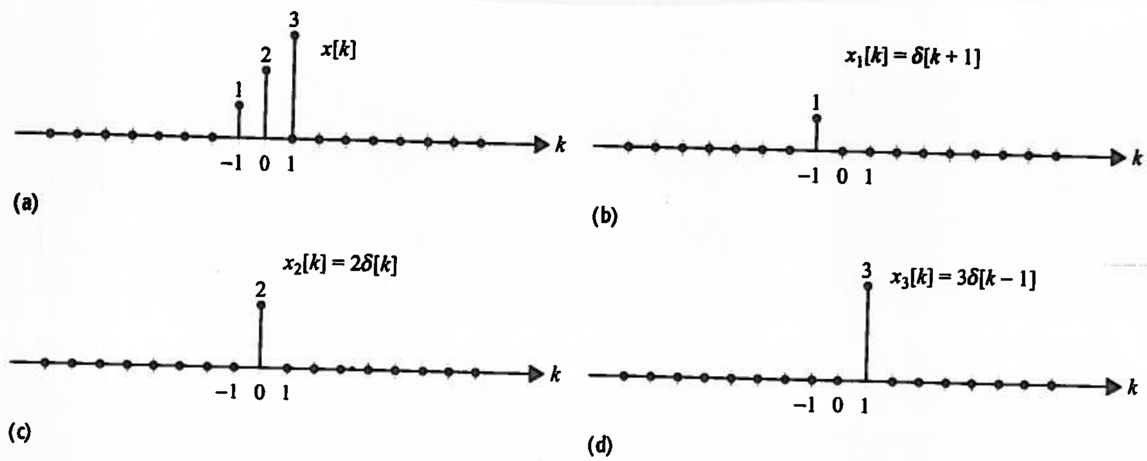


Fig. 1.20. The DT functions in Example 1.13: (a) $x[k]$, (b), $x[k]$, (c) $x_2[k]$, and (d) $x_3[k]$. The DT function in (a) is the sum of the shifted DT impulse functions shown in (b), (c), and (d).

Example 1.13

Represent the DT sequence shown in Fig. 1.20(a) as a function of time-shifted DT unit impulse functions.

Solution

The DT signal $x[k]$ can be represented as the summation of three functions, $x_1[k]$, $x_2[k]$, and $x_3[k]$, as follows:

$$x[k] = x_1[k] + x_2[k] + x_3[k],$$

where $x_1[k]$, $x_2[k]$, and $x_3[k]$ are time-shifted impulse functions,

$$x_1[k] = \delta[k+1], \quad x_2[k] = 2\delta[k], \quad \text{and} \quad x_3[k] = 4\delta[k-1],$$

and are plotted in Figs. 1.20(b), (c), and (d), respectively. The DT sequence $x[k]$ can therefore be represented as follows:

$$x[k] = \delta[k+1] + 2\delta[k] + 4\delta[k-1].$$

Operations on Signals

1) Time-shifting:

$$y[k] = x[k+m]$$

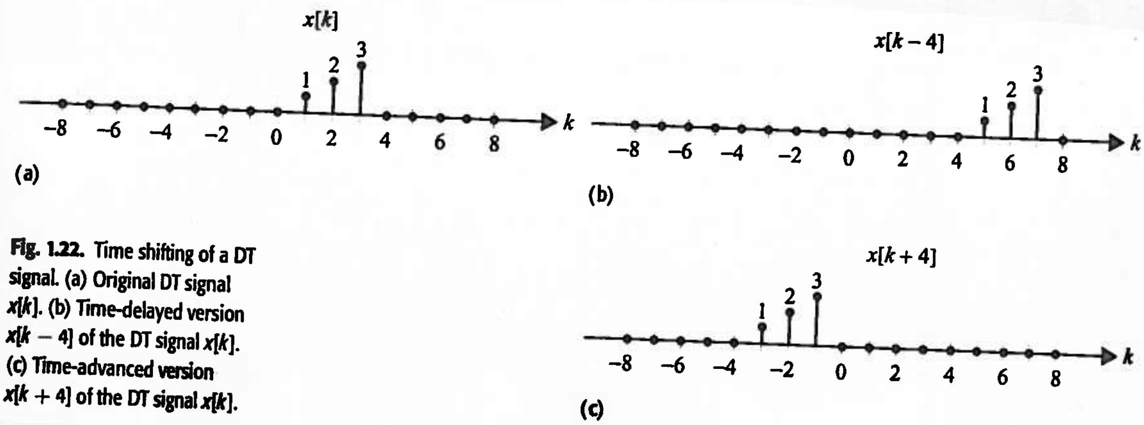


Fig. 1.22. Time shifting of a DT signal. (a) Original DT signal $x[k]$. (b) Time-delayed version $x[k-4]$ of the DT signal $x[k]$. (c) Time-advanced version $x[k+4]$ of the DT signal $x[k]$.

Example 1.15

Consider the signal $x[k]$ defined as follows:

$$x[k] = \begin{cases} 0.2k & 0 \le k \le 5 \\ 0 & \text{elsewhere.} \end{cases} \tag{1.52}$$

Determine and plot signals $p[k] = x[k-2]$ and $q[k] = x[k+2]$.

Solution

The signal $x[k]$ is plotted in Fig. 1.24(a). To calculate the expression for $p[k]$, substitute $k = m - 2$ in Eq. (1.52). The resulting equation is given by

$$x[m-2] = \begin{cases} 0.2(m-2) & 0 \le (m-2) \le 5 \\ 0 & \text{elsewhere.} \end{cases}$$

By changing the independent variable from m to k and simplifying, we obtain

$$p[k] = x[k-2] = \begin{cases} 0.2(k-2) & 2 \le k \le 7 \\ 0 & \text{elsewhere.} \end{cases}$$

The non-zero values of $p[k]$ for $-2 \le k \le 7$, are shown in Table 1.1, and the stem plot $p[k]$ is plotted in Fig. 1.24(b). To calculate the expression for $q[k]$, substitute $k = m + 2$ in Eq. (1.52). The resulting equation is as follows:

$$x[m+2] = \begin{cases} 0.2(m+2) & 0 \le (m+2) \le 5 \\ 0 & \text{elsewhere.} \end{cases}$$

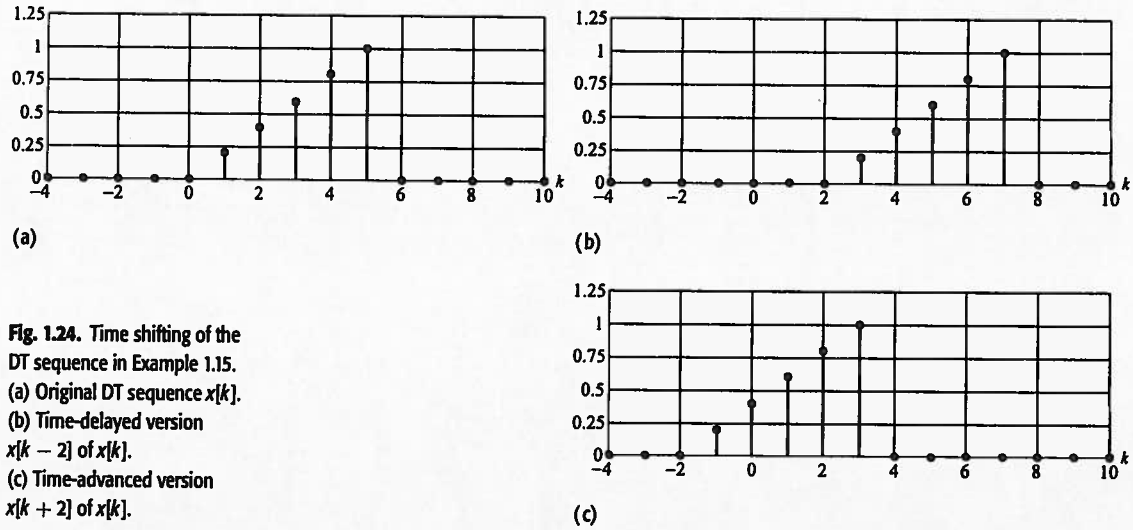


Fig. 1.24. Time shifting of the DT sequence in Example 1.15. (a) Original DT sequence $x[k]$. (b) Time-delayed version $x[k-2]$ of $x[k]$. (c) Time-advanced version $x[k+2]$ of $x[k]$.

Table 1.1. Values of the signals $p[k]$ and $q[k]$

k	−2	−1	0	1	2	3	4	5	6	7
$p[k]$	0	0	0	0	0	0.2	0.4	0.6	0.8	1
$q[k]$	0	0.2	0.4	0.6	0.8	1	0	0	0	0

By changing the independent variable from m to k and simplifying, we obtain

$$q[k] = x[k+2] = \begin{cases} 0.2(k+2) & -2 \le k \le 3 \\ 0 & \text{elsewhere.} \end{cases}$$

Values of $q[k]$, for $-2 \le k \le 7$, are shown in Table 1.1, and the stem plot for $q[k]$ is plotted in Fig. 1.24(c).

As in Example 1.14, we observe that the waveform for $p[k] = x[k-2]$ can be obtained directly by shifting the waveform of $x[k]$ towards the right-hand side by two time units. Similarly, the waveform for $q[k] = x[k+2]$ can be obtained directly by shifting the waveform of $x[k]$ towards the left-hand side by two time units.

1.3.2 Time scaling

The *time-scaling* operation compresses or expands the input signal in the time domain. A CT signal $x(t)$ scaled by a factor c in the time domain is denoted by $x(ct)$. If $c > 1$, the signal is compressed by a factor of c. On the other hand, if $0 < c < 1$ the signal is expanded. We illustrate the concept of time scaling of CT signals with the help of a few examples.

2) Decimation and Interpolation

A continuous time signal can be expanded or compressed in time by letting $y(t) = x(ct)$ where $c > 1$ means <u>compression</u> and $c < 1$ corresponds to expansion in time.

In the case of DT signals if a sequence $x[k]$ is compressed then some data samples are lost.

1.3.2.1 Decimation

If a sequence $x[k]$ is compressed by a factor c, some data samples of $x[k]$ are lost. For example, if we decimate $x[k]$ by 2, the decimated function $y[k] = x[2k]$ retains only the alternate samples given by $x[0]$, $x[2]$, $x[4]$, and so on. Compression (referred to as decimation for DT sequences) is, therefore, an irreversible process in the DT domain as the original sequence $x[k]$ cannot be recovered precisely from the decimated sequence $y[k]$.

1.3.2.2 Interpolation

In the DT domain, expansion (also referred to as interpolation) is defined as follows:

$$x^{(m)}[k] = \begin{cases} x\left[\dfrac{k}{m}\right] & \text{if } k \text{ is a multiple of integer } m \\ 0 & \text{otherwise.} \end{cases} \tag{1.54}$$

The interpolated sequence $x^{(m)}[k]$ inserts $(m-1)$ zeros in between adjacent samples of the DT sequence $x[k]$. Interpolation of the DT sequence $x[k]$ is a reversible process as the original sequence $x[k]$ can be recovered from $x^{(m)}[k]$.

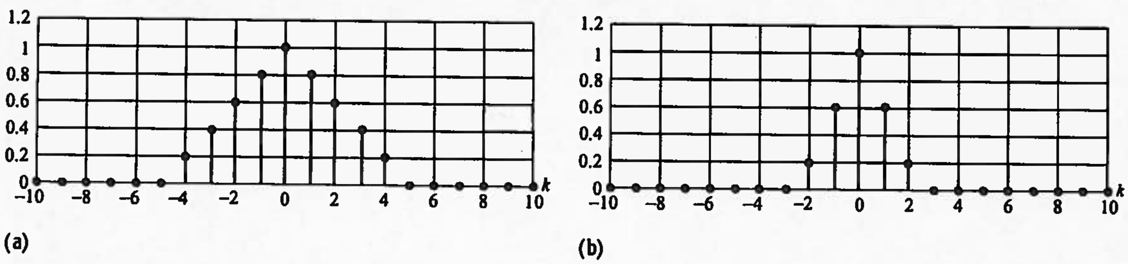
Example 1.17
Consider the DT sequence $x[k]$ plotted in Fig. 1.26(a). Calculate and sketch $p[k] = x[2k]$ and $q[k] = x[k/2]$.

Table 1.2. Values of the signal $p[k]$ for $-3 \le k \le 3$

k	-3	-2	-1	0	1	2	3
$p[k]$	$x[-6] = 0$	$x[-4] = 0.2$	$x[-2] = 0.6$	$x[0] = 1$	$x[2] = 0.6$	$x[4] = 0.2$	$x[6] = 0$

Table 1.3. Values of the signal $q[k]$ for $-10 \le k \le 10$

k	-10	-9	-8	-7	-6	-5	-4
$q[k]$	$x[-5] = 0$	0	$x[-4] = 0.2$	0	$x[-3] = 0.4$	0	$x[-2] = 0.6$
k	-3	-2	-1	0	1	2	3
$q[k]$	0	$x[-1] = 0.8$	0	$x[0] = 1$	0	$x[1] = 0.8$	0
k	4	5	6	7	8	9	10
$q[k]$	$x[2] = 0.6$	0	$x[3] = 0.4$	0	$x[4] = 0.2$	0	$x[5] = 0$



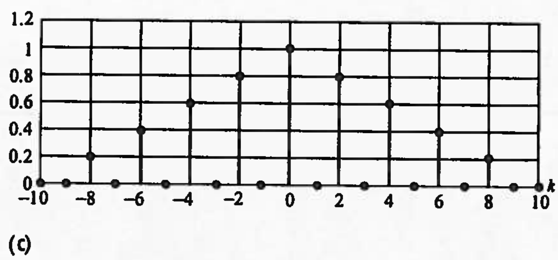


Fig. 1.26. Time scaling of the DT signal in Example 1.17. (a) Original DT sequence $x[k]$. (b) Decimated version $x[2k]$, of $x[k]$. (c) Interpolated version $x[0.5k]$ of signal $x[k]$.

Solution

Since $x[k]$ is non-zero for $-5 \le k \le 5$, the non-zero values of the decimated sequence $p[k] = x[2k]$ lie in the range $-3 \le k \le 3$. The non-zero values of $p[k]$ are shown in Table 1.2. The waveform for $p[k]$ is plotted in Fig. 1.26(b).

The waveform for the decimated sequence $p[k]$ can be obtained by directly *compressing* the waveform for $x[k]$ by a factor of 2 about the y-axis. While performing the compression, the value of $x[k]$ at $k = 0$ is retained in $p[k]$. On both sides of the $k = 0$ sample, every second sample of $x[k]$ is retained in $p[k]$.

To determine $q[k] = x[k/2]$, we first determine the range over which $x[k/2]$ is non-zero. The non-zero values of $q[k] = x[k/2]$ lie in the range $-10 \le k \le 10$ and are shown in Table 1.3. The waveform for $q[k]$ is plotted in Fig. 1.26(c).

The waveform for the decimated sequence $q[k]$ can be obtained by directly *expanding* the waveform for $x[k]$ by a factor of 2 about the y-axis. During

Table 1.4. Values of the signal $q_2[k]$ for $-10 \le k \le k$

k	-10	-9	-8	-7	-6	-5	-4
$q_2[k]$	$x[-5]=0$	0.1	$x[-4]=0.2$	0.3	$x[-3]=0.4$	0.5	$x[-2]=0.6$
k	-3	-2	-1	0	1	2	3
$q_2[k]$	0.7	$x[-1]=0.8$	0.9	$x[0]=1$	0.9	$x[1]=0.8$	0.7
k	4	5	6	7	8	9	10
$q_2[k]$	$x[2]=0.6$	0.5	$x[3]=0.4$	0.3	$x[4]=0.2$	0.1	$x[5]=0$

expansion, the value of $x[k]$ at $k = 0$ is retained in $q[k]$. The even-numbered samples, where k is a multiple of 2, of $q[k]$ equal $x[k/2]$. The odd-numbered samples in $q[k]$ are set to zero.

While determining the interpolated sequence $x[mk]$, Eq. (1.54) inserts $(m-1)$ zeros in between adjacent samples of the DT sequence $x[k]$, where $x[k]$ is not defined. Instead of inserting zeros, we can possibly interpolate the undefined values from the neighboring samples where $x[k]$ is defined. Using linear interpolation, an interpolated sequence can be obtained using the following equation:

$$x^{(m)}[k] = \begin{cases} x\left[\dfrac{k}{m}\right] & \text{if } k \text{ is a multiple of integer } m \\ (1-\alpha)x\left[\left\lfloor\dfrac{k}{m}\right\rfloor\right] + \alpha x\left[\left\lceil\dfrac{k}{m}\right\rceil\right] & \text{otherwise,} \end{cases} \tag{1.55}$$

where $\lfloor \frac{k}{m} \rfloor$ denotes the nearest integer less than or equal to (k/m), $\lceil \frac{k}{m} \rceil$ denotes the nearest integer greater than or equal to (k/m), and $\alpha = (k \bmod m)/m$. Note that mod is the modulo operator that calculates the remainder of the division k/m. For $m = 2$, Eq. (1.55) simplifies to the following:

$$x^{(2)}[k] = \begin{cases} x\left[\dfrac{k}{2}\right] & \text{if } k \text{ is even} \\ 0.5\left(x\left[\dfrac{k-1}{2}\right] + x\left[\dfrac{k+1}{2}\right]\right) & \text{if } k \text{ is odd.} \end{cases}$$

Although, Eq. (1.55) is useful in many applications, we will use Eq. (1.54) to denote an interpolated sequence throughout the book unless explicitly stated otherwise.

Example 1.18

Repeat Example 1.17 to obtain the interpolated sequence $q_2[k] = x[k/2]$ using the alternative definition given by Eq. (1.55).

Solution

The non-zero values of $q_2[k] = x[k/2]$ are shown in Table 1.4, where the values of the odd-numbered samples of $q_2[k]$, highlighted with the gray background, are obtained by taking the average of the values of the two neighboring

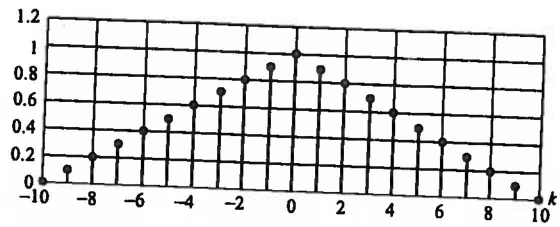


Fig. 1.27. Interpolated version $x[0.5k]$ of signal $x[k]$, where unknown sample values are interpolated.

samples at k and $k-1$ obtained from $x[k]$. The waveform for $q_2[k]$ is plotted in Fig. 1.27.

1.3.3 Time inversion

The *time inversion* (also known as *time reversal* or *reflection*) operation reflects the input signal about the vertical axis ($t = 0$). When a CT signal $x(t)$ is time-reversed, the inverted signal is denoted by $x(-t)$. Likewise, when a DT signal $x[k]$ is time-reversed, the inverted signal is denoted by $x[-k]$. In the following we provide examples of time inversion in both CT and DT domains.

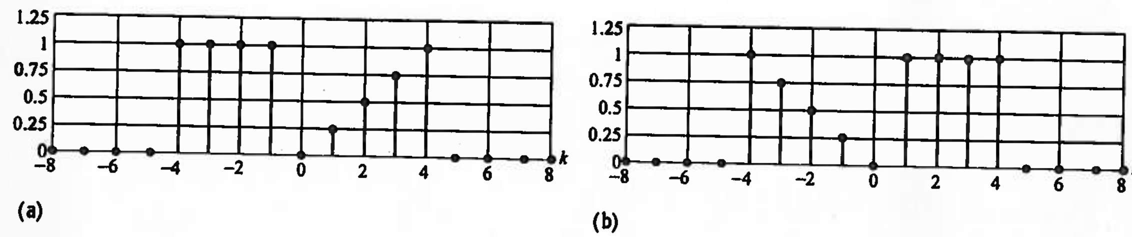


Fig. 1.29. Time inversion of the DT signal in Example 1.20. (a) Original CT sequence $x[k]$. (b) Time-inverted version $x[-k]$.

The time-reversed signal $x(-t)$ is plotted in Fig. 1.28(b). Signal inversion can also be performed graphically by simply flipping the signal $x(t)$ about the y-axis.

Example 1.20

Sketch the time-inverted version of the following DT sequence:

$$x[k] = \begin{cases} 1 & -4 \le k \le -1 \\ 0.25k & 0 \le k \le 4 \\ 0 & \text{elsewhere,} \end{cases} \tag{1.57}$$

which is plotted in Fig. 1.29(a).

Solution

To derive the expression for the time-inverted signal $x[-k]$, substitute $k = -m$ in Eq. (1.57). The resulting expression is given by

$$x[-m] = \begin{cases} 1 & -4 \le -m \le -1 \\ -0.25m & 0 \le -m \le 4 \\ 0 & \text{elsewhere.} \end{cases}$$

Simplifying the above expression and expressing it in terms of the independent variable k yields

$$x[-m] = \begin{cases} 1 & 1 \le m \le 4 \\ -0.25m & -4 \le -m \le 0 \\ 0 & \text{elsewhere.} \end{cases}$$

The time-reversed signal $x[-k]$ is plotted in Fig. 1.29(b).

Systems

A system is an entity, say a circuit, a processor, etc. that accepts one or more signals as the iput(s) and outputs one or more signals:

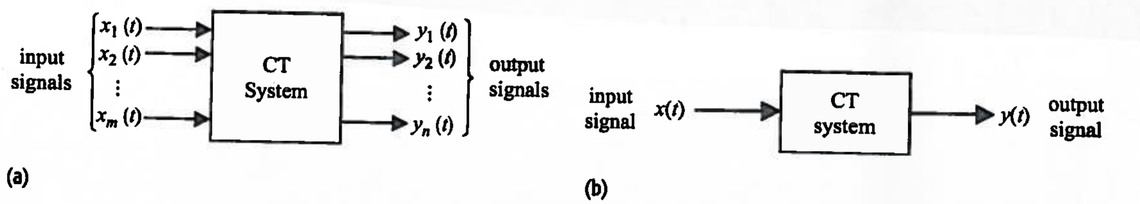


Fig. 2.1. General schematics of CT systems. (a) Multiple-input, multiple-output (MIMO) CT system with m inputs and n outputs. (b) Single-input, single-output CT system.

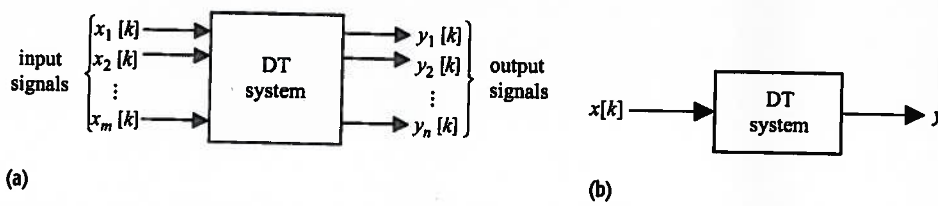


Fig. 2.2. General schematics of DT systems. (a) Multiple-input, multiple-output (MIMO) DT system with m inputs and n outputs. (b) Single-input, single-output DT system.

notation:

CT system $\quad x(t) \rightarrow y(t);$ (2.1)

DT system $\quad x[k] \rightarrow y[k].$ (2.2)

Classification of systems:

Systems can be classified as:

(i) linear and non-linear systems;
(ii) time-invariant and time-varying systems;
(iii) systems with and without memory;
(iv) causal and non-causal systems;
(v) invertible and non-invertible systems;
(vi) stable and unstable systems.

1) Linear and non-linear systems

Definition:

A DT system is linear iff:

$x_1[k] \rightarrow y_1[k]$ and $x_2[k] \rightarrow y_2[k]$

then

$$\alpha x_1[k] + \beta x_2[k] \longrightarrow \alpha y_1[k] + \beta y_2[k]$$

Example 2.2

Consider two DT systems with the following input–output relationships:

(a) differencing system $\quad y[k] = 3(x[k] - x[k-2]);$ (2.37)

(b) sinusoidal system $\quad y[k] = \sin(x[k]).$ (2.38)

Determine if the DT systems are linear.

Solution

(a) From Eq. (2.37), it follows that:

$$x_1[k] \rightarrow 3x_1[k] - 3x_1[k-2] = y_1[k]$$

and

$$x_2[k] \rightarrow 3x_2[k] - 3x_2[k-2] = y_2[k],$$

giving

$$\alpha x_1[k] + \beta x_2[k] \rightarrow 3\alpha x_1[k] - 3\alpha x_1[k-2] + 3\beta x_2[k] - 3\beta x_2[k-2].$$

Since

$$3\alpha x_1[k] - 3\alpha x_1[k-2] + 3\beta x_2[k] - 3\beta x_2[k-2] = \alpha y_1[k] + \beta y_2[k],$$

the differencing system, Eq. (2.37), is linear.

b) $x_1[k] \rightarrow \sin(x_1[k]) = y_1[k]$

and

$x_2[k] \rightarrow \sin(x_2[k]) = y_2[k]$

But

$\alpha x_1[k] + \beta x_2[k] \longrightarrow \sin(\alpha x_1[k] + \beta x_2[k])$

which is not equal to $y_1[k] + y_2[k]$

2) Time-varying and Time-invariant Systems

Definition:

A system is <u>time-invariant</u> iff:

when $x[k] \rightarrow y[k]$

then $x[k-k_0] \rightarrow y[k-k_0]$

Example 2.5

Consider two DT systems with the following input–output relationships:

(i) system I $\quad y[k] = 3(x[k] - x[k-2]);$ (2.44)

(ii) system II $\quad y[k] = k\,x[k].$ (2.45)

Determine if the systems are time-invariant.

Solution

(i) From Eq. (2.44), it follows that:

$$x[k] \rightarrow 3(x[k] - x[k-2]) = y[k]$$

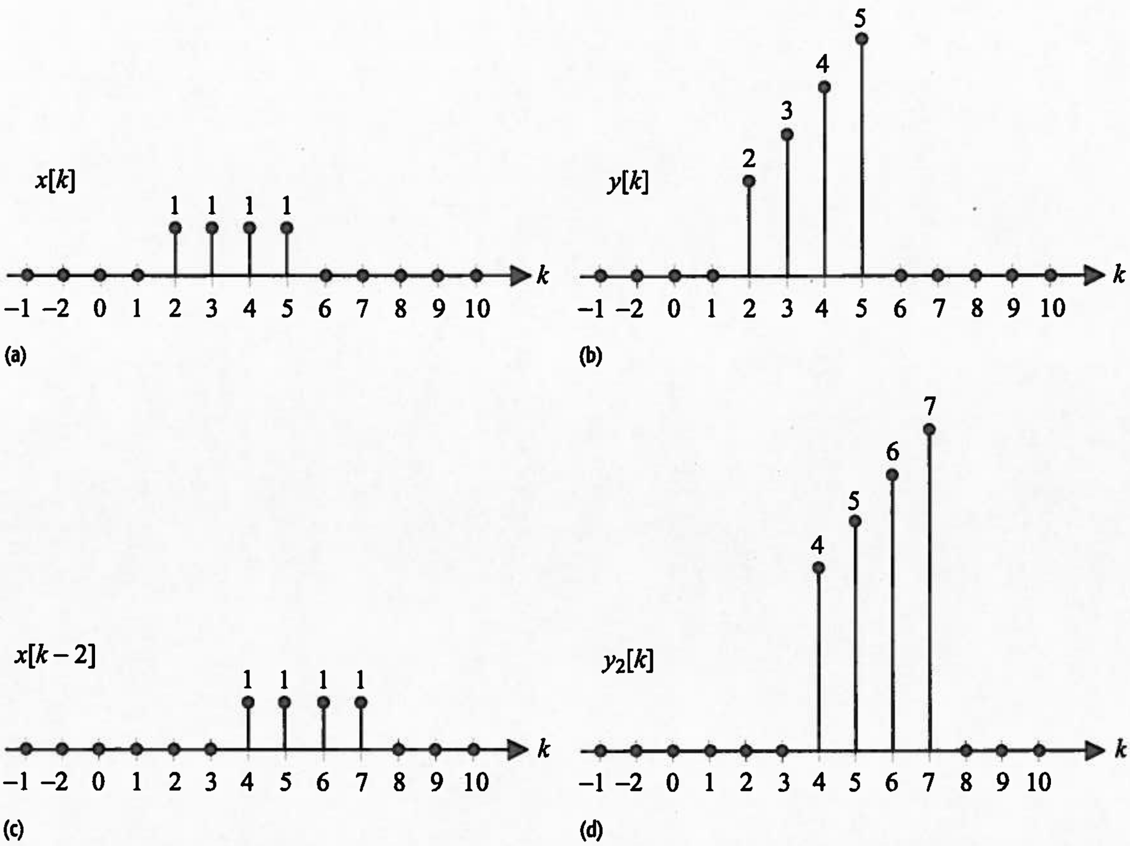


Fig. 2.15. Input–output pairs of the DT time-varying system specified in Example 2.5(ii). The output $y_2[k]$ for the time-shifted input $x_2[k] = x[k-2]$ is different in shape from the output $y[k]$ obtained for input $x[k]$. Therefore the system is time-variant. Parts (a)–(d) are discussed in the text.

and

$$x[k-k_0] \to 3(x[k-k_0] - x[k-k_0-2]) = y[k-k_0].$$

Therefore, the system in Eq. (2.44) is a time-invariant system.

(ii) From Eq. (2.45), it follows that:

$$x[k] \to kx[k] = y[k]$$

and

$$x[k-k_0] \to kx[k-k_0] \neq y[k-k_0] = (k-k_0)x[k-k_0].$$

Therefore, system II is not time-invariant. In Fig. 2.15, we plot the outputs of the DT system in Eq. (2.45) for input $x[k]$, shown in Fig. 2.15(a) and a shifted version $x[k-2]$ of the input, shown in Fig. 2.15(c). The resulting outputs are plotted, respectively, in Figs. 2.15(b) and (d). As expected, the Fig. 2.15(d) is not a delayed version of Fig. 2.15(b) since the system is time-variant.

3) Systems with memory and memoryless systems

Definition : A system is memoryless if the output at time $k=k_0$ depends only on the input at time $k=k_0$.

Table 2.1. Examples of CT and DT systems with and without memory

Continuous-time		Discrete-time	
Memoryless systems	Systems with memory	Memoryless systems	Systems with memory
$y(t)=3x(t)+5$	$y(t)=x(t-5)$	$y[k]=3x[k]+7$	$y[k]=x[k-5]$
$y(t)=\sin\{x(t)\}+5$	$y(t)=x(t+2)$	$y[k]=\sin(x[k])+3$	$y[k]=x[k+3]$
$y(t)=e^{x(t)}$	$y(t)=x(2t)$	$y[k]=e^{x[k]}$	$y[k]=x[2k]$
$y(t)=x^2(t)$	$y(t)=x(t/2)$	$y[k]=x^2[k]$	$y[k]=x[k/2]$

4) **Causal and non-causal systems**

A CT system is *causal* if the output at time t_0 depends only on the input $x(t)$ for $t \le t_0$. Likewise, a DT system is *causal* if the output at time instant k_0 depends only on the input $x[k]$ for $k \le k_0$. A system that violates the causality condition is called a *non-causal* (or *anticipative*) system. Note that all memoryless systems are causal systems because the output at any time instant depends only on the input at that time instant. Systems with memory can either be causal or non-causal.

Example 2.7

(i) CT time-delay system $y(t)=x(t-2) \Rightarrow$ causal system;

(ii) CT time-forward system $y(t)=x(t+2) \Rightarrow$ non-causal system;

(iii) DT time-delay system $y[k]=x[k-2] \Rightarrow$ causal system;

(iv) DT time-advance system $y[k]=x[k+2] \Rightarrow$ non-causal system;

(v) DT linear system $y[k]=x[k-2]+x[k+10] \Rightarrow$ non-causal system.

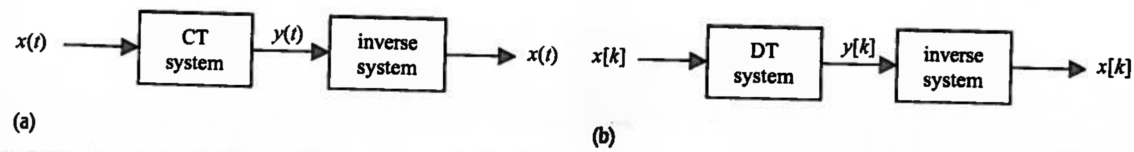
Table 2.2. Examples of causal and non-causal systems
The CT and DT systems are represented using their input–output relationships. Note that all systems in the table have memory.

CT systems		DT systems	
Causal	Non-causal	Causal	Non-causal
$y(t)=x(t-5)$	$y(t)=x(t+2)$	$y[k]=3x[k-1]+7$	$y[k]=x[k+3]$
$y(t)=\sin\{x(t-4)\}+3$	$y(t)=\sin\{x(t+4)\}+3$	$y[k]=\sin(x[k-4])+3$	$y[k]=\sin(x[k+4])+3$
$y(t)=e^{x(t-2)}$	$y(t)=x(2t)$	$y[k]=e^{x[k-2]}$	$y[k]=x[2k]$
$y(t)=x^2(t-2)$	$y(t)=x(t/2)$	$y[k]=x^2[k-5]$	$y[k]=x[k/2]$
$y(t)=x(t-2)+x(t-5)$	$y(t)=x(t-2)+x(t+2)$	$y[k]=x[k-2]+x[k-8]$	$y[k]=x[k+2]+x[k-8]$

5) **Invertible and non-invertible systems**

A CT system is *invertible* if the input signal $x(t)$ can be uniquely determined from the output $y(t)$ produced in response to $x(t)$ for all time $t \in (-\infty, \infty)$. Similarly, a DT system is called *invertible* if, given an arbitrary output response $y[k]$ of the system for $k \in (-\infty, \infty)$, the corresponding input signal $x[k]$ can be uniquely determined for all time $k \in (-\infty, \infty)$. To be invertible, two different inputs cannot produce the same output since, in such cases, the input signal cannot be uniquely determined from the output signal.

A direct consequence of the invertibility property is the determination of a second system that restores the original input. A system is said to be invertible if the input to the system can be recovered by applying the output of the original system as input to a second system. The second system is called the inverse of the original system. The relationship between the original system and its inverse is shown in Fig. 2.17.



Example 2.9

Determine if the following DT systems are invertible.

(i) Incrementally linear system:

$$y[k] = 2x[k] + 7.$$

The input–output relationship is expressed as follows:

$$x[k] = \frac{1}{2}\{y[k] - 7\}.$$

The above expression shows that given an output signal, the input can be uniquely determined. Therefore, the system is invertible.

(ii) Exponential output:

$$y[k] = e^{x[k]}.$$

The input–output relationship is expressed as follows:

$$x[k] = \ln\{y[k]\}.$$

The above expression shows that given an output signal, the input can be uniquely determined. Therefore, the system is invertible.

(iii) Increasing ramped output:

$$y[k] = k\,x[k].$$

The input–output relationship is expressed as follows:

$$x[k] = \frac{1}{k} y[k].$$

The input signal can be uniquely determined for all time instant k, except at $k = 0$. Therefore, the system is not invertible.

(iv) Summer:

$$y[k] = x[k] + x[k-1].$$

Following the procedure used in Example 2.8(iv), the input signal is expressed as an infinite sum of the output $y[k]$ as follows:

$$x[k] = y[k] - y[k-1] + y[k-2] - y[k-3] + - \cdots$$
$$= \sum_{m=0}^{\infty} (-1)^m y[k-m].$$

The input signal $x[k]$ can be reconstructed if $y[m]$ is known for all $m \leq k$. Therefore, the system is invertible.

(v) Accumulator:

$$y[k] = \sum_{m=-\infty}^{k} x[m].$$

We express the accumulator as follows:

$$y[k] = x[k] + \sum_{m=-\infty}^{k-1} x[m] = x[k] + y[k-1]$$

or

$$x[k] = y[k] - y[k-1].$$

Therefore, the system is invertible.

6) stable and un-stable systems

a system is called Bounded-Input-Bounded output (BIBO) stable if:

$$|x[k]| \leq B_x < \infty$$

results in

$$|y[k]| \leq B_y < \infty$$

That is a bounded input results in a bounded output.

Example 2.11

Determine if the following DT systems are stable.

(i) $$y[k] = 50\sin(x[k]) + 10. \tag{2.54}$$

Note that $\sin(x[k])$ is bounded between $[-1, 1]$ for any arbitrary choice of $x[k]$. The output $y[k]$ is therefore bounded within the interval $[-40, 60]$. Therefore, system (i) is stable.

(ii) $$y[k] = e^{x[k]}. \tag{2.55}$$

Assume $|x[k]| \leq B_x$ for all t. Based on Eq. (2.52), it follows that:

$$y[k] \leq e^{B_x} = B_y \quad \text{for all } k.$$

Therefore, system (ii) is stable.

(iii) $$y[k] = \sum_{m=-2}^{2} x[k-m]. \tag{2.56}$$

The output is expressed as follows:

$$y[k] = x[k-2] + x[k-1] + x[k] + x[k+1] + x[k+2].$$

If $|x[k]| \leq B_x$ for all k, then $|y[k]| \leq 5B_x$ for all k. Therefore, the system is stable.

(iv) $$y[k] = \sum_{m=-\infty}^{k} x[m]. \tag{2.57}$$

The output is calculated by summing an infinite number of input signal values. Hence, there is no guarantee that the output will be bounded even if all the input values are bounded. System (iv) is, therefore, not a stable system.