ELEC 342

Lecture 2

Now that we are familiar with the basic concepts of Signals and Systems and classification of each, let's start talking about the examples of Systems.

Instead of talking about the general case of an LTI discrete-time System and giving a generic difference equation formula, we will start with simple examples and move to some more elaborate one. This way the general formulation will make more sense.

A System S is defined as,

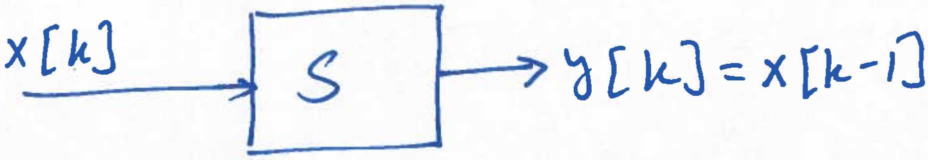
$$x[k] \longrightarrow \boxed{S} \longrightarrow y[k]$$

where $y[k]$ is the output resulting from applying the input $x[k]$.

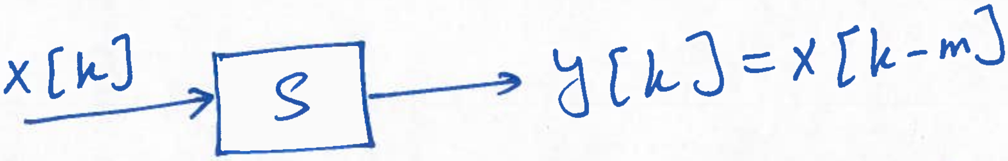
The simples (and somehow trivial) System is

$$x[k] \longrightarrow \boxed{S} \longrightarrow y[k] = x[k]$$

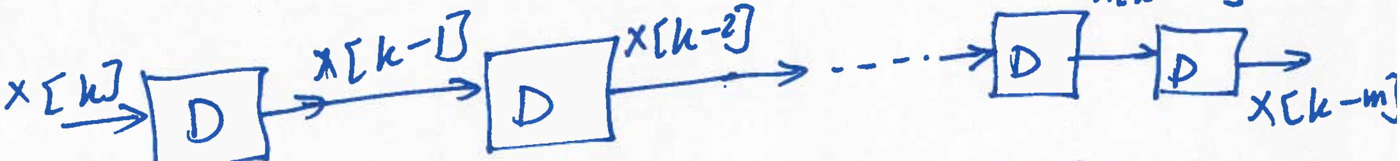
This is an identity operator. Physically, it represen
a wire or a buffer.

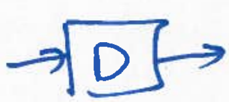


This is simply a one-unit delay where the output is the input in the previous sampling time. Physically, it is a shift register, i.e., a set of flip-flops like what you have seen in COE 212.



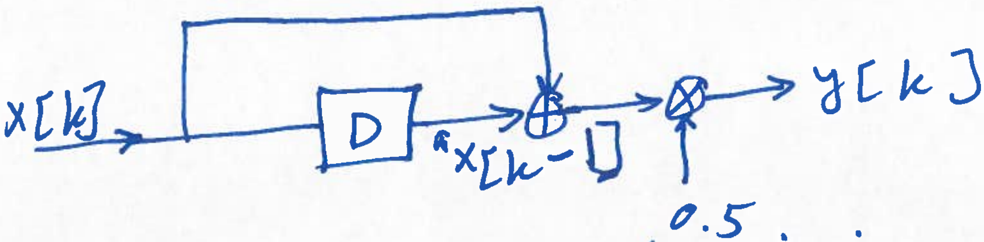
This is multi stage delay where the input is delayed by m time units. It can be implemented by cascading m stages of shift registers:



where  is a unit delay.

$$x[k] \to \boxed{S} \to y[k] = \frac{1}{2}(x[k] + x[k-1])$$

This system averages the input over two consecutive times. It can be implemented using a shift register and an adder and

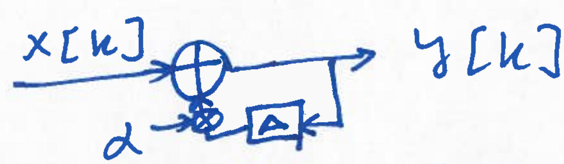


a multiplier. The multiplier is just a right-shift of the output of the adder (refresh your COEN 212 materials).

A system with feedback

$$x[k] \to \boxed{S} \to y[k] = \alpha y[k-1] + x[k]$$

The implementation is



In general, the output can depend on the input (at the current time) as well as M past inputs and M past outputs.

For the above generic LTI System can be represented as

$$y[k] = x[k] + \sum_{m=1}^{M} \alpha_m x[k-m] + \sum_{n=1}^{N} \beta_n y[k-n]$$

or

$$y[k] = \sum_{m=0}^{M} \alpha_m x[k-m] + \sum_{n=1}^{N} \beta_n y[k-n]$$

where $\alpha_0 = 1$.

Example 10.1

The DT sequence $x[k] = 2ku[k]$ is applied at the input of a DT system described by the following difference equation:

$$y[k+1] - 0.4y[k] = x[k].$$

By iterating the difference equation from the ancillary condition $y[-1] = 4$, compute the output response $y[k]$ of the DT system for $0 \le k \le 5$.

Solution

Express $y[k+1] - 0.4y[k] = x[k]$ as follows:

$$\begin{aligned} y[k] &= 0.4y[k-1] + x[k-1] \\ &= 0.4y[k-1] + 2(k-1)\,u(k-1) \quad \{\because x[k] = 2k\,u[k]\}, \end{aligned}$$

which can alternatively be expressed as

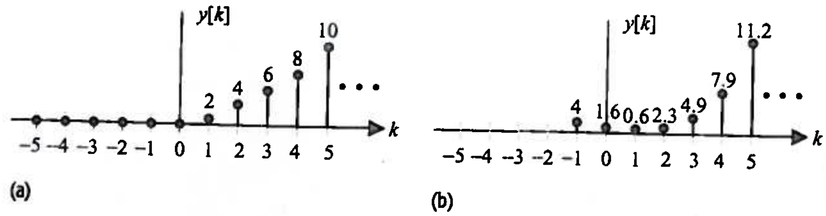
$$y[k] = \begin{cases} 0.4y[k-1] & k = 0 \\ 0.4y[k-1] + 2(k-1) & k \ge 1. \end{cases}$$

By iterating from $k = 0$, the output response is computed as follows:

$$\begin{aligned} y[0] &= 0.4y[-1] = 1.6, \\ y[1] &= 0.4y[0] + 2 \times 0 = 0.64, \\ y[2] &= 0.4y[1] + 2 \times 1 = 2.256, \\ y[3] &= 0.4y[2] + 2 \times 2 = 4.902, \\ y[4] &= 0.4y[3] + 2 \times 3 = 7.961, \\ y[5] &= 0.4y[4] + 2 \times 4 = 11.184. \end{aligned}$$

Additional values of the output sequence for $k > 5$ can be similarly evaluated from further iterations with respect to k. The input and output sequences are plotted in Fig. 10.1 for $0 \le k \le 5$.

Fig. 10.1. Input and output sequences for Example 10.1. (a) Input sequence $x[k]$; (b) output sequence $y[k]$.



Zero-state and zero-input response.

The response of a DT system can be divided into two parts:

- Zero-input response: $y_{zi}[k]$.
  This output is due to the initial conditions of the system. It is also called <u>natural response</u>.

- Zero-state response: $y_{zs}[k]$
  also called <u>forced</u> response is the result of applying the given input.

$$y[k] = y_{zi}[k] + y_{zs}[k]$$

To be completely accurate, we can say that the system is LTI if the initial condition of the system is zero. Otherwise, it is incrementally linear.

Example: Repeat example 10.1 above to Calculate i) the zero-input response $y_{zi}[k]$,
ii) the zero-state response $y_{zs}[k]$ and:
iii) the overall response $y[k]$ for $0 \le k \le 5$.

Solution

(i) The zero-input response of the system is obtained by solving the following difference equation:

$$y[k+1] - 0.4y[k] = x[k],$$

with input $x[k] = 0$ and ancillary condition $y[-1] = 4$. The difference equation reduces to

$$y_{zi}[k] = 0.4y_{zi}[k-1],$$

with ancillary condition $y_{zi}[-1] = 4$. Iterating for $k = 0, 1, 2, 3, 4$, and 5 yields

$$\begin{aligned}
y_{zi}[0] &= 0.4y_{zi}[-1] = 1.6,\\
y_{zi}[1] &= 0.4y_{zi}[0] = 0.64,\\
y_{zi}[2] &= 0.4y_{zi}[1] = 0.256,\\
y_{zi}[3] &= 0.4y_{zi}[2] = 0.1024,\\
y_{zi}[4] &= 0.4y_{zi}[3] = 0.0410,\\
y_{zi}[5] &= 0.4y_{zi}[4] = 0.0164.
\end{aligned}$$

(ii) The zero-state response of the system is calculated by solving the following difference equation:

$$y_{zs}[k] = 0.4y_{zs}[k-1] + 2(k-1)u[k-1],$$

with ancillary condition $y_{zs}[-1] = 0$. Iterating the difference equation for $k = 0, 1, 2, 3, 4$, and 5 yields

$$\begin{aligned}
y_{zs}[0] &= 0.4y_{zs}[-1] + 2 \times (-1) \times 0 = 0,\\
y_{zs}[1] &= 0.4y_{zs}[0] + 2 \times 0 \times 1 = 0,\\
y_{zs}[2] &= 0.4y_{zs}[1] + 2 \times 1 \times 1 = 2,\\
y_{zs}[3] &= 0.4y_{zs}[2] + 2 \times 2 \times 1 = 4.8,\\
y_{zs}[4] &= 0.4y_{zs}[3] + 2 \times 3 \times 1 = 7.92,\\
y_{zs}[5] &= 0.4y_{zs}[4] + 2 \times 4 \times 1 = 11.168.
\end{aligned}$$

(iii) Adding the zero-input and zero-state components obtained in parts (i) and (ii), yields

$$\begin{aligned}
y[0] &= y_{zi}[0] + y_{zs}[0] = 1.6,\\
y[1] &= y_{zi}[1] + y_{zs}[1] = 0.64,\\
y[2] &= y_{zi}[2] + y_{zs}[2] = 2.256,\\
y[3] &= y_{zi}[3] + y_{zs}[3] = 4.902,\\
y[4] &= y_{zi}[4] + y_{zs}[4] = 7.961,\\
y[5] &= y_{zi}[5] + y_{zs}[5] = 11.184.
\end{aligned}$$

Note that the overall output response $y[k]$ is identical to the output response obtained in Example 10.1. By iterating with respect to k, additional values for the output response $y[k]$ for $k > 5$ can be computed.