Noise in Analog Communication Systems:

Random Processes:

A random Process $X(t)$ is a function of time whose value at any given time $t = t_1, t_2, \ldots$ is a random variable.

example 1: A function generator whose output has an amplitude varying around a nominal value (due, say, to themal variations of the circuitry and their surrounding).

example 2: A binary source (a computer) generating $+A$ and $-A$ (or $+A$ and 0) volts representing logic 1 and 0, respectively. The probability of 1 and 0 being according to a probability law.

Transmission of Random processes through Linear Time-invariant (LTI) Systems

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = X(t) * h(t) \Longleftrightarrow Y(\omega) = X(\omega) H(\omega)$$

It can be shown that

$$R_y(\tau) = h(\tau) * h(-\tau) * R_x(\tau)$$

and

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega)$$

where $R_y(\tau)$ and $R_x(\tau)$ are the autocorrelation functions of $y(t)$ and $x(t)$, respectively; $S_y(\omega)$ and $S_x(\omega)$ are the power spectral densities (PSD) of $y(t)$ and $x(t)$, respectively

That is:

$$R_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha) h(\beta) R_x(\tau + \alpha - \beta)\, d\alpha\, d\beta$$

Definition of the autocorrelation and PSD function:

$$R_x(\tau) = E[x(t)x(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)x(t+\tau)\, p_{X(t)X(t+\tau)}(x_1,x_2)\, dx_1\, dx_2$$

and

$$S_x(\omega) = \mathcal{F}[R_x(\tau)] = \int_{-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau}\, d\tau$$

$$P_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\, d\omega = R_x(0)$$

and

$$R_x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega) e^{j\omega\tau}\, d\omega$$

Thermal Noise

Thermal noise is the result of thermal motion of electrons in a resistor. The power spectral density of thermal noise is flat over a large band of frequencies and given as:

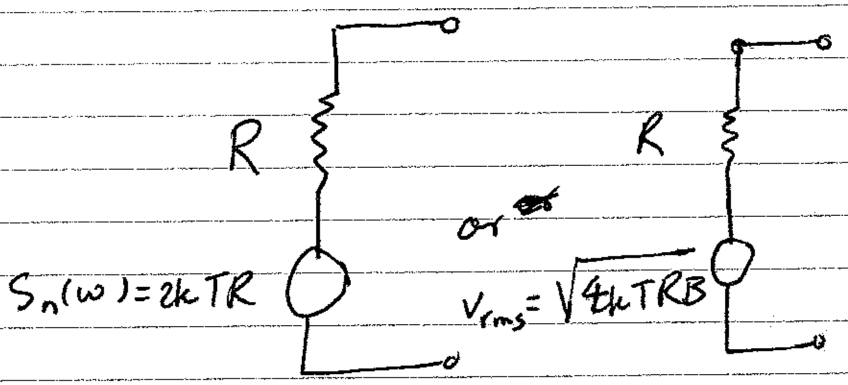
$$S_n(\omega) = 2kTR \Rightarrow$$ Power over a band of freq. Δf is $4kTR(\Delta f)$ or $4kTRB$

where k is the Bolzman Constant $= 1.38 \times 10^{-23}$

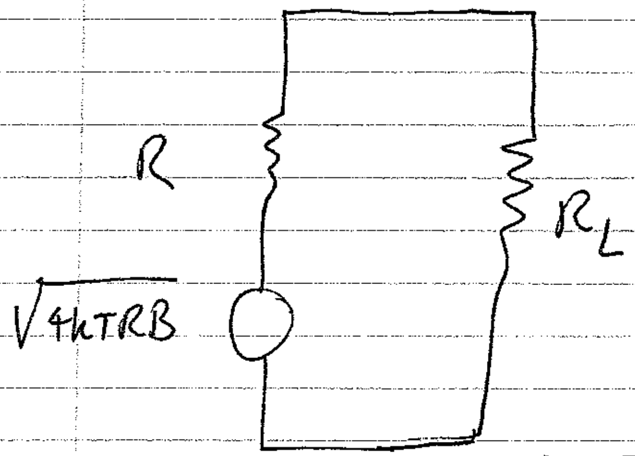
T is the ambient temperature in Kelvins (K)

and R is the resistance in Ohms.

We can present thermal noise (also called White noise) as:



Maximum Thermal noise delivered to a load



$$P_{out} = i^2 R_L = \frac{\left(\sqrt{4kTBR}\right)^2}{(R+R_L)^2} R_L = \frac{4kTBRR_L}{(R+R_L)^2}$$

P_{out} is maximized if $R_L = R$; i.e.,

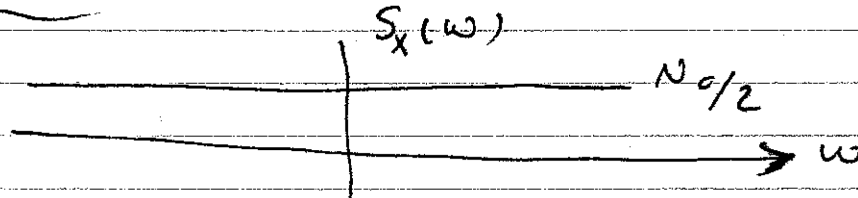
$$P_{max} = \frac{4kTBR^2}{4R^2} = kTB$$

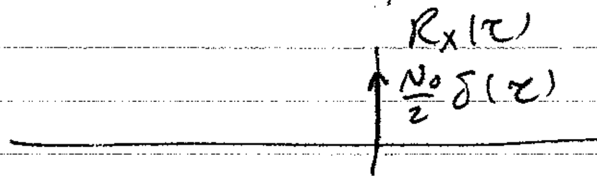
or maximum noise density is $\frac{kT}{2} \left(= \frac{kTB}{2B}\right)$

(this is two-sided noise density)

Usually one denotes noise density as

$\frac{N_0}{2}$, i.e., $\frac{N_0}{2} = \frac{kT}{2}$ or $N_0 = kT$.





Power Spectral Density of a modulated random signal (noise):

$y(t) = X(t)\cos(\omega_c t + \theta)$ where $\theta \in [0, 2\pi]$ and is uniformly distributed

$$S_y(\omega) = \frac{1}{4}\left[S_X(\omega - \omega_c) + S_X(\omega + \omega_c)\right]$$

$$R_y(\tau) = E[y(t)y(t+\tau)] = E[X(t)\cos(\omega_c t + \theta)X(t+\tau)\cos(\omega_c t + \omega_c\tau + \theta)]$$

$$R_y(\tau) = \frac{1}{2} R_X(\tau)\cos\omega_c\tau$$

so

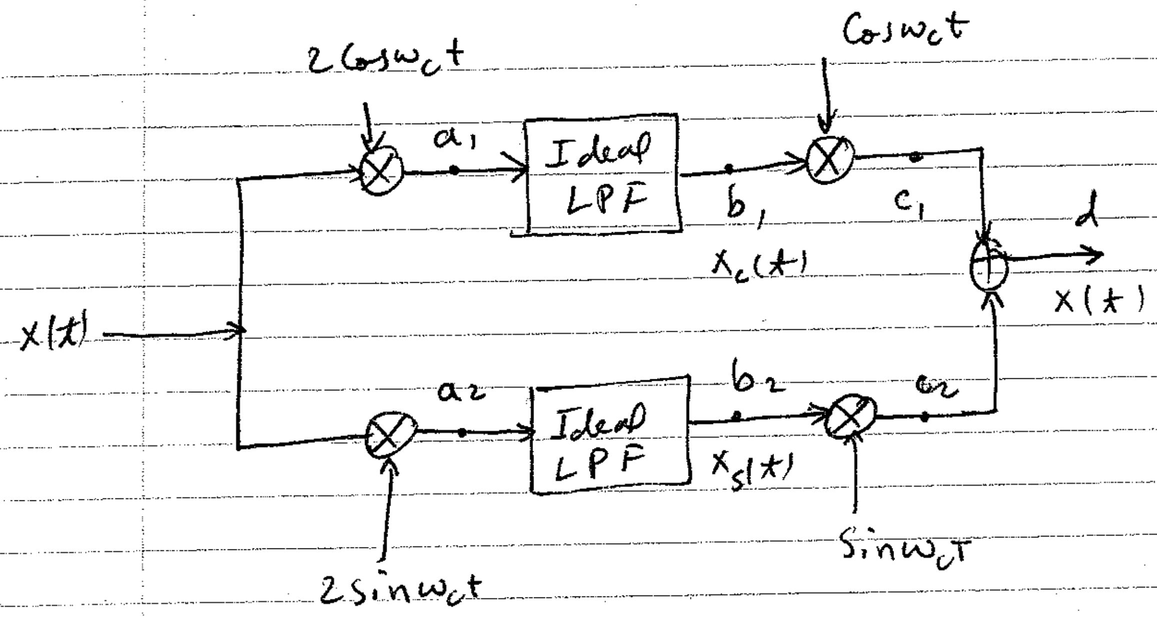
$$S_y(\omega) = \frac{1}{4}\left[S_X(\omega + \omega_c) + S_X(\omega - \omega_c)\right]$$

Bandpass Noise representation

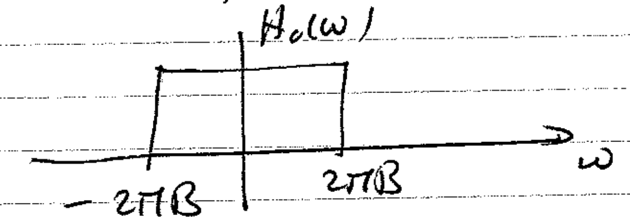
If the noise is bandpass, we can represent it as a combination of $X_c(t)$ and $X_s(t)$ as:

$$X(t) = X_c(t)\cos\omega_c t + X_s(t)\sin\omega_c t$$

To see this consider the following block diagram. This block Diagram corresponds to a BPF generated from a LPF.



where the ideal LPF has and impulse response $h_o(t)$ and a transfer function



let's find the impulse response of the "BPF":

if $x(t) = \delta(t-\alpha)$ then

at a_1 we have $2\delta(t-\alpha)\cos w_c \alpha$

" a_2 " " $2\delta(t-\alpha)\sin w_c \alpha$

" b_1 " " $2\cos w_c \alpha\, h_o(t-\alpha)$

" b_2 " " $2\sin w_c \alpha\, h_o(t-\alpha)$

" c_1 " " $2\cos w_c t \cos w_c \alpha\, h_o(t-\alpha)$

" c_2 " " $2\sin w_c t \sin w_c \alpha\, h_o(t-\alpha)$

at d the output is $2h_o(t-\alpha)\cos w_c(t-\alpha)$

So the response of the system to $\delta(t-\alpha)$ is

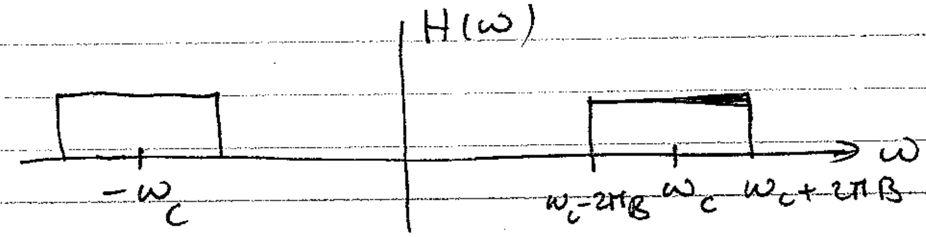
$$2h_c(t-\alpha)\cos\omega_c(t-\alpha)$$

So the impulse respusense of the system is

$$2h_o(t)\cos\omega_c t$$

or

$$H(\omega) = H_o(\omega-\omega_c) + H_o(\omega+\omega_c)$$



So, if we input a signal $x(t)$ with spectrum between $\omega_c - 2\pi B$ and $\omega_c + 2\pi B$ we get it out un-altered. So, if we call ~~$x_c(t)$~~ the signal at point b_1, $X_c(t)$ and simal at b_2 as $X_s(t)$, we have

$$X(t) = X_c(t)\cos\omega_c t + X_s(t)\sin\omega_c t$$

But $X_c(t) = \left[2X(t)\cos\omega_c t\right]_{LPF}$

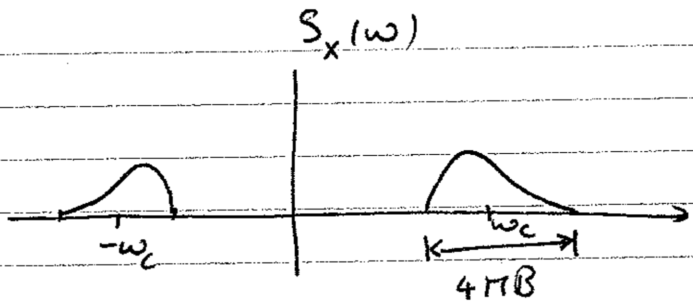
So:

$$S_{X_c}(\omega) = \begin{cases} S_X(\omega+\omega_c) + S_X(\omega-\omega_c) & |\omega| < 2\pi B \\ 0 & |\omega| > 2\pi B \end{cases}$$

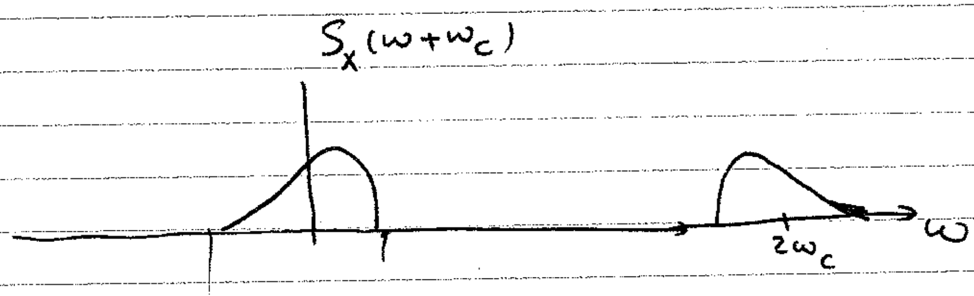
also

$$S_{X_s}(\omega) = S_{X_c}(\omega)$$

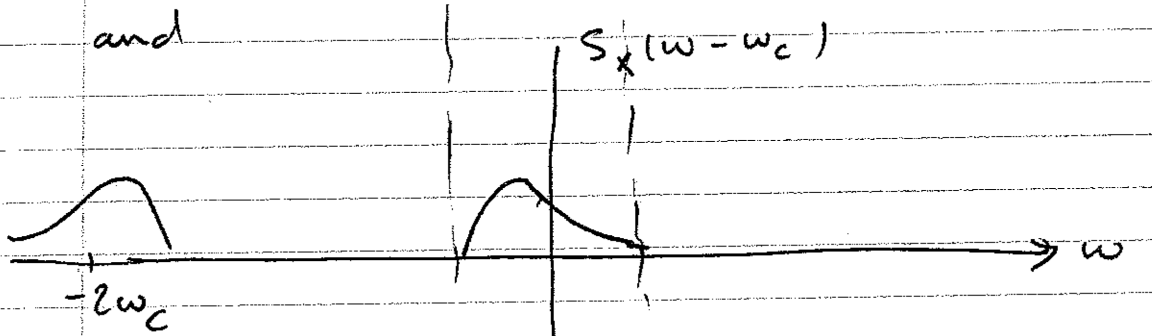
Let :



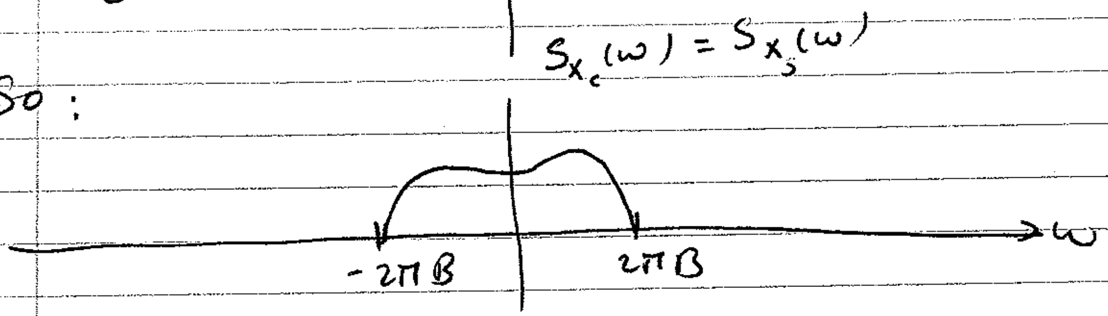
Then



and

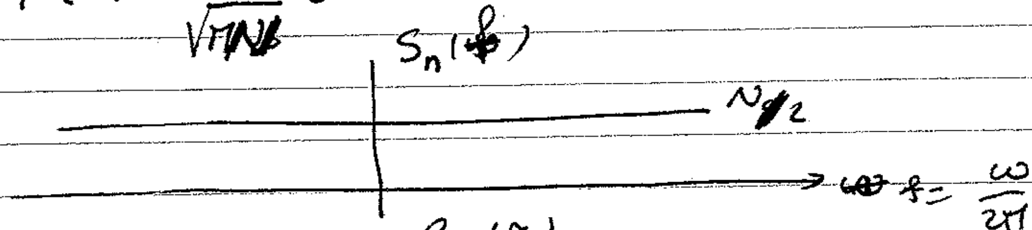


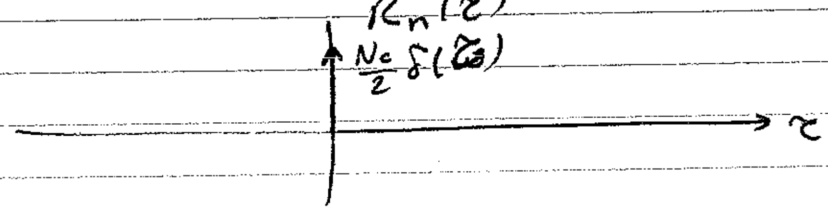
So :



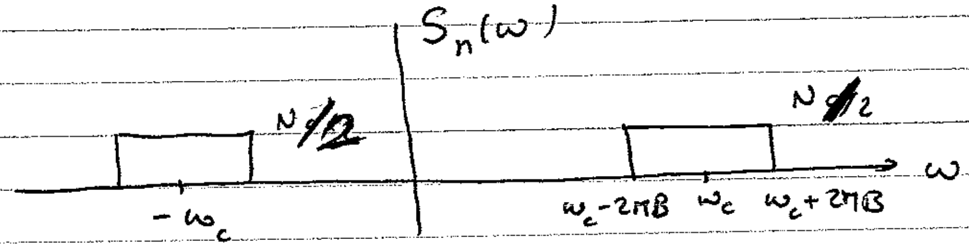
Gaussian White Random Process

$$P(n) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{n^2}{N_0}}$$





Bandpass (filtered) Gaussian Random Process



we can represent $n(t)$ as

$$n(t) = n_c(t)\cos\omega_c t + n_s(t)\sin\omega_c t$$

where

$$S_{n_c}(\omega) = S_{n_s}(\omega) = \begin{cases} N & |\omega| \leq 2\pi B \\ 0 & |\omega| > 2\pi B \end{cases}$$

$$\overline{n_c^2} = \overline{n_s^2} = \overline{n^2} = 2NB$$

we can also show $n(t)$ as

$$n(t) = E(t)\cos[\omega_c t + \theta(t)]$$

where

$$E(t) = \sqrt{n_c^2(t) + n_s^2(t)}$$

and

$$\theta(t) = -\tan^{-1}\frac{n_s(t)}{n_c(t)}$$

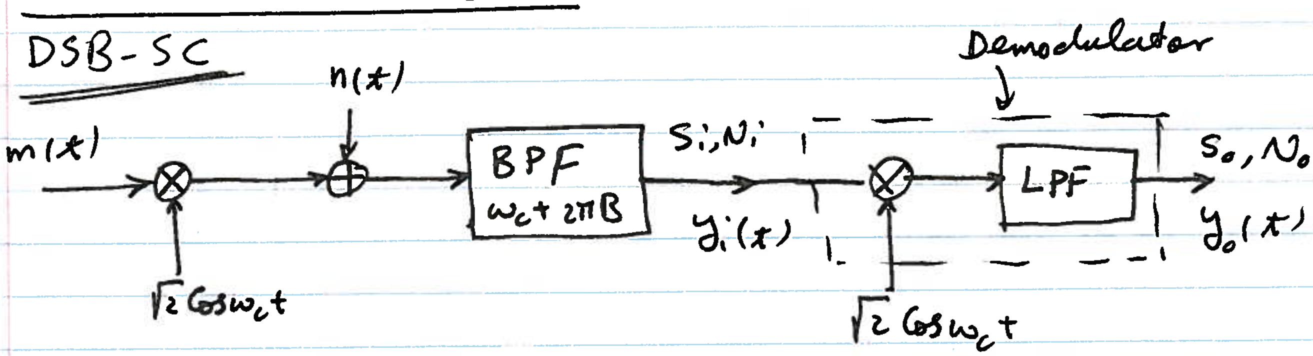
$$p_{n_c}(\alpha) = p_{n_s}(\alpha) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{\alpha^2}{2\sigma^2}}$$

Rayleigh Distribution

where $\sigma = 2NB$ and $P_E(E) = \dfrac{E}{\sigma^2}\, e^{-E^2/2\sigma^2}$

Noise in AM systems

DSB-SC



$$S_i = \overline{\left[\sqrt{2}\, m(t)\cos\omega_c t\right]^2} = 2\,\overline{\left(m(t)\cos\omega_c t\right)^2} = \overline{m^2(t)} = \overline{m^2}$$

$$y_i(t) = \sqrt{2}\, m(t)\cos\omega_c t + n(t)$$

$$= \left[\sqrt{2}\, m(t) + n_c(t)\right]\cos\omega_c t + n_s(t)\sin\omega_c t$$

$$y_o(t) = m(t) + \frac{1}{\sqrt{2}} n_c(t)$$

So:

$$S_o = \overline{m^2} = S_i$$

$$N_o = \frac{1}{2}\overline{n_c^2(t)}$$

but $\overline{n_c^2(t)} = \overline{n^2(t)} = 2NB$

So

$$N_o = NB$$

therefore, the output SNR is

$$\frac{S_o}{N_o} = \frac{S_i}{NB} = \gamma$$

But $N_i = \overline{n^2(t)} = 2NB$

So, the input SNR is

$$\frac{S_i}{N_i} = \frac{S_i}{2NB} = \frac{\overline{m^2}}{2NB}$$

or,

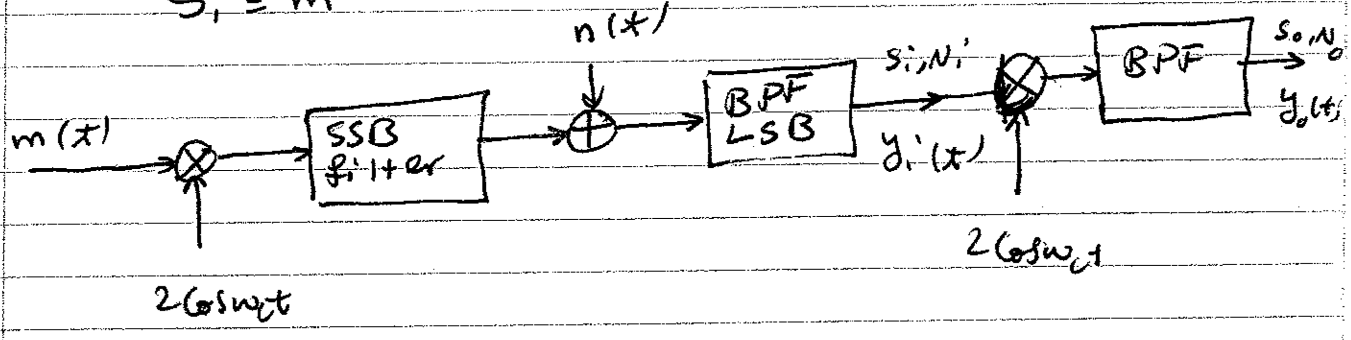
$$\frac{S_o}{N_o} = 2\left(\frac{S_i}{N_i}\right)$$

that is DSB-SC improves the SNR by a factor of 2 (3 dB).

SSB-SC

$$\varphi_{SSB}(t) = m(t)\cos w_c t + m_h(t)\sin w_c t$$

$$S_i = \overline{m^2}$$



$$y_i(t) = [m(t) + n_c(t)]\cos w_c t + [m_h(t) + n_s(t)]\sin w_c t$$

$$N_i = \frac{\overline{n_c^2(t)}}{2} + \frac{\overline{n_s^2(t)}}{2} = NB$$

$$\frac{S_i}{N_i} = \frac{\overline{m^2}}{NB}$$

$y_o(t) = m(t) + n_c(t)$

So: $S_o = \overline{m^2} = S_i$

$N_o = \overline{n_c^2} = NB$

So

$$\frac{S_o}{N_o} = \frac{S_i}{NB} = \frac{S_i}{N_i} = \gamma$$

Example:

In a DSB-SC System, $f_c = 500\,kHz$, $m(t)$ has a uniform PSD bandlimited to $4\,kHz$. The modulated signal is transmitted over a distortionless channel with a noise PSD of $S_n(\omega) = \dfrac{1}{\omega^2 + a^2}$ where $a = 10^6\pi$. The useful Signal Power at the receiver input is $1\,\mu W$. The received signal is bandpass filtered, multiplied by $2\cos\omega_c t$ and then low pass filtered to give $S_o(t) + n_o(t)$. Determine the output SNR.

$y_i(t) = [km(t) + n_c(t)]\cos\omega_c t + n_s(t)\sin\omega_c t$

Solution: $y_o(t) = S_o(t) + n_o(t) = km(t) + n_c(t)$

So: $S_o = k^2 \overline{m^2}$, and $N_o = \overline{n_c^2}$

But

$$S_i = \frac{k^2 \overline{m^2}}{2} = 10^{-6} \qquad (1\ \mu w)$$

So:

$$S_o = k^2 \overline{m^2} = 2 \times 10^{-6}$$

also

$$\overline{n_c^2} = \overline{n^2}$$

$$\overline{n^2} = 2 \int_{496,000}^{504,000} \frac{1}{\omega^2 + a^2}\, df = \frac{1}{\pi} \int_{(496000)(2\pi)}^{(504,000)(2\pi)} \frac{1}{\omega^2 + a^2}\, d\omega$$

$$\overline{n^2} = 8.25 \times 10^{-10} = N_o$$

Therefore,

$$\frac{S_o}{N_o} = \frac{2 \times 10^{-6}}{8.25 \times 10^{-10}} = 2.42 \times 10^{3}$$

or $$\frac{S_o}{N_o} = 33.83 \text{ dB}$$

Effect of Noise in FM

An FM signal is represented as

$$\varphi_{FM}(t) = A\cos[\omega_c t + k_f a(t)]$$

where $a(t) = \int_{-\infty}^{t} m(\alpha)\,d\alpha$

at the receiver side

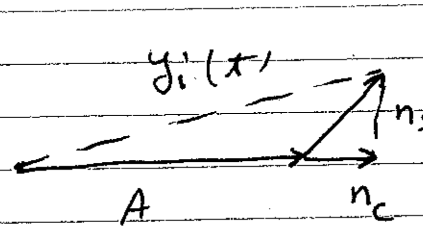
$$y_i(t) = A\cos[\omega_c t + k_f a(t)] + n_c(t)\cos\omega_c t + n_s(t)\sin\omega_c t$$

let's consider the effect of noise on the amplitude and angle of the FM signal.

For simplicity, let's ignor $k_f a(t)$ for the time being then, then

$$y_i(t) = A\cos\omega_c t + n_c(t)\cos\omega_c t + n_s(t)\sin\omega_c t$$

$$= [A + n_c(t)]\cos\omega_c t + n_s(t)\sin\omega_c t$$



$$y_i(t) = \sqrt{[A + n_c(t)]^2 + n_s^2(t)}\,\cos\left[\omega_c t + \operatorname{atan}\frac{n_s(t)}{A + n_c(t)}\right]$$

the amplitude is $\simeq A$ if $n_c(t) \ll A$ and $n_s(t) \ll A$

The receiver consists of:

$\rightarrow$ limiter $\rightarrow$ BPF $\rightarrow$ d/dt $\rightarrow$ envelope Detector $\rightarrow$

The output of limiter is

$$\cos\left[\omega_c t + \operatorname{atan}\frac{n_s(t)}{A+n_c(t)}\right] + \text{other terms}$$

output of BPF is

$$\cos\left[\omega_c t + \operatorname{atan}\frac{n_s(t)}{A+n_c(t)}\right]$$

The output of $\frac{d}{dt}$ {differentiator} is

$$-\left[\omega_c + \frac{d}{dt}\operatorname{atan}\frac{n_s(t)}{A+n_c(t)}\right]\sin\left[\omega_c t + \operatorname{atan}\frac{n_s(t)}{A+n_c(t)}\right]$$

and the output of the envelope detector is

$$n_{out} = \frac{d}{dt}\left\{\operatorname{atan}\frac{n_s(t)}{A+n_c(t)}\right\} \cong \frac{[A+n_c(t)]\frac{dn_s}{dt} - n_s(t)\frac{dn_c(t)}{dt}}{[A+n_c(t)]^2}$$

we have assumed $\frac{n_s(t)}{A+n_c(t)}$ is very small

so $\operatorname{atan}\frac{n_s(t)}{A+n_c(t)} \cong \frac{n_s(t)}{A+n_c(t)}$.

Using again the fact that $n_c(t) \ll A$ and $n_s(t) \ll A$

we get

$$n_{out} = \frac{\frac{dn_s(t)}{dt}}{A}$$

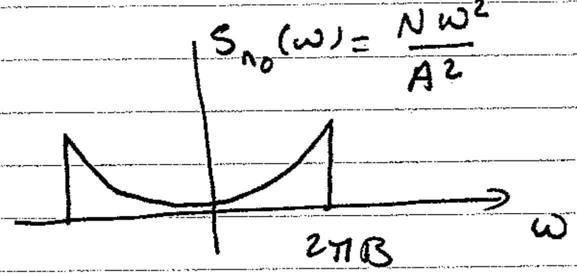
$$n_s \longrightarrow \boxed{j\omega/A} \longrightarrow n_{out}$$

It is equivalent to passing noise component $n_s(t)$ through a filter with transfer function

$$H(\omega) = j\frac{\omega}{A}$$

The power spectral density of the noise at the output is, then,

$$S_{n_0}(\omega) = S_{n_i}(\omega)|H(\omega)|^2 = \frac{N\omega^2}{A^2} \quad |\omega| \leq 2\pi B$$



Then the power of noise at the output is:

$$N_o = 2\int_0^B \frac{N}{A^2}(2\pi f)^2 df = \frac{8\pi^2 N B^3}{3A^2}$$

Signal Power at the output is

$$S_o = k_f^2 \overline{m^2}$$

since the output is $k_f m(t)$.

So:

$$\frac{S_o}{N_o} = 3\left(\frac{k_f^2 \overline{m^2}}{(2\pi B)^2}\right)\frac{A^2/2}{NB} = 3\left(\frac{k_f^2 \overline{m^2}}{(2\pi B)^2}\right)\gamma$$

where $\gamma = \frac{S_i}{N_i} = \frac{S_i}{NB} = \frac{A^2/2}{NB}$

since $\Delta\omega = k_f m_p$

or $2\pi \Delta f = k_f m_p \Rightarrow \Delta f = \dfrac{k_f m_p}{2\pi}$

then
$$\frac{S_o}{N_o} = 3\left[\frac{\Delta f}{B}\right]^2 \left(\frac{\overline{m^2}}{m_p^2}\right)\gamma$$
$$\frac{S_o}{N_o} = 3\beta^2 \gamma \left(\frac{\overline{m^2}}{m_p^2}\right)$$

Note the tradeoff between ~~frequency~~ bandwidth requirement and SNR improvement.

For tone modulation, i.e., when

$$m(t) = a \cos\omega_m t$$
$$m_p = a \quad \text{and} \quad \overline{m^2} = \frac{a^2}{2}$$

So:
$$\frac{S_o}{N_o} = \frac{3}{2}\beta^2 \gamma$$

Example:

for a zero-mean Gaussian Noise and also zero-mean Gaussian $m(t)$ find the SNR_o for FM.

$m_p = \infty$. but since for $\geq 3\sigma_m$ only 0.0027 of the signal probability is there

take $m_p = 3\sigma_m$

Then $\overline{m^2} = \sigma_m^2$ and $m_p^2 = (3\sigma_m)^2$

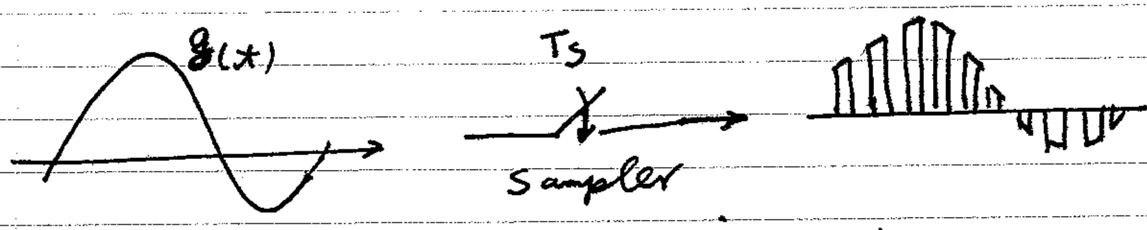
So:

$$\frac{S_o}{N_d} = \frac{1}{3}\beta^2\gamma$$

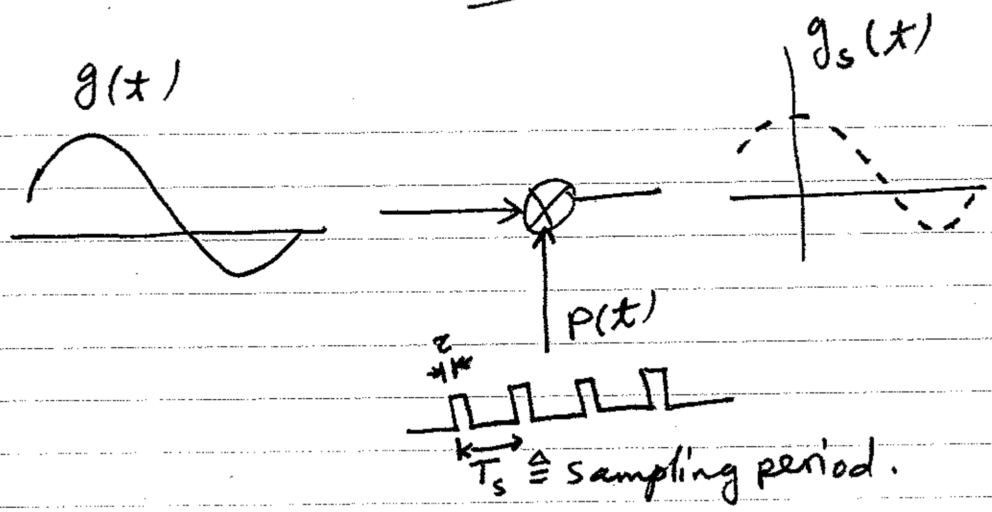
Sampling Theorem

An anolg Signal is time-Continuous and amplitude continuous. It is transformed into a discrete-time Signal by Sampling and then into a discrete-value signal by Quantization. The end result is a digital Signal.

Sampling is like gating, i.e., only keeping the Signal for some of the time instants and, hoping that the missed times can be accounted for through interpolation.



The function of sampler is equivalent to multiplying $g(t)$ by a pulse train



$f_s = \frac{1}{T_s}$ is called the sampling rate or sampling frequency.

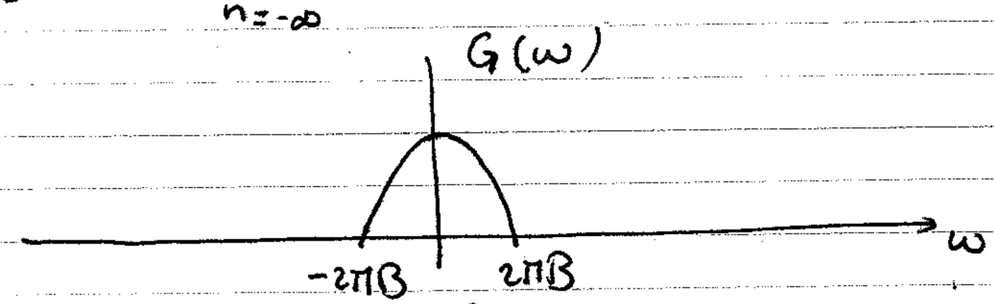
$p(t)$ is a periodic signal, so it can be represented as a Fourier series:

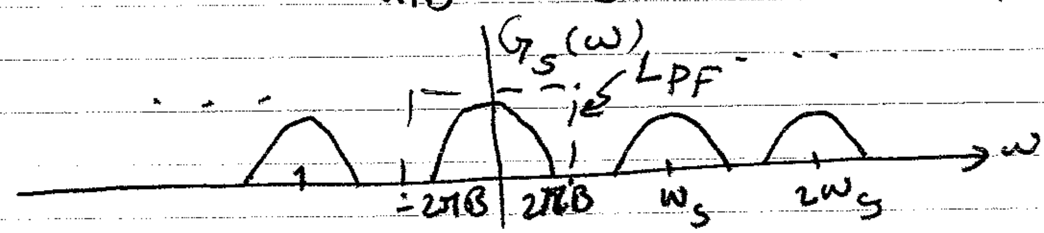
$$p(t) = \sum_{n \geq 0} C_n \cos n\omega_s t = \sum_{n \geq 0} C_n \cos 2\pi n f_s t$$

where $\omega_s = 2\pi f_s = \frac{2\pi}{T_s}$

$$g_s(t) = g(t) \sum_n C_n \cos(n\omega_s t) = \sum_n C_n\, g(t) \cos(n\omega_s t)$$

$$G_s(\omega) = \sum_{n=-\infty}^{\infty} C_n\, G(\omega - n\omega_s)$$





if $\tau \simeq 0$ then the sampling pulse tends to a delta (impulse) function

and $$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

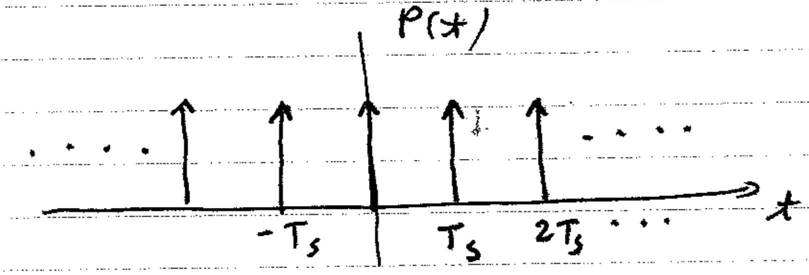
and

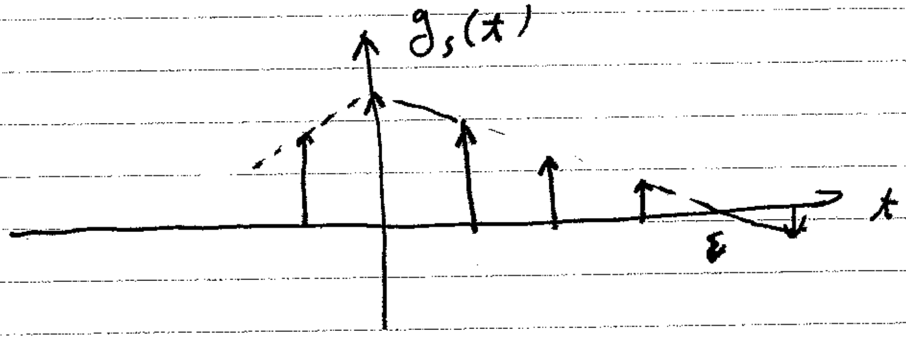
$$g_s(t) = g(t)p(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t-nT_s)$$

Fourier Series of $p(t)$ is

$$p(t) = \frac{1}{T_s}[1 + 2\cos\omega_s t + 2\cos 2\omega_s t + \cdots]$$

where $\omega_s = \frac{2\pi}{T_s} = 2\pi f_s$





$$g_s(t) = g(t)p(t) = \frac{1}{T_s}[g(t) + 2g(t)\cos\omega_s t + 2g(t)\cos 2\omega_s t + \cdots]$$

In order to be able to recover $g(t)$ from $g_s(t)$ [or $G(\omega)$ from $G_s(\omega)$] we need to have

$$\omega_s - 2\pi B \geq 2\pi B$$

or

$$\omega_s \geq 2\pi(2B) \quad \text{or} \quad \boxed{f_s \geq 2B}$$

2B is called the Nyquist rate and it is the minimum sampling rate required for sampling a signal with bandwidth B so that, we can recover the original signal by low pass filtering.

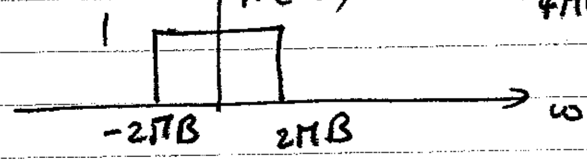
if $f_s < 2B$ or $T_s > \frac{1}{2B}$

there would be aliasing.

Reconstruction:

Low pass filtering by

$$H(\omega) = \text{rect}\left(\frac{\omega}{4\pi B}\right)$$



$$G(\omega) = G_s(\omega) H(\omega)$$

therefore

$$g(t) = g_s(t) * h(t)$$

where

$$h(t) = \mathcal{F}^{-1}[H(\omega)] = \frac{2\pi B}{\pi} \operatorname{Sinc}(2\pi B t)$$

$$= 2B \operatorname{Sinc}(2\pi B t)$$

$$= f_s \operatorname{Sinc}(2\pi B t)$$

$$g_s(t) = \sum_k g(kT_s) \operatorname{rect}\left(\frac{t-kT_s}{\tau}\right)$$

if $\tau \approx 0$ then $\operatorname{rect}\left(\frac{t-kT_s}{\tau}\right) \to \delta(t-kT_s)$

or $$g_s(t) = \sum_k g(kT_s)\delta(t-kT_s)$$

So:

$$g(t) = \left[\sum_k g(kT_s)\delta(t-kT_s)\right] * h(t)$$

$$= \sum_k g(kT_s) h(t-kT_s)$$

$$= f_s \sum_k g(kT_s) \operatorname{Sinc}(2\pi B(t-kT_s))$$

$$= f_s \sum g(kT_s) \operatorname{Sinc}(2\pi B t - k\pi)$$

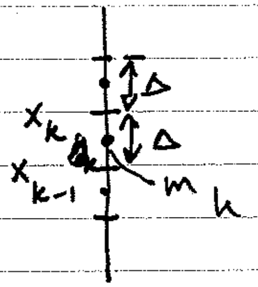
Example: Voice Signals ~~maximum have~~ occupy a frequency band from 0 to 3.2 kHz. So, the minimum sampling rate required is 6.4 kSamples/sec. However, in order to make the low pass filtering easier a sampling rate of 8 kSamples/sec. is used, i.e., one sample every $\frac{1}{8000} = 0.125$ msec. or $125\ \mu sec. = T_s$ one sample is taken.

Quantization

Each sample taken (every T_s seconds) can have any value (in the dynamic range of the signal). However, a digital system, e.g., a computer, can only ~~tak~~ take finite number of bits. So, we ~~need to~~ can assign only a limited number, say R, bits/sample. That is, we need to quantize the samples using $L = 2^R$ levels.

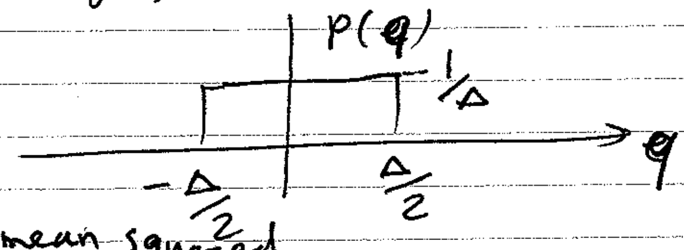
So, if we have a dynamic range of $\pm m_p$, we divide (in uniform quantization) this range into L sections of length:

$$\Delta = \frac{2m_p}{L} = \frac{2m_p}{2^R}$$



any ~~sam~~ sample falling in the rang $[x_{k-1}, x_k]$ will be reproduced as m_k, so, the error will be $q = (m - m_k)$ and will be between $\pm \frac{\Delta}{2}$

if we assume that q is uniformly distributed in this range, i.e.



then the mean squared quantization noise will be:

$$\overline{q^2} = \int_{-\Delta/2}^{\Delta/2} p(q)\, q^2\, dq = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} q^2\, dq = \frac{\Delta^2}{12}$$

or
$$\overline{q^2} = \frac{\left(\frac{2m_p}{L}\right)^2}{12} = \frac{m_p^2}{3L^2} = \frac{m_p^2}{3 \times 2^{2R}}$$

SQNR (Signal-to-Quantization-Noise-Ratio) is:

$$SQNR = \frac{\overline{m^2}}{\overline{q^2}} = \frac{\overline{m^2}}{m_p^2} \times 3 \times 2^{2R}$$

in dB, we have

$$SQNR_{dB} = 10 \log SQNR = 10 \log\left(\frac{3\overline{m^2}}{m_p^2}\right) + 20R \log 2$$

$$SQNR_{dB} = 10 \log\left(\frac{3\overline{m^2}}{m_p^2}\right) + 6R = 10 \log 3\frac{m_{rms}^2}{m_p^2} + 6R$$

that is, adding each extra bit gains 6 dB in SQNR.

<u>example</u>: for a sinusoidal signal

$$\overline{m^2} = m_{rms}^2 = \frac{m_p^2}{2}$$

so,

$$SQNR_{dB} = 10 \log\left(\frac{3}{2}\right) + 6R = 1.76 + 6R$$

X A uniform quantizer is optimal for a uniformly distributed input. However, its simplicity results in using it for other sources.

Example:

Using a uniform quantizer for a Gaussian source: Take a Gaussian source with zero-mean and variance σ^2. Then

$$p(m) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{m^2}{2\sigma^2}}$$

$$\overline{m^2} = \int_{-\infty}^{\infty} m^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{m^2}{2\sigma^2}} dm = \sigma^2 \Rightarrow m_{rms} = \sigma$$

m takes values from $-\infty$ to ∞ so, $m_p = \infty$ (in theory), however, in practice probability that $|m| > 3\sigma$ is

$$P_r(|m| > 3\sigma) = 2\int_{3\sigma}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{m^2}{2\sigma^2}} dm = 0.0027$$

let $x = \frac{m}{\sigma}$ then

$$P_r(|m| > 3\sigma) = 2\int_{3}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 2Q(3) = 0.0027$$

See page 454 for tabulation of $Q(x)$.

use: $Q(x) \simeq \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}$ for $x \gg 1$

or tables for finding $Q(x)$ values.

if we take $m_p \simeq 3\sigma$ then

$$SQNR = 10\log\left(3\frac{\sigma^2}{9\sigma^2}\right) + 6R = -4.77 + 6R$$

(this is called <u>granular</u> quantization noise)

The distortion resulting from saturation, called the <u>overload</u> noise is:

$$D_{OL} = 2\int_{m_p}^{\infty} (x - m_p)^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx$$

<u>Example</u>: Digital voice:

sampled at 8ks/sec

each sample being assigned 8 bits

So the bit rate for a single voice channel, called a PCM channel, a DS0, T_0, is <u>64kbps</u>.

The SQNR $\simeq$ 48 dB (theoretical)

Example: Digital Audio

For music (studio quality), it is assumed that the frequencies upto 18 or 20 kHz is important for having good quality for (most) sensitive ears. So, the sampling rate is

44k samples/sec.

and 16 bits/sample is used for ADC.

So the bit rate is // SQNR is better than 90 dB (96 dB)

$2 \times 44 \times 16 = 1408$ kbps or 1.4 Mbps.

a CD-ROM can take 640 Mbytes

So the amount of music on a CD-ROM is

$$\frac{640 \times 8}{1.4} = 3636 \text{ sec.} \simeq 1 \text{ Hr.}$$

(not compressed using, e.g., MP3).

MP3 $\approx$ 128 kbps so,

$$\frac{640 \times 8}{0.128} \simeq 11 \text{ Hrs.}$$

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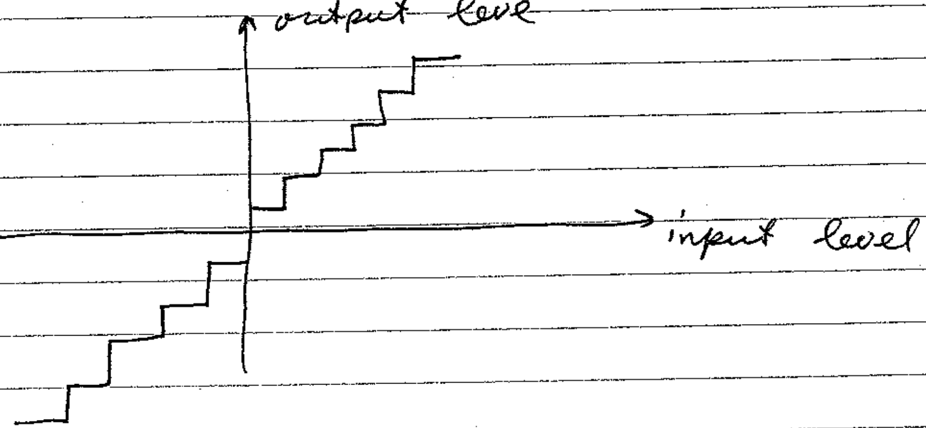
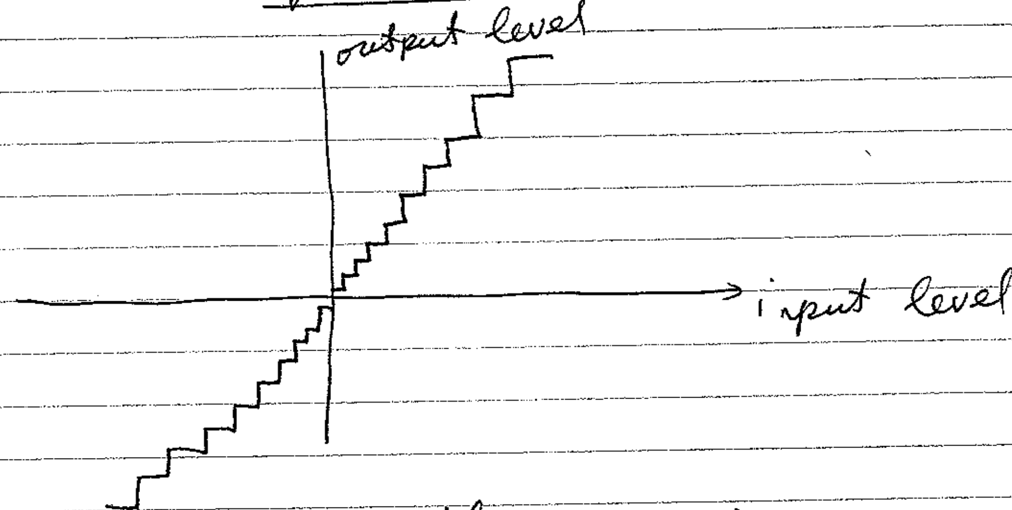
Table 10.2
$Q(x)$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0000	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
.1000	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
.2000	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
.3000	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
.4000	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
.5000	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
.6000	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
.7000	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
.8000	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
.9000	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.000	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.100	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.200	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.9853E-01
1.300	.9680E-01	.9510E-01	.9342E-01	.9176E-01	.9012E-01	.8851E-01	.8691E-01	.8534E-01	.8379E-01	.8226E-01
1.400	.8076E-01	.7927E-01	.7780E-01	.7636E-01	.7493E-01	.7353E-01	.7215E-01	.7078E-01	.6944E-01	.6811E-01
1.500	.6681E-01	.6552E-01	.6426E-01	.6301E-01	.6178E-01	.6057E-01	.5938E-01	.5821E-01	.5705E-01	.5592E-01
1.600	.5480E-01	.5370E-01	.5262E-01	.5155E-01	.5050E-01	.4947E-01	.4846E-01	.4746E-01	.4648E-01	.4551E-01
1.700	.4457E-01	.4363E-01	.4272E-01	.4182E-01	.4093E-01	.4006E-01	.3920E-01	.3836E-01	.3754E-01	.3673E-01
1.800	.3593E-01	.3515E-01	.3438E-01	.3362E-01	.3288E-01	.3216E-01	.3144E-01	.3074E-01	.3005E-01	.2938E-01
1.900	.2872E-01	.2807E-01	.2743E-01	.2680E-01	.2619E-01	.2559E-01	.2500E-01	.2442E-01	.2385E-01	.2330E-01
2.000	.2275E-01	.2222E-01	.2169E-01	.2118E-01	.2068E-01	.2018E-01	.1970E-01	.1923E-01	.1876E-01	.1831E-01
2.100	.1786E-01	.1743E-01	.1700E-01	.1659E-01	.1618E-01	.1578E-01	.1539E-01	.1500E-01	.1463E-01	.1426E-01
2.200	.1390E-01	.1355E-01	.1321E-01	.1287E-01	.1255E-01	.1222E-01	.1191E-01	.1160E-01	.1130E-01	.1101E-01
2.300	.1072E-01	.1044E-01	.1017E-01	.9903E-02	.9642E-02	.9387E-02	.9137E-02	.8894E-02	.8656E-02	.8424E-02
2.400	.8198E-02	.7976E-02	.7760E-02	.7549E-02	.7344E-02	.7143E-02	.6947E-02	.6756E-02	.6569E-02	.6387E-02
2.500	.6210E-02	.6037E-02	.5868E-02	.5703E-02	.5543E-02	.5386E-02	.5234E-02	.5085E-02	.4940E-02	.4799E-02
2.600	.4661E-02	.4527E-02	.4396E-02	.4269E-02	.4145E-02	.4025E-02	.3907E-02	.3793E-02	.3681E-02	.3573E-02
2.700	.3467E-02	.3364E-02	.3264E-02	.3167E-02	.3072E-02	.2980E-02	.2890E-02	.2803E-02	.2718E-02	.2635E-02
2.800	.2555E-02	.2477E-02	.2401E-02	.2327E-02	.2256E-02	.2186E-02	.2118E-02	.2052E-02	.1988E-02	.1926E-02
2.900	.1866E-02	.1807E-02	.1750E-02	.1695E-02	.1641E-02	.1589E-02	.1538E-02	.1489E-02	.1441E-02	.1395E-02
3.000	.1350E-02	.1306E-02	.1264E-02	.1223E-02	.1183E-02	.1144E-02	.1107E-02	.1070E-02	.1035E-02	.1001E-02
3.100	.9676E-03	.9354E-03	.9043E-03	.8740E-03	.8447E-03	.8164E-03	.7888E-03	.7622E-03	.7364E-03	.7114E-03
3.200	.6871E-03	.6637E-03	.6410E-03	.6190E-03	.5976E-03	.5770E-03	.5571E-03	.5377E-03	.5190E-03	.5009E-03
3.300	.4834E-03	.4665E-03	.4501E-03	.4342E-03	.4189E-03	.4041E-03	.3897E-03	.3758E-03	.3624E-03	.3495E-03
3.400	.3369E-03	.3248E-03	.3131E-03	.3018E-03	.2909E-03	.2802E-03	.2701E-03	.2602E-03	.2507E-03	.2415E-03
3.500	.2326E-03	.2241E-03	.2158E-03	.2078E-03	.2001E-03	.1926E-03	.1854E-03	.1785E-03	.1718E-03	.1653E-03
3.600	.1591E-03	.1531E-03	.1473E-03	.1417E-03	.1363E-03	.1311E-03	.1261E-03	.1213E-03	.1166E-03	.1121E-03
3.700	.1078E-03	.1036E-03	.9961E-04	.9574E-04	.9201E-04	.8842E-04	.8496E-04	.8162E-04	.7841E-04	.7532E-04
3.800	.7235E-04	.6948E-04	.6673E-04	.6407E-04	.6152E-04	.5906E-04	.5669E-04	.5442E-04	.5223E-04	.5012E-04
3.900	.4810E-04	.4615E-04	.4427E-04	.4247E-04	.4074E-04	.3908E-04	.3747E-04	.3594E-04	.3446E-04	.3304E-04
4.000	.3167E-04	.3036E-04	.2910E-04	.2789E-04	.2673E-04	.2561E-04	.2454E-04	.2351E-04	.2252E-04	.2157E-04
4.100	.2066E-04	.1978E-04	.1894E-04	.1814E-04	.1737E-04	.1662E-04	.1591E-04	.1523E-04	.1458E-04	.1395E-04
4.200	.1335E-04	.1277E-04	.1222E-04	.1168E-04	.1118E-04	.1069E-04	.1022E-04	.9774E-05	.9345E-05	.8934E-05
4.300	.8540E-05	.8163E-05	.7801E-05	.7455E-05	.7124E-05	.8807E-05	.6503E-05	.6212E-05	.5934E-05	.5668E-05
4.400	.5413E-05	.5169E-05	.4935E-05	.4712E-05	.4498E-05	.4294E-05	.4098E-05	.3911E-05	.3732E-05	.3561E-05
4.500	.3398E-05	.3241E-05	.3092E-05	.2949E-05	.2813E-05	.2682E-05	.2558E-05	.2439E-05	.2325E-05	.2216E-05
4.600	.2112E-05	.2013E-05	.1919E-05	.1828E-05	.1742E-05	.1660E-05	.1581E-05	.1506E-05	.1434E-05	.1366E-05
4.700	.1301E-05	.1239E-05	.1179E-05	.1123E-05	.1069E-05	.1017E-05	.9680E-06	.9211E-06	.8765E-06	.8339E-06
4.800	.7933E-06	.7547E-06	.7178E-06	.6827E-06	.6492E-06	.6173E-06	.5869E-06	.5580E-06	.5304E-06	.5042E-06
4.900	.4792E-06	.4554E-06	.4327E-06	.4111E-06	.3906E-06	.3711E-06	.3525E-06	.3448E-06	.3179E-06	.3019E-06
5.000	.2867E-06	.2722E-06	.2584E-06	.2452E-06	.2328E-06	.2209E-06	.2096E-06	.1989E-06	.1887E-06	.1790E-06
5.100	.1698E-06	.1611E-06	.1528E-06	.1449E-06	.1374E-06	.1302E-06	.1235E-06	.1170E-06	.1109E-06	.1051E-06

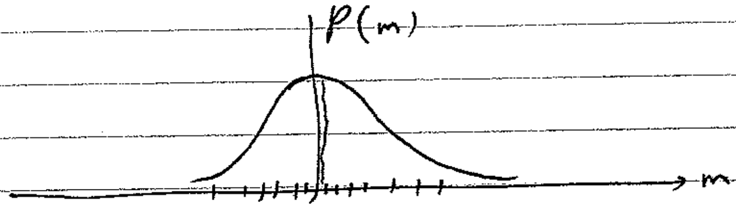
Non-uniform Quantization



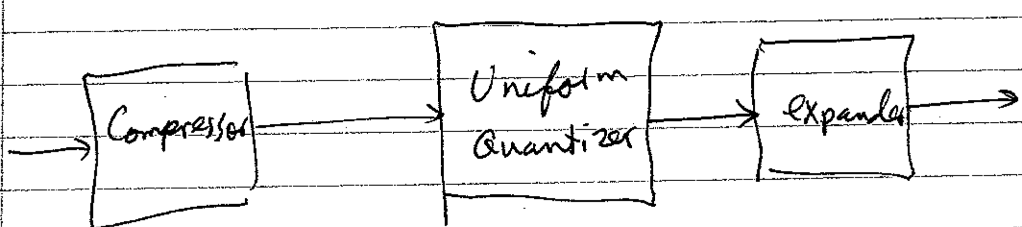
Uniform Quantizer

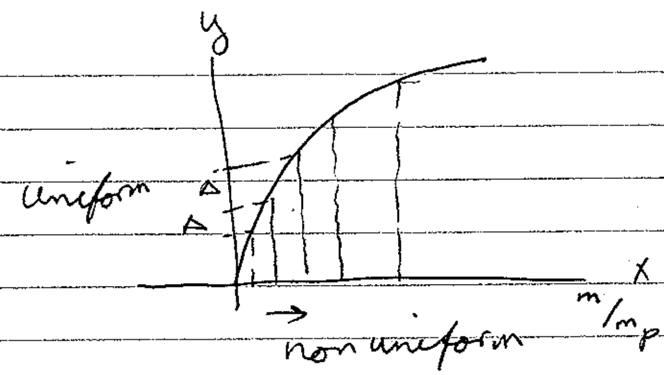


non-uniform Quantizer



COMPANDING : Compressing / Expanding





Therere two Companding laws:

μ - Law North America

A - Law rest of the world

μ - Law:

$$y = \frac{1}{\ln(1+\mu)} \ln\left(1 + \frac{\mu m}{m_p}\right) \qquad 0 \leq \frac{m}{m_p} \leq 1$$

A - law

$$y = \begin{cases} \dfrac{A}{1+\ln A}\left(\dfrac{m}{m_p}\right) & 0 \leq \dfrac{m}{m_p} \leq \dfrac{1}{A} \\ \dfrac{1}{1+\ln A}\left(1+\ln\dfrac{Am}{m_p}\right) & \end{cases}$$

For μ - Law the SNR is approximated as:

$$SNR \cong \frac{3L^2}{[\ln(1+\mu)]^2} \qquad \mu^2 \gg \frac{m_p^2}{m^2}$$

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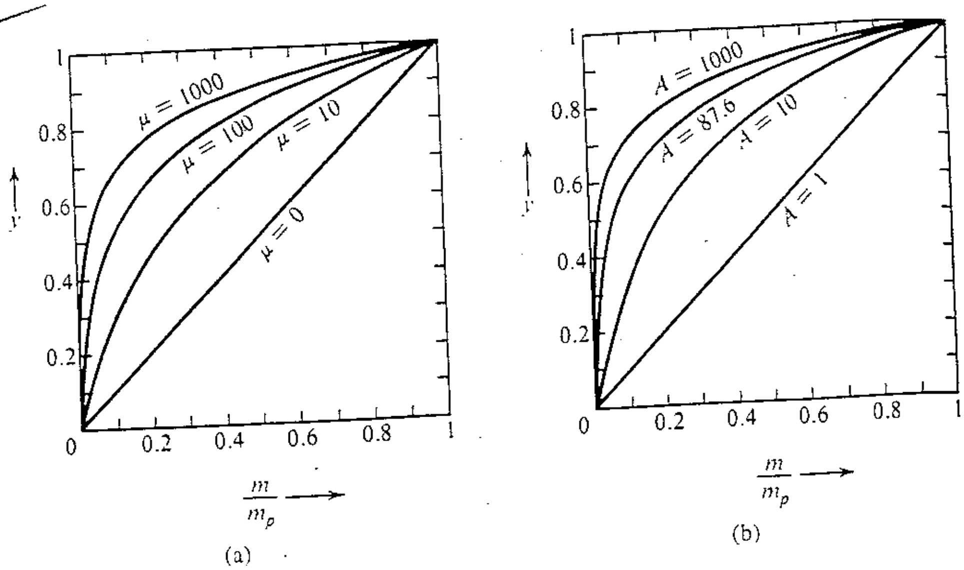


Figure 6.12 (a) μ-law characteristic. (b) A-law characteristic.

Figure 6.14 Piecewise linear compressor characteristic.

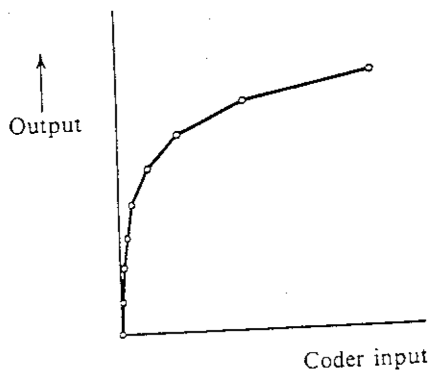
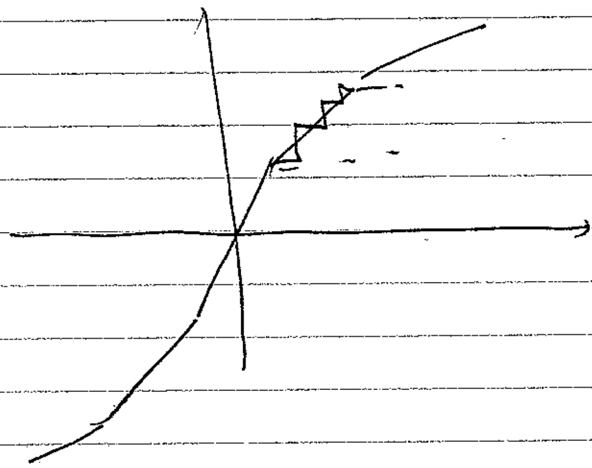


TABLE 3.4 *The 15-segment companding characteristic ($\mu = 255$)*

Linear Segment Number	*Step-Size*	*Projections of Segment End Points onto the Horizontal Axis*
0	2	±31
1a, 1b	4	±95
2a, 2b	8	±223
3a, 3b	16	±479
4a, 4b	32	±991
5a, 5b	64	±2015
6a, 6b	128	±4063
7a, 7b	256	±8159

Piece-wise linear approximation of PCM



each level is represented by 8-bits

$$\underbrace{b_7}_{\text{sign}}\ \underbrace{b_6\, b_5\, b_4}_{\text{segment}}\ \underbrace{b_3\, b_2\, b_1\, b_0}_{\text{place on the segment}}$$

example: Find the SNR for an 8-bit PCM quantizer, take $\mu = 255$

$$R = 8 \Rightarrow L = 2^R = 256$$

$$SNR = \frac{3(256)^2}{[\ln(1+255)]^2} \approx 6399 \approx 38.06 \text{ dB}$$

Do the same for $R = 6$ bits/sample.

$$R = 6 \Rightarrow L = 2^6 = 64$$

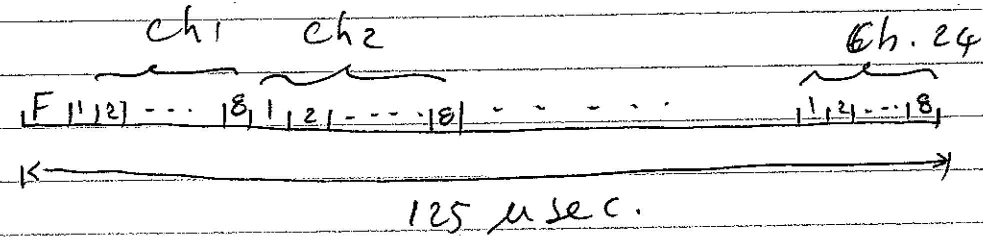
$$SNR = \frac{3(64)^2}{(\ln 256)^2} = 399.63 \approx 26 \text{ dB}$$

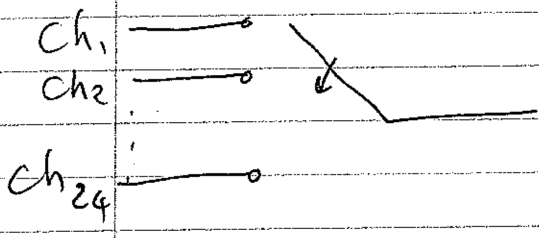
12 dB worse (6 dB degradation per bit omitted)

but the bit rate is only $\frac{2}{8} = \frac{1}{4}$ or 25% less (not wise saving), e.g., from 64kbps to 48kbps.

T_1 Carrier

24 PCM channels





$$\text{Total Rate of } T_1 = \frac{193}{125\times10^{-6}} = 1.544 \text{ Mbps}$$

$$\text{Useful rate} = \frac{192}{125\times10^{-6}} = 1.536$$

$$\text{efficiency } \eta = \frac{1.536}{1.544} = 99.5\%$$

→ $\boxed{F:}$ represent framing inf as:

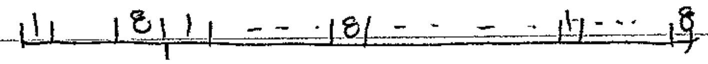
100011011100

Signaling: The LSB of each channel is robbed every 6th frame. So the number of bits/sample is actually $7\frac{5}{6}$ instead of 8.

So, the real efficiency is

$\boxed{188 = 193 - 1 - 4}$ $\quad \eta = \frac{188}{193} = 97.4\%$ // and actual rate is 1.504 Mbps.

E1 or G703 Standard



$32 \times 8 = 256$ bits/frame

$30 \times 8 = 240$ bits voice (information)

16 bits synchronization and framing.

Total rate $\dfrac{256}{125 \times 10^{-6}} = 2.048$ Mbps

effective rate $= \dfrac{240}{125 \times 10^{-6}} = 1.92$ Mbps

efficiency

$$\eta = \frac{240}{256} = \frac{1.92}{2.048} = 93.75\%$$

Digital Transmission Hierarchy

		PCM
DS0 (PCM or T_0)	64 kbps	1
DS1 (T1)	1.544 Mbps	24
DS2 (T2)	6.312 Mbps	96 ($4T_1$)
DS3 (T3)	44.736 Mbps	672
DS4 (T4)	274.176 Mbps	4032

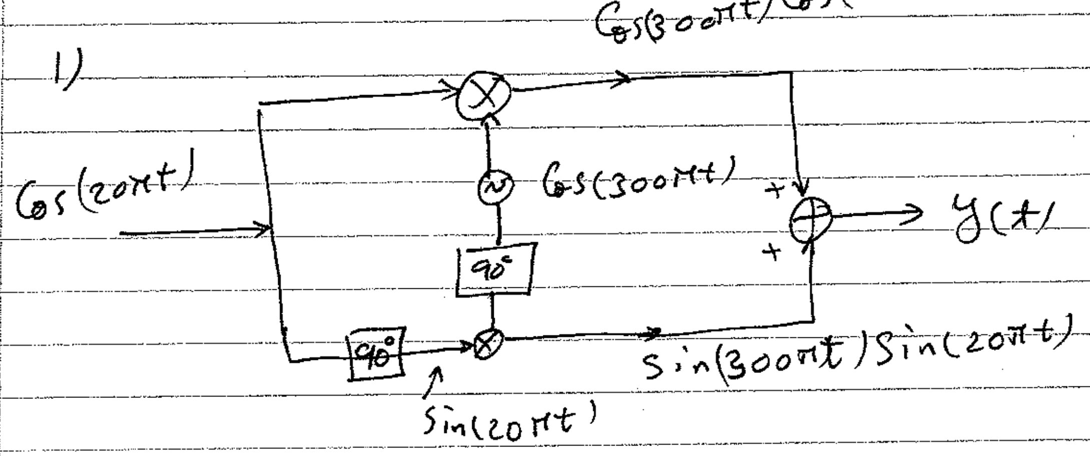
optical Hierarchy

OC-1	51.84 Mbps
OC-3	155.52 "
OC-12	622.08 Mbps
OC-48	2488.32 Mbps (2 Gbps)
OC-192	9953.28 Mbps (10 Gbps)

Review Problems:

Last year's Midterm:

1)



$$y(t) = \cos 300\pi t \cos 20\pi t + \sin 300\pi t \sin 20\pi t$$

$$y(t) = \cos(300\pi - 20\pi)t = \cos(280\pi t)$$

2)

a) $2 \times 5 = 10$ kHz.

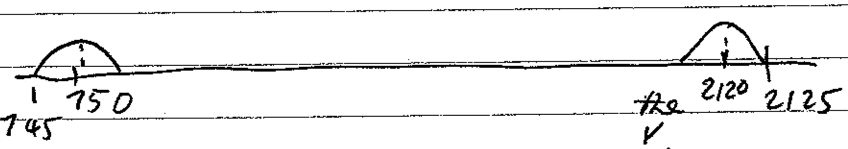
$$\frac{2125 - 745}{10} = 138 \text{ stations}$$

b) $750 + 625 = 1375$ kHz or $750 - 625 = 125$ kHz

$2120 + 625 = 2745$ kHz. or $2120 - 625 = 1495$ kHz

answer 1375 to 2745 kHz.

c)



$750 + 2 \times 625 = 2000$ for the lowest channel

$2120 + 2 \times 625 = 3370$ for the highest "

3)

a) $BW = 2(\beta+1) f_m$

$f_m = 50$

$BW = 1200 - 800 = 400$

$400 = 2(\beta+1) \times 50 \Rightarrow \beta = 3$

$\beta = \dfrac{a k_f}{f_m}$

$3 = \dfrac{2 \times k_f}{50} \Rightarrow k_f = 75$ Hz./V

b) $BW = 1350 - 650 = 700$

$700 = 2(\beta+1) \times 50 \Rightarrow \beta = 6$

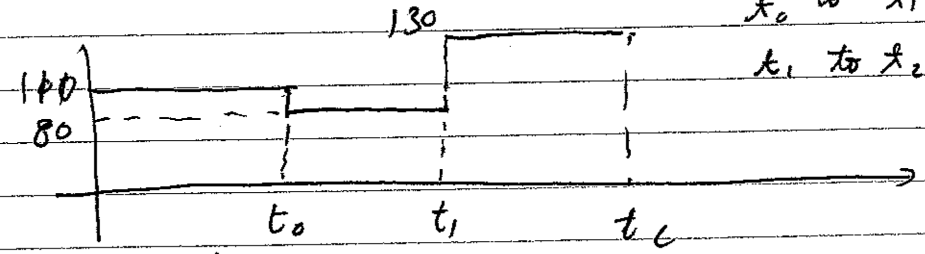
$\beta = \dfrac{a k_f}{f_m} \Rightarrow 6 = \dfrac{a(75)}{50} \Rightarrow a = \dfrac{300}{75} = 4$ V

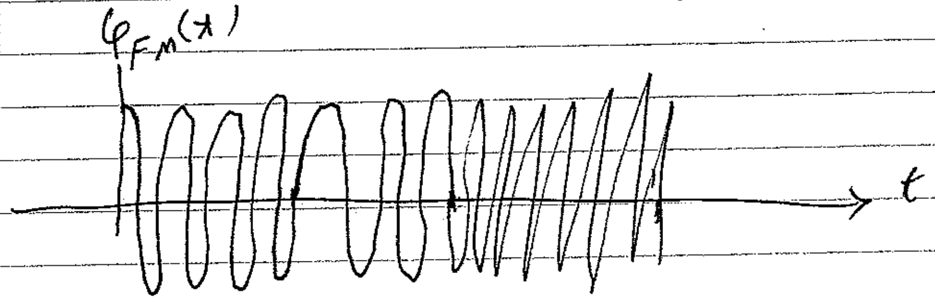
4) $f(i) = f_c + k_f\, s(t)$

0 to t_0 : $10^5 + 10^4 = 110$ kHz

t_0 to t_1 : $10^5 - 2\times10^4 = 80$ kHz

t_1 to t_2 : $10^5 + 3\times10^4 = 130$



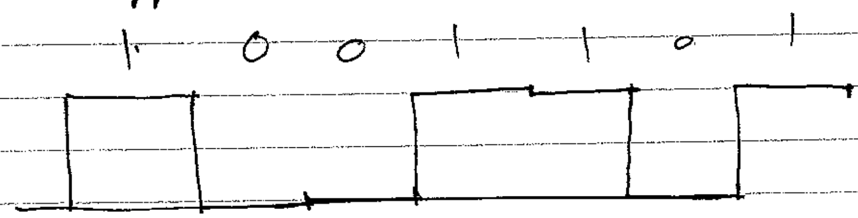


Note:
with k_f
pay attention
to unit -
whether it is
in Hz./V
or
Rad/V

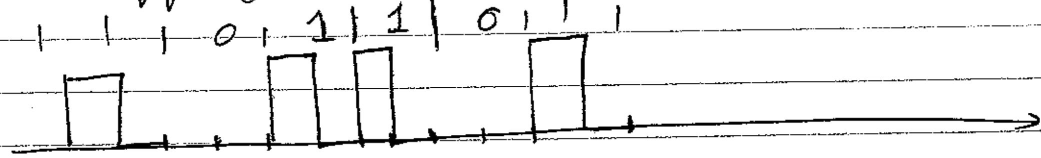
Digital Modulation:

Line Coding:

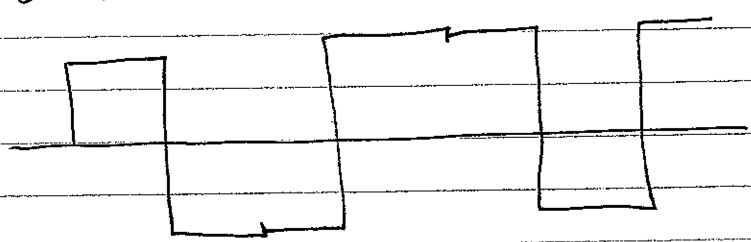
On-off (NRZ) Codes



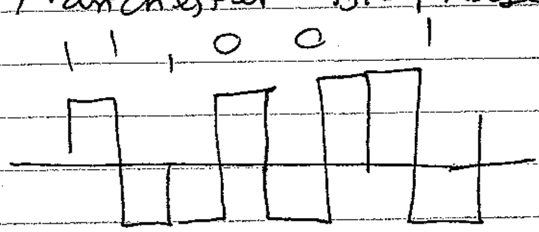
on-off (RZ) Code (or unipolar RZ)



Bi-polar (NRZ)



Manchester Bi-phase Code



Properties of Line Codes:

1) Transmission Bandwidth

2) Power efficiency.

3) Nice Power spectral density, e.g., no power at low frequencies

4) Adequate timing content

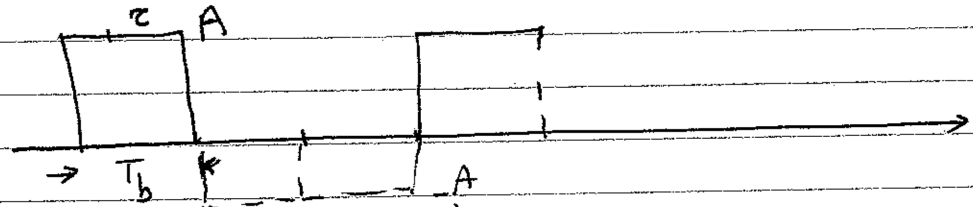
Power Spectral Density:

$$S_x(\omega) = \mathcal{F}[R_x(\tau)]$$

where

$$R_x(\tau) = E[x(t)x(t+\tau)]$$

example: on-off (NRZ)



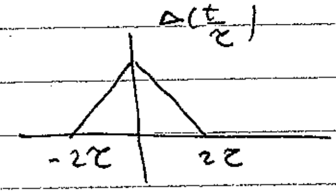
$E[x(t)x(t+\tau)] = 0$ if $\tau > T_b$

because $\frac{1}{2}(A \times A) + (\frac{1}{2})(-A \times A) = 0$

for $\tau \leq T$

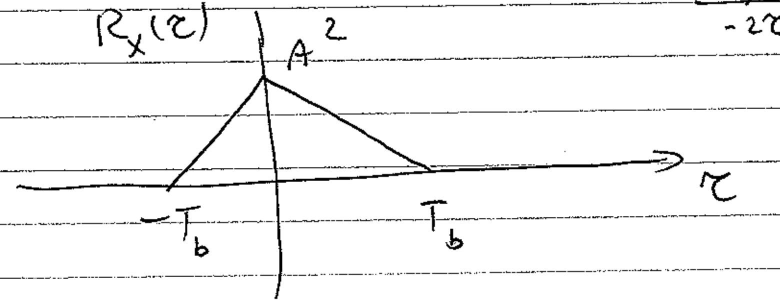
$$\frac{\tau}{T}(-A^2/2 + A^2/2) + (1 - \frac{\tau}{T})A^2$$

$$\Delta\left(\frac{t}{\tau}\right) \Longleftrightarrow \frac{\tau}{2}\,\text{sinc}^2\left(\frac{\omega\tau}{4}\right)$$

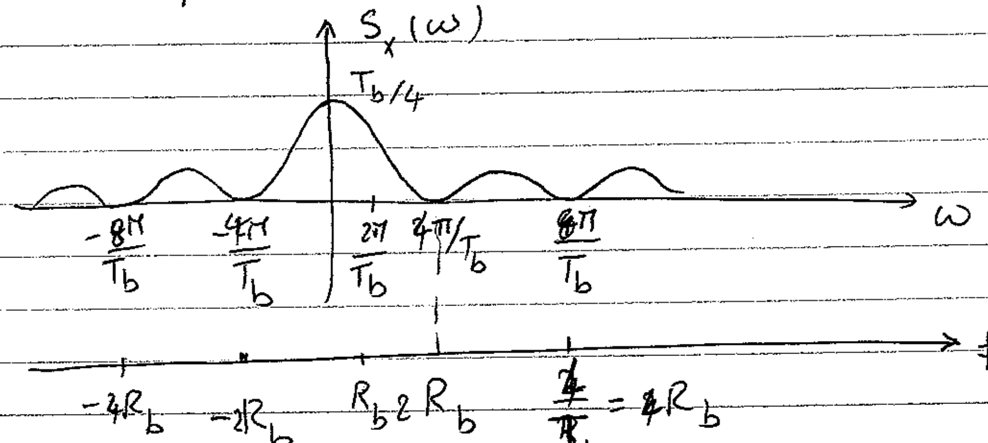


So:

$$R_X(\tau) = \left(1 - \frac{|\tau|}{T_b}\right) A^2$$



$$S_x(\omega) = A^2\frac{T_b}{4}\,\text{sinc}^2\left(\frac{\omega T_b}{4}\right)$$



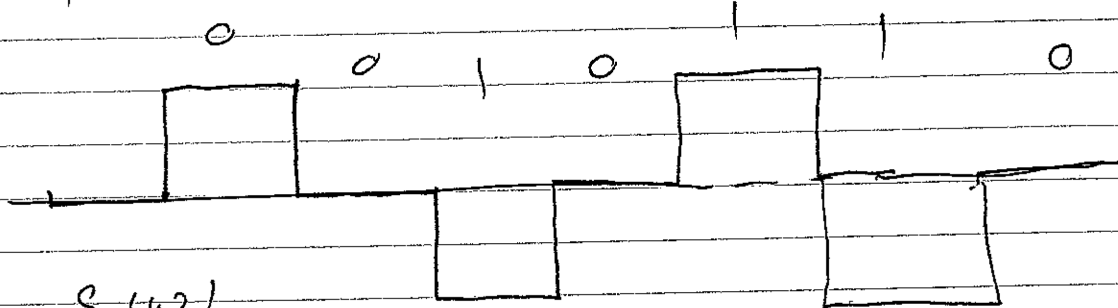
- This is not a BW efficient scheme, since it occupies $2R_b$ of BW which is four times the Nyquist BW requirement.
- It also has strong dc component.
- On the positive side it is power efficient, i.e., it gives lowest BER for the same power as other line codes.

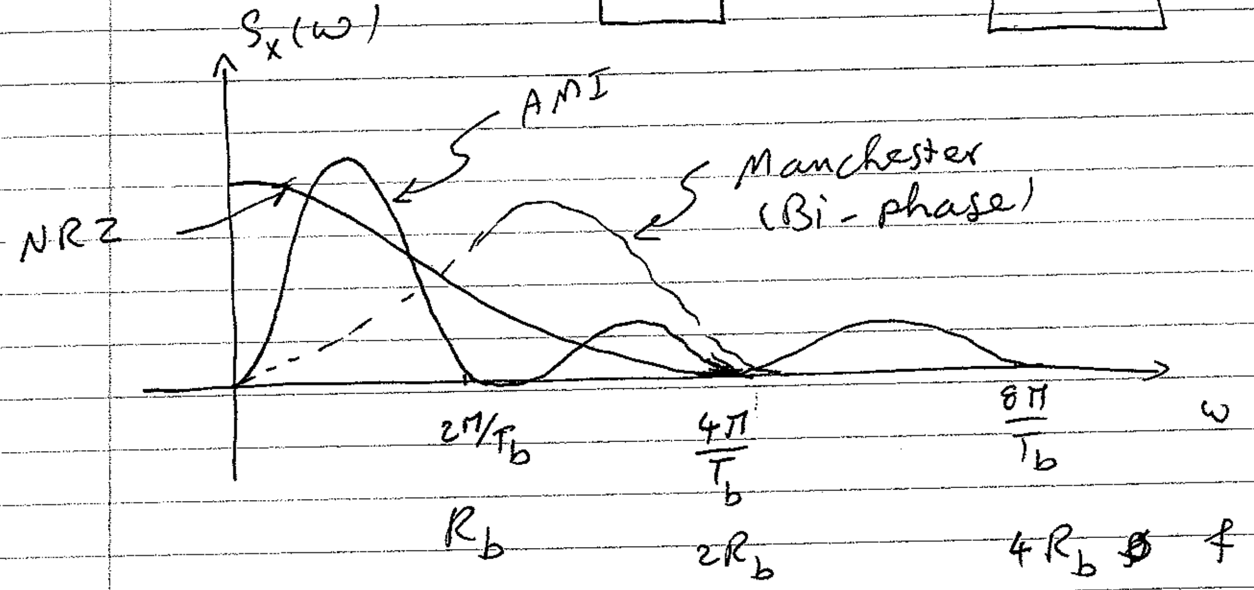
~~Bipolar~~

Alternative Mark Inversion (AMI)

0 no pulse

1 +A or −A alternatively





High Density Bipolar Signaling (HDBN)

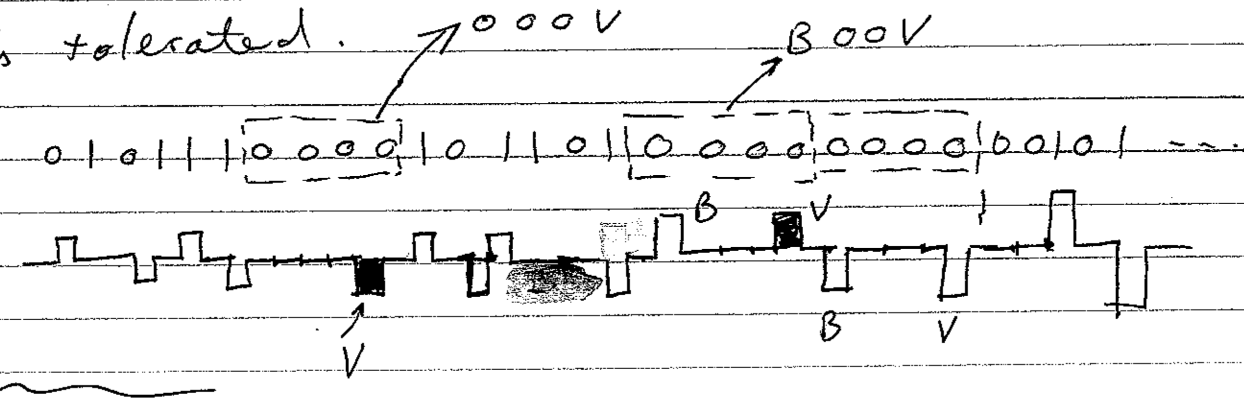
in AMI, when there ~~are~~ is a long ~~runs~~ run of 0's, the timing is lost (no timing content). In order to avoid this, level changes can be introduced artificially.

In the Standard HDBN, if there is a run of N+1 zeros, it is replaced by either 000...0V or B00...V where B and V represent level changes (i.e., 1's), B represents valid level change, i.e., if previous has been negative now it is positive and vice versa and V represents a violation. So, the net effect is violation and at the receiving end, it can be detected and replaced by N+1 zeros.

B0...0V is used when the number of ones and zeros to the present are not equal and 00...0V is used when the number 1's and zeros so far are equal.

Hilroy

example: HDB3, i.e., no run of 4 zeros is tolerated.



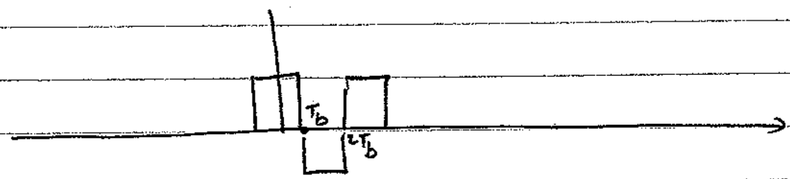
Another scheme is: Binary with N zero Substitution (BNZS).

For example B8ZS used in T1

and ~~B6Z~~ B6ZS in T2.

PULSE Shaping:

Nyquist criterion for no ISI.



The pulses used so far wer chosen in such a way that they ~~have~~ extend over one bit duration and are zero outside T_b. This results in large (theoretically infinite) Bandwidth. But, since we are only interested in the value of the signal at time T_b, $2T_b$, ... etc., we do not need to have pulses that completely diminish at T_b but those that are zero at T_b, $2T_b$, etc. In otherwords the pulse can extend in time and take any value between the points ~~t~~ 0 to T_b, T_b to $2T_b$, ... and the only condition is:

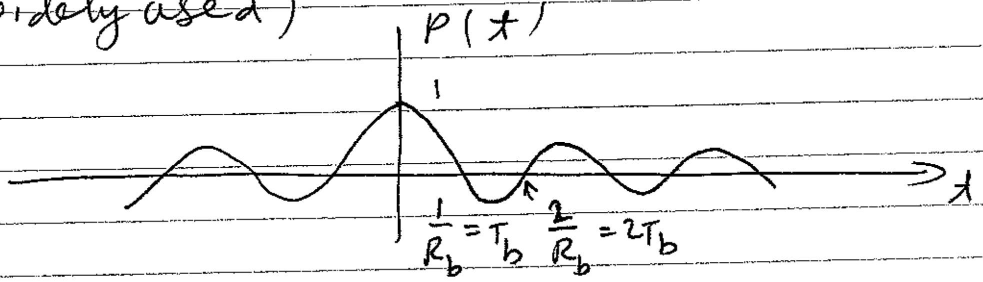
$$p(t) = \begin{cases} 1 & t = 0 \\ 0 & t = nT_b \end{cases} \qquad T_b = \frac{1}{R_b}$$

Hilroy

One signal (pulse) that has this property is:

$$p(t) = \text{Sinc}(\pi R_b t) = \frac{\sin(\pi R_b t)}{\pi R_b t} = \frac{\sin\left(\frac{\pi t}{T_b}\right)}{\frac{\pi t}{T_b}}$$

(Note: this is usually denoted $\text{sinc}\left(\frac{\pi}{T_b}\right)$ or $\text{sinc}(R_b t)$. The notation is the book is not widely used)

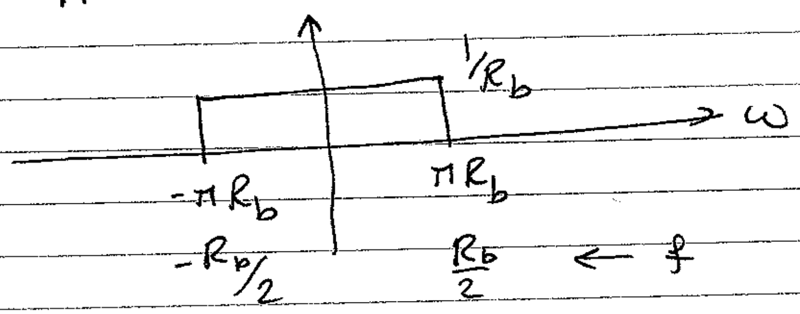


The spectrum of this pulse is

$$P(\omega) = \mathcal{F}[p(t)] = \frac{1}{R_b}\,\text{rect}\left(\frac{\omega}{2\pi R_b}\right) = T_b\,\text{rect}\left(\frac{\omega T_b}{2\pi}\right)$$

(using:

$$\frac{W}{\pi}\,\text{Sinc}(Wt) \Leftrightarrow \text{rect}\left(\frac{\omega}{2W}\right) \qquad \text{page 85} \;)$$



So, for transmitting R_b bps, we require, at least $R_b/2$ Hz of BW in the baseband.

The problem with Nyquist pulse, i.e., $\mathrm{sinc}\left(\frac{\pi t}{T_b}\right)$ is:

1) Non-causality,

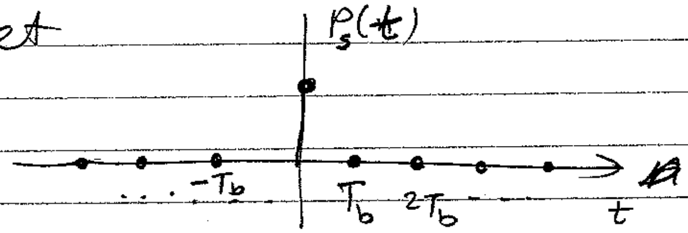
2) slow decay (as $\frac{1}{t}$)

The condition for no ISI:

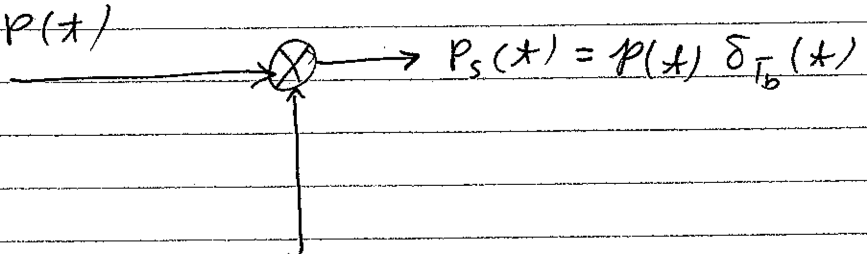
When we write

$$p(t) = \begin{cases} 1 & t = 0 \\ 0 & t = nT_b \end{cases}$$

it means that, if we sample $p(t)$ with a sampling frequency $\frac{1}{T_b} = R_b$, we should get



We can represent this as



$$\delta_{T_b}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_b)$$

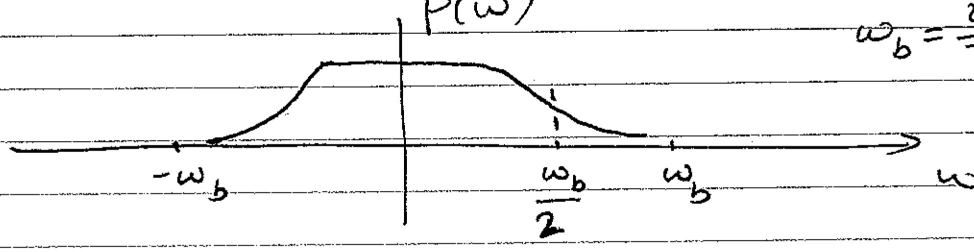
$$\mathcal{F}[p_s(t)] = \mathcal{F}\left[p(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_b)\right]$$

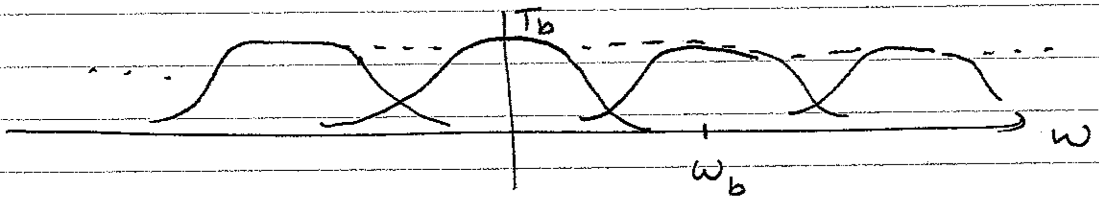
$$= \frac{1}{T_b}\sum_{n=-\infty}^{\infty} P\left(\omega - n\frac{2\pi}{T_b}\right) = 1$$

So, in order not to have ISI, we need:

$$\sum_{n=-\infty}^{\infty} P\left(\omega - \frac{n\,2\pi}{T_b}\right) = T_b \quad \text{or} \quad \sum_{n=-\infty}^{\infty} P(\omega - n\omega_b) = T_b$$

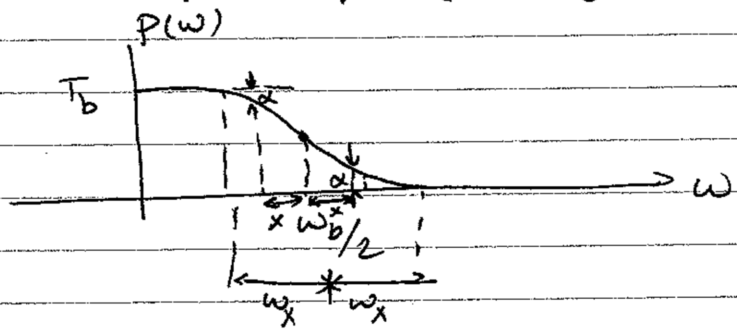
$$\omega_b = \frac{2\pi}{T_b} = 2\pi R_b$$





It means that

$$P(\omega) + P(\omega - \omega_b) = T_b$$



i.e., the spectrum $P(\omega)$ has to have odd symmetry around $\omega_b/2$

The roll-off factor r is given as

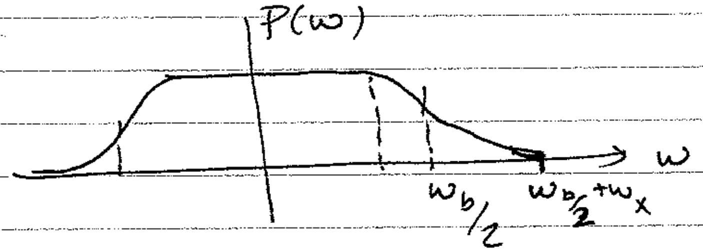
$$r = \frac{\text{excess bandwidth}}{\text{theoretical min. BW}} = \frac{\omega_x}{\omega_b/2} = \frac{2\omega_x}{\omega_b}$$

$$0 < r < 1$$

so, the required BW using a pulse (or filter) with roll-off r, is

$$B_T = \frac{R_b}{2} + \frac{rR_b}{2} = \frac{(1+r)R_b}{2}$$

Raised Cosine Pulse:



$$P(\omega) = \begin{cases} 1 & |\omega| < \frac{\omega_b}{2} - \omega_x \\ 0 & |\omega| > \frac{\omega_b}{2} + \omega_x \\ \frac{1}{2}\left\{1 - \sin\left(\frac{\pi[\omega - \omega_b/2]}{2\omega_x}\right)\right. & |\omega - \frac{\omega_b}{2}| < \omega_x \end{cases}$$

or

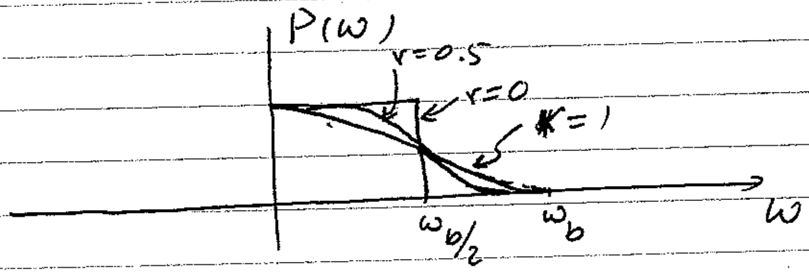
$$P(\omega) = \begin{cases} 1 & |\omega| < \frac{\omega_b}{2} - \omega_x \\ 0 & |\omega| > \frac{\omega_b}{2} + \omega_x \\ \frac{1}{2}\left\{1 - \sin\left[\frac{\pi}{r}\left(\frac{\omega}{\omega_b} - \frac{1}{2}\right)\right]\right. & |\omega - \frac{\omega_b}{2}| < \omega_x \end{cases}$$

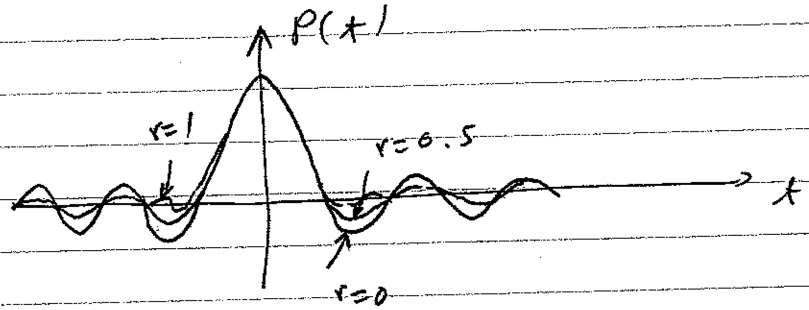
The pulse, in time domain is:

$$p(t) = R_b \frac{\cos(\pi r R_b t)}{1-4r^2R_b^2 t^2} \cdot \frac{\sin(\pi t R_b)}{\pi t R_b}$$

or

$$p(t) = R_b \frac{\cos(\pi r \frac{t}{T_b})}{1-4r^2 \frac{t^2}{T_b^2}} \cdot \frac{\sin(\pi \frac{t}{T_b})}{\pi t/T_b}$$





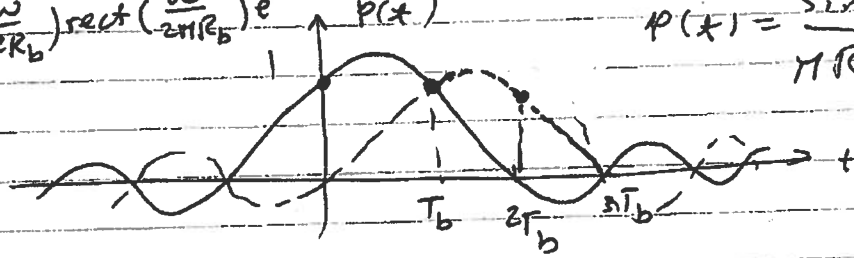
<u>Example</u>: A T_1 carrier is being transmitted using baseband binary modulation using a raised cosine filter (pulse) with roll-off factor 0.4. What is the required Bandwidth

$$B = \frac{R_b}{2}(1+r) = \frac{1.544}{2}(1+0.4) = 1.08 \text{ MHz.}$$

Controlled ISI: Partial response signalling:

$$p(t) = \begin{cases} 1 & t = 0, T_b \\ 0 & t = nT_b \quad n = \pm 2, \pm 3, \ldots \end{cases}$$

$$P(\omega) = \frac{2}{R_b} \cos\left(\frac{\omega}{2R_b}\right) \operatorname{rect}\left(\frac{\omega}{2\pi R_b}\right) e^{-j\omega/2R_b}$$



$$p(t) = \frac{\sin(\pi R_b t)}{\pi R_b t(1 - R_b t)}$$

That is, we have ISI with the adjacent bit.

The received level is either $1 + 1 = 2$

$-1 - 1 = -2$

or

$+1 - 1 = -1 + 1 = 0$

At the receiver we can make our decisions

based on:

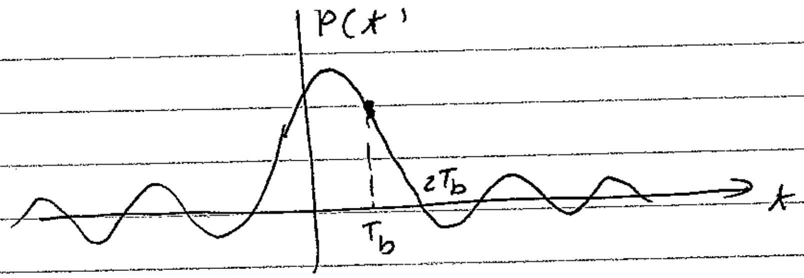
if received level > 0 decide 1 (also previous one)

" " " < 0 " 0 (")

if " " $= 0$ then decide $a_n = \overline{a}_{n-1}$

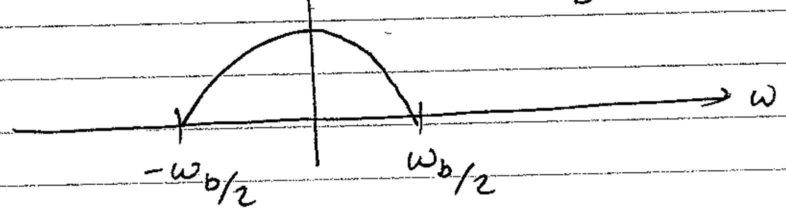
example:

transmitted	1	1	0	1	1	0	0	0	1	0	1	1	1		
received	1	2	0	0	2	0	-2	-2	0	0	0	2	2		
Detected	1	1	0	1	1	0	0	0	1	0	1	ϕ	1		



$$p(t) = \frac{\sin(\pi R_b t)}{\pi R_b t(1 - R_b t)}$$

$$|P(\omega)| = \frac{2}{R_b}\cos\left(\frac{\omega}{2R_b}\right)$$



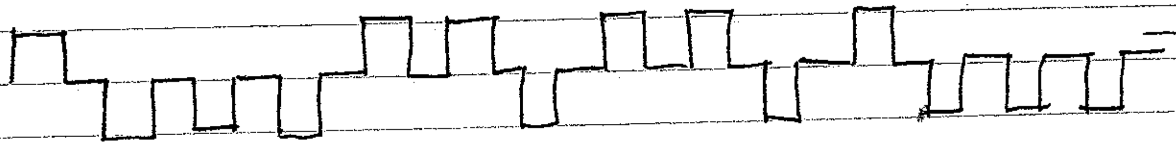
Differential Coding

In partial response signaling when 0 is received (or a level closer to zero than -2 or $+2$) then the decision depends on the previously detected bit. This makes error propagation possible. To avoid this one uses differential coding, here:

- A 1 is transmitted as a pulse similar to the previous one and,
- A 0 is transmitted as a pulse different from the previous one.

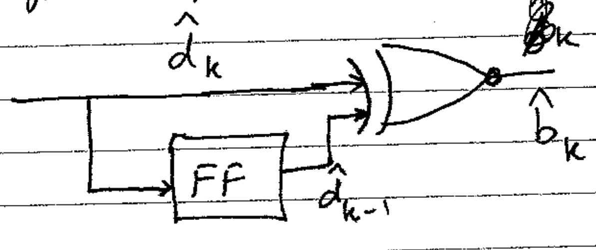
example 1:

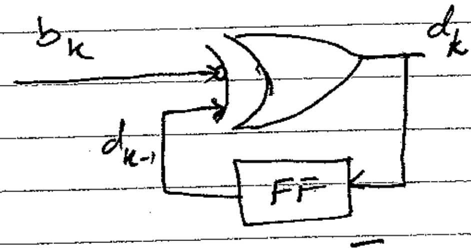
1 0 1 1 0 1 0 0 1 0 0 0 1 1



Here if two pulses are identical, i.e., −1 & −1 or +1 and +1 resulting in −2 or +2, respectively this indicates that the present bit is 1.
But, if the two are different, resulting in 0 to be received then the decision is 0.

Digital implementation





$d_k = d_{k-1} \oplus \overline{b_k}$

Receiver

$\hat{b}_k = \hat{d}_k \oplus \hat{d}_{k-1}$

b_k 1 0 1 1 0 1 0 0 1 0 0 0 1 1

d_k 1 1 0 0 0 1 1 0 1 1 0 1 0 0 0

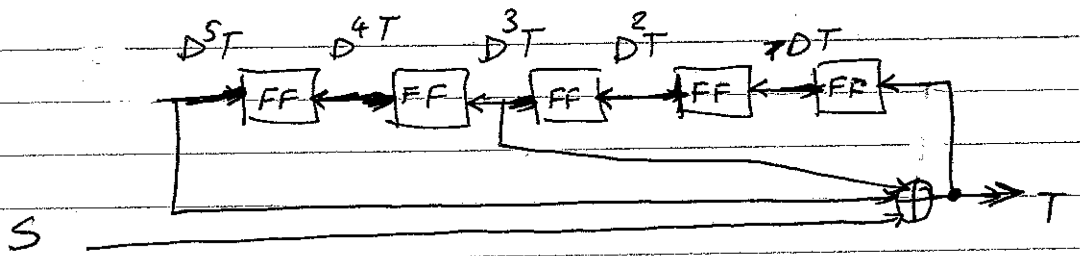
+1 +1 −1 −1 −1 +1 +1 −1 +1 +1 −1 +1 −1 −1 −1

2 0 −2 −2 0 +2 0 0 +2 0 0 0 −2 −2

1 0 1 1 0 1 0 0 1 0 0 0 1 1

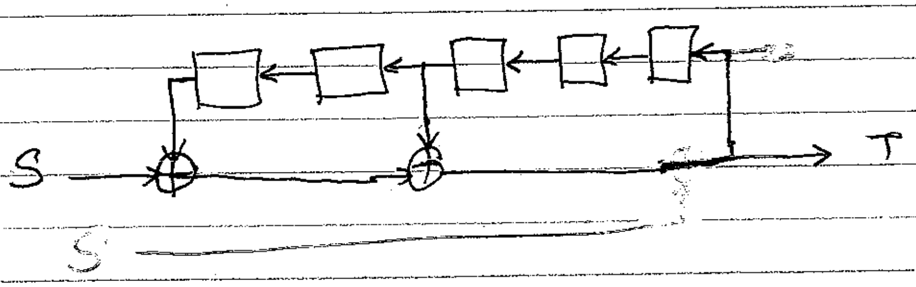
Scrambling

Scramblers are use to randomize data i.e., not to allow long runs of 1's or 0's. They are usually implemented using feedback shift registers.



$$T = S + D^3T + D^5T$$

or $$S = T + D^3T + D^5T$$



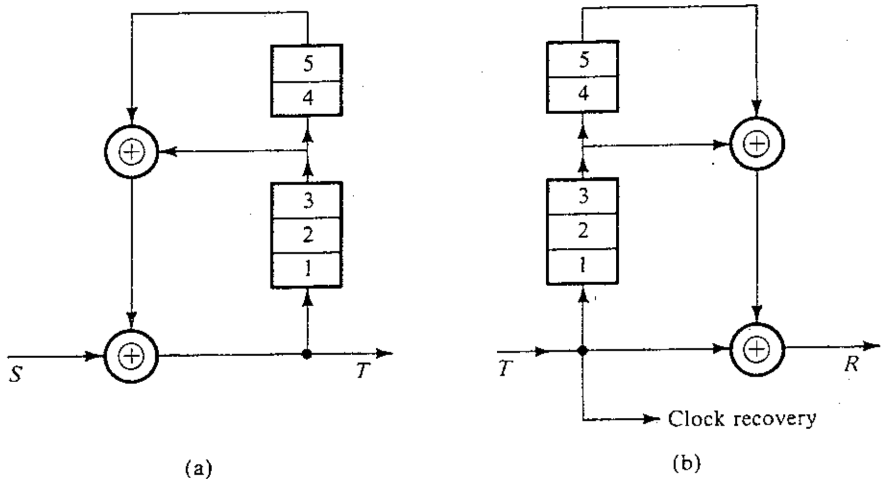


Figure 7.19 Scrambler and descrambler.

where D represents the delay operator; that is, $D^n T$ is the sequence T delayed by n units. The symbol $\oplus$ indicates modulo 2 sum. Now, recall that the modulo 2 sum of any sequence with itself gives a sequence of all **0**'s. Adding $(D^3 \oplus D^5)T$ to both sides of Eq. (7.39), we get

$$\begin{aligned} S &= T \oplus (D^3 \oplus D^5)T \\ &= [1 \oplus (D^3 \oplus D^5)]T \\ &= (1 \oplus F)T \qquad \text{where } F = D^3 \oplus D^5 \end{aligned} \tag{7.40}$$

To design the descrambler at the receiver, we start with T, the sequence received at the descrambler. From Eq. (7.40), it follows that

$$S = T \oplus FT = T \oplus (D^3 \oplus D^5)T$$

This equation, where we regenerate the input sequence S from the received sequence T, is readily implemented by the descrambler shown in Fig. 7.19b.

Note that a single detection error in the received sequence T will affect three output bits in R. Hence, scrambling has the disadvantage of causing multiple errors for a single received bit error.

EXAMPLE 7.2 The data stream **101010100000111** is fed to the scrambler in Fig. 7.19a. Find the scrambler output T, assuming the initial content of the registers to be zero.

From Fig. 7.19a we observe that initially $T = S$, and the sequence S enters the register and is returned as $(D^3 \oplus D^5)S = FS$ through the feedback path. This new sequence FS again enters the register and is returned as F^2S, and so on. Hence,

$$\begin{aligned} T &= S \oplus FS \oplus F^2S \oplus F^3S \oplus \cdots \\ &= (1 \oplus F \oplus F^2 \oplus F^3 \oplus \cdots)S \end{aligned} \tag{7.41}$$

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Recognizing that

$$F = D^3 \oplus D^5$$

we have

$$F^2 = (D^3 \oplus D^5)(D^3 \oplus D^5) = D^6 \oplus D^{10} \oplus D^8 \oplus D^8$$

Because modulo-2 addition of any sequence with itself is zero, $D^8 \oplus D^8 = 0$, and

$$F^2 = D^6 \oplus D^{10}$$

Similarly,

$$F^3 = (D^6 \oplus D^{10})(D^3 \oplus D^5) = D^9 \oplus D^{11} \oplus D^{13} \oplus D^{15}$$

and so on. Hence [see Eq. (7.41)],

$$T = (1 \oplus D^3 \oplus D^5 \oplus D^6 \oplus D^9 \oplus D^{10} \oplus D^{11} \oplus D^{12} \oplus D^{13} \oplus D^{15} \oplus \cdots)S$$

Because $D^n S$ is simply the sequence S delayed by n bits, various terms in the preceding equation correspond to the following sequences:

$$
\begin{aligned}
S &= \mathbf{101010100000111} \\
D^3 S &= \mathbf{000101010100000111} \\
D^5 S &= \mathbf{00000101010100000111} \\
D^6 S &= \mathbf{000000101010100000111} \\
D^9 S &= \mathbf{000000000101010100000111} \\
D^{10} S &= \mathbf{0000000000101010100000111} \\
D^{11} S &= \mathbf{00000000000101010100000111} \\
D^{12} S &= \mathbf{000000000000101010100000111} \\
D^{13} S &= \mathbf{0000000000000101010100000111} \\
\underline{D^{15} S} &= \underline{\mathbf{000000000000000}}\mathbf{101010100000111} \\
T &= 101110001101001
\end{aligned}
$$

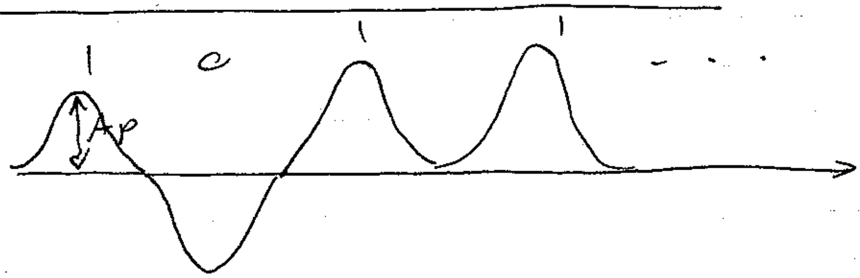
Note that the input sequence contains the periodic sequence **10101010** . . . , as well as a long string of **0**'s. The scrambler output effectively removes the periodic component as well as the long string of **0**'s. The input sequence has 15 digits. The scrambler output up to the 15th digit only is shown, because all the output digits beyond 15 depend on the input digits beyond 15, which are not given.

We can verify that the descrambler output is indeed S when this sequence T is applied at its input (see Prob. 7.4-1).

7.5 REGENERATIVE REPEATER

Basically, a regenerative repeater performs three functions: (1) reshaping incoming pulses by means of an equalizer, (2) the extraction of timing information required to sample incoming

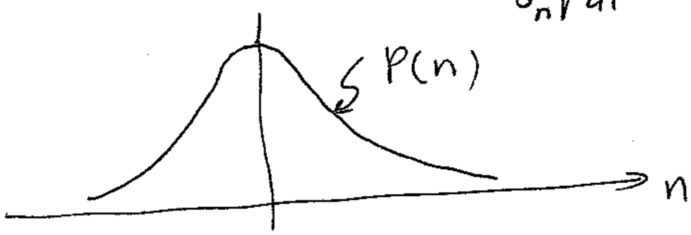
Detection Error Probability



$r_k = s_k + n_k$

s_k is the sample of the k-th pulse at its peak so, it has the value A_p.

n_k is the sample of noise having Gaussian (Normal) density. $p(n) = \frac{1}{\sigma_n\sqrt{2\pi}} e^{-n^2/2\sigma_n^2}$



if we consider a polar signal (polar NRZ)

then for 0 $\quad s_k = -A_p$

and

for 1 $\quad s_k = +A_p$

so, we make an error when 0 is transmitted if $n_k > A_p$, i.e., if $r_k > 0$. so,

$$P(\varepsilon|0) = P_r(n > A_p)$$

Similarly,

$$P(\varepsilon|1) = Pr(n < -A_p)$$

$$P(\varepsilon|0) = Pr(n > A_p) = \int_{A_p}^{\infty} p(n)\,dn$$

$$= \frac{1}{\sigma_n\sqrt{2\pi}} \int_{A_p}^{\infty} e^{-\frac{n^2}{2\sigma_n^2}}\,dn$$

let $x = \frac{n}{\sigma_n}$ then

$$P(\varepsilon|0) = \frac{1}{\sqrt{2\pi}} \int_{\frac{A_p}{\sigma_n}}^{\infty} e^{-\frac{x^2}{2}}\,dx = Q\left(\frac{A_p}{\sigma_n}\right)$$

where

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_{y}^{\infty} e^{-x^2/2}\,dx$$

Similarly,

$$P(\varepsilon|1) = \int_{-\infty}^{-A_p} \frac{1}{\sigma_n\sqrt{2\pi}} e^{-\frac{n^2}{2\sigma_n^2}}\,dn$$

let $x = \frac{n}{\sigma_n}$

$$P(\varepsilon|1) = \int_{-\infty}^{-A_p/\sigma_n} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = \int_{\frac{A_p}{\sigma_n}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = Q\left(\frac{A_p}{\sigma_n}\right)$$

$$P(\varepsilon) = \frac{1}{2}P(\varepsilon|0) + \frac{1}{2}P(\varepsilon|1) = Q\left(\frac{A_p}{\sigma_n}\right)$$

$Q(x)$ is tabulated in page 454. It can also be approximated as:

$$Q(x) \cong \frac{1}{x\sqrt{2\pi}}\left(1 - \frac{0.7}{x^2}\right) e^{-x^2/2} \qquad x > 2$$

$P(\varepsilon) = Q\left(\frac{A_p}{\sigma_n}\right)$ indicates that the BER depends on the ratio of the peak pulse amplitude to noise rms value (or to the square root of signal power to noise power $\left(\sqrt{\frac{A_p^2}{\sigma_n^2}}\right)$. Let $\frac{A_p}{\sigma_n} = k$ then

$$P(\varepsilon) = Q\left(\frac{A_p}{\sigma_n}\right) = Q(k)$$

for

$k =$	1	2	3	4	5	6
$P(\varepsilon) =$	0.1587	0.0227	0.00135	3.16×10^{-5}	2.87×10^{-7}	9.9x

Probability of error for on-off signalling:

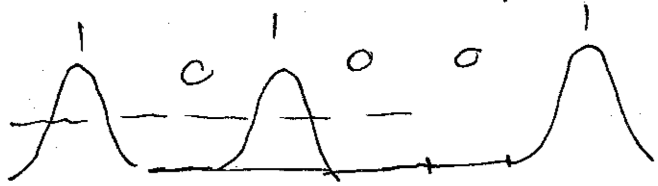
here

$$r_k = s_k + n_k$$

where for 0 $\quad s_k = 0$

and

for 1 $\quad s_k = A_p$



now:

$$P(\varepsilon|0) = P_r\left(n > \frac{A_p}{2}\right) = \int_{\frac{A_p}{2}}^{\infty} \frac{1}{\sigma_n\sqrt{2\pi}} e^{-\frac{n^2}{2\sigma_n^2}}\, dn = Q\left(\frac{A_p}{2\sigma_n}\right)$$

and

$$P(\varepsilon|1) = P\left(n < -\frac{A_p}{2}\right) = Q\left(\frac{A_p}{2\sigma_n}\right)$$

so

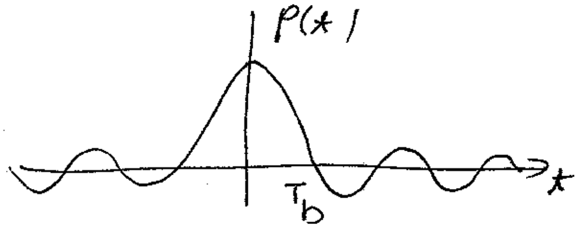
$$P(\varepsilon) = \frac{1}{2} Q\left(\frac{A_p}{2\sigma_n}\right) + \frac{1}{2} Q\left(\frac{A_p}{2\sigma_n}\right) = Q\left(\frac{A_p}{2\sigma_n}\right)$$

This represent a worse performance compared to polar case, since in order to get the same $P(\varepsilon)$ (with the same noise σ_n), we need to double A_p. This means that the average power is $\frac{1}{2}(2A_p)^2 + \frac{1}{2}(0) = 2A_p^2$ as opposed to A_p^2 (in the case of polar signaling).

M-ary Communications

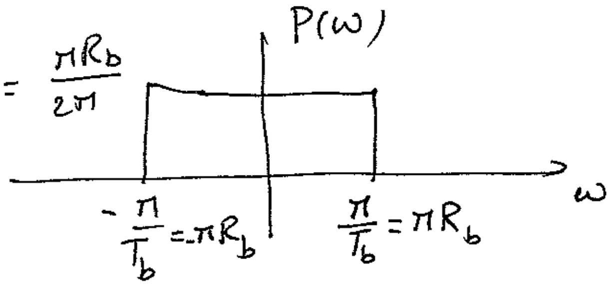
We saw that using a Nyquist pulse

$$p(t) = \frac{\sin(\frac{\pi t}{T_b})}{(\frac{\pi t}{T_b})}$$



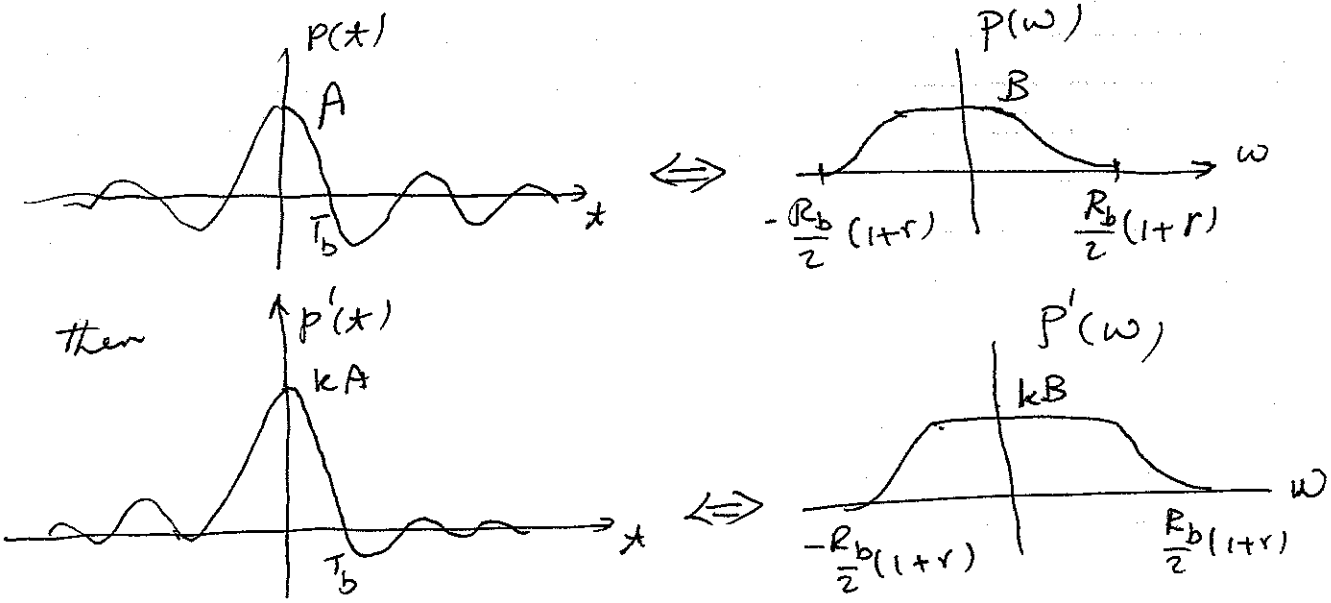
we can avoid ISI and we require a Bandwidth

of $\frac{R_b}{2} = \frac{\pi R_b}{2\pi}$



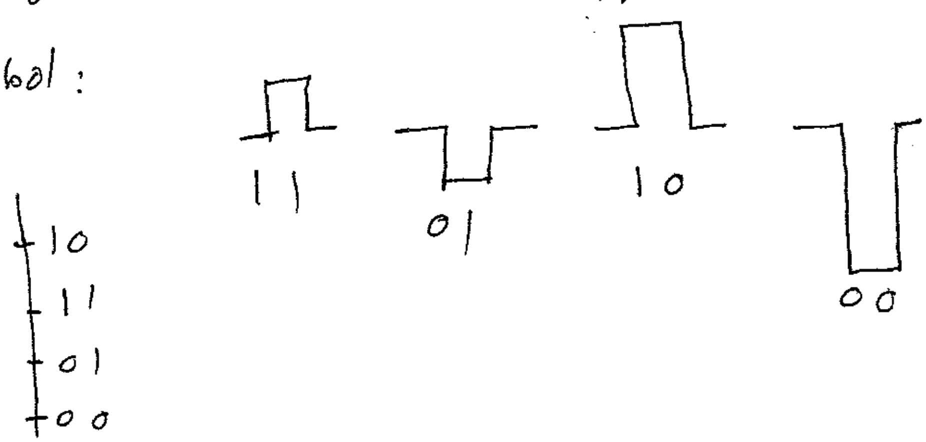
To avoid "difficulties" of implementing an ideal pulse like this, we can use pulses with roll-off fact such as a raised cosine pulse. Then we require $B_T = \frac{R_b}{2}(1+r) = \frac{1}{2T_b}(1+r)$.

Since these pulses are zero at $T_b, 2T_b, \ldots,$ another symbol can be send at $t = T_b$, another at $t = 2T_b$, etc. This leads to no ISI and with the bandwidth $\frac{1}{2T_b}(1+r)$. This is independent of the amplitude of the pulse. That is if



This fact can be used to increase the bit rate while keeping the bandwidth constant, i.e., instead of considering two levels for 0 and 1, respectively, one can consider M levels for $\log_2 M$ bits.

Example - 2B1Q signaling used for ISDN every two bits are mapped as 1 quaternary symbol:



Digital Carrier Systems:

The need for upconversion (frequency shifting)

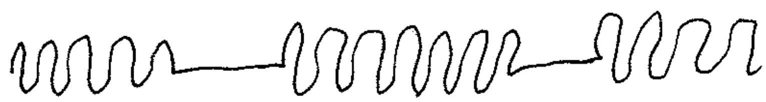
Types of digital Modulation:

- ASK : Amplitude Shift Keying
- PSK : Phase Shift Keying
- FSK : Frequency Shift Keying.

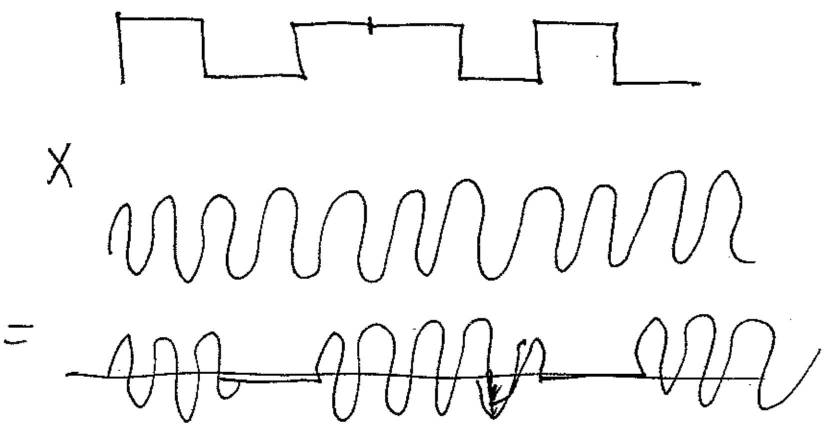
ASK	1	$A \cos \omega_c t$
	0	0

example

1 0 1 1 0 1



it is similar to using on-off binary signalling and then modulating by sinusoidal



This also called OOK = on-off keying.

Phase Shift Keying:

1 $\quad A \cos \omega_c t$

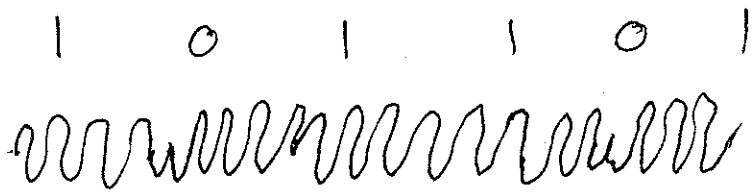
0 $\quad A \cos(\omega_c t + \varphi)$

usually $\varphi = \pi$, so:

1 $\quad A \cos \omega_c t$

0 $\quad A \cos(\omega_c t + \pi) = -A \cos \omega_c t$

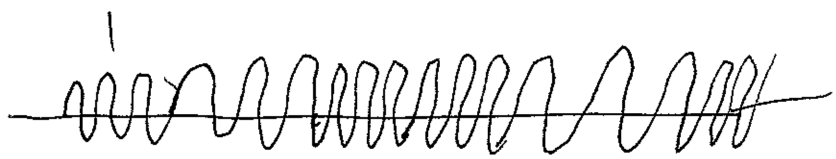
example



Frequency Shift Keying:

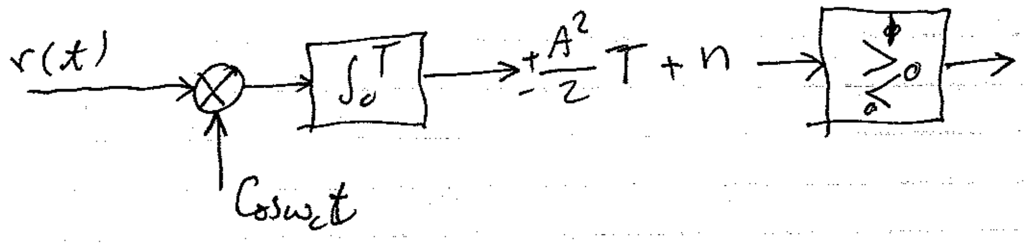
1 $\quad A \cos \omega_1 t$

0 $\quad A \cos \omega_0 t$



Demodulation:

example of PSK



$r(t) = \pm A \cos w_c t + n(t)$

$$A\int_0^T \cos w_c t \cos w_c t \, dt + \int_0^T n(t) \cos w_c t \, dt$$

$$= \frac{A^2}{2} T + n$$

or if $-A \cos w_c t$

$$-\frac{A^2}{2} T + n$$

QPSK : Quadrature Phase shift Keying

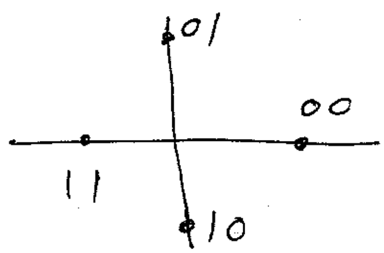
or 4 - PSK :

00 $\quad A \cos w_c t$

01 $\quad A \cos(w_c t + \frac{\pi}{2})$

11 $\quad A \cos(w_c t + \pi)$

10 $\quad A \cos(w_c t + 3\pi/2)$



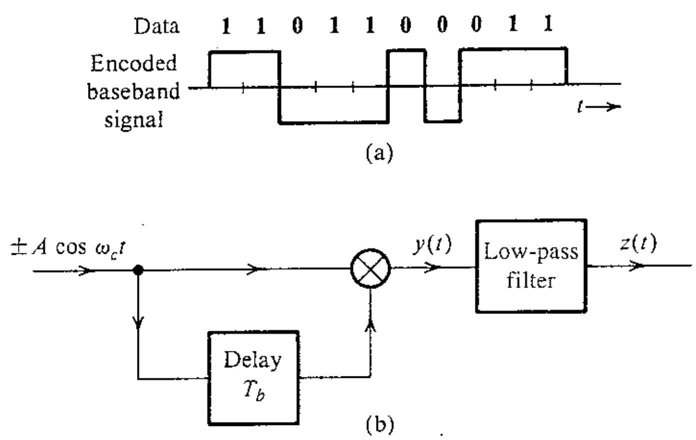


Figure 7.30 (a) Differential coding. (b) Differential PSK receiver.

0, the present pulse and the previous pulse are of opposite polarities or phases; if the present pulse is $A\cos\omega_c t$, the previous pulse is $-A\cos\omega_c t$, and vice versa.

In the demodulation of DPSK (Fig. 7.30b), we avoid generation of a local carrier by observing that the received modulated signal itself is a carrier ($\pm A\cos\omega_c t$) with a possible sign ambiguity. For demodulation, in place of the carrier, we use the received signal delayed by T_b (1-bit interval). If the received pulse is identical to the previous pulse, the product $y(t) = A^2\cos^2\omega_c t = (A^2/2)(1+\cos 2\omega_c t)$, and the low-pass filter output $z(t) = A^2/2$. We immediately detect the present bit as **1**. If the received pulse and the previous pulse are of opposite polarity, $y(t) = -A^2\cos^2\omega_c t$ and $z(t) = -A^2/2$, and the present bit is detected as **0**.

The FSK can be viewed as two interleaved ASK signals with carrier frequencies ω_{c_0} and ω_{c_1}, respectively (Fig. 7.28c). Therefore, FSK can be detected coherently or noncoherently.

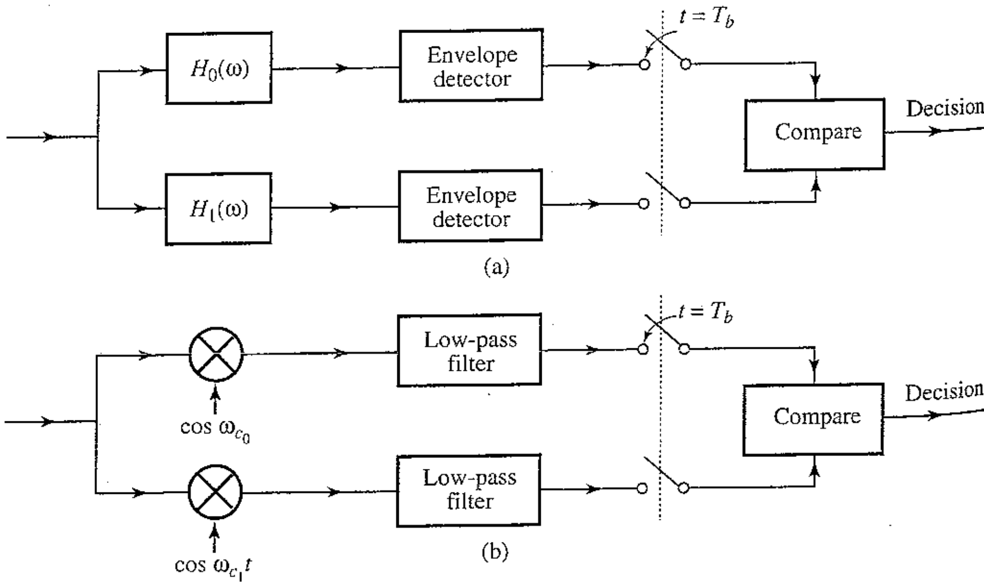


Figure 7.31 (a) Noncoherent detection of FSK. (b) Coherent detection of FSK.

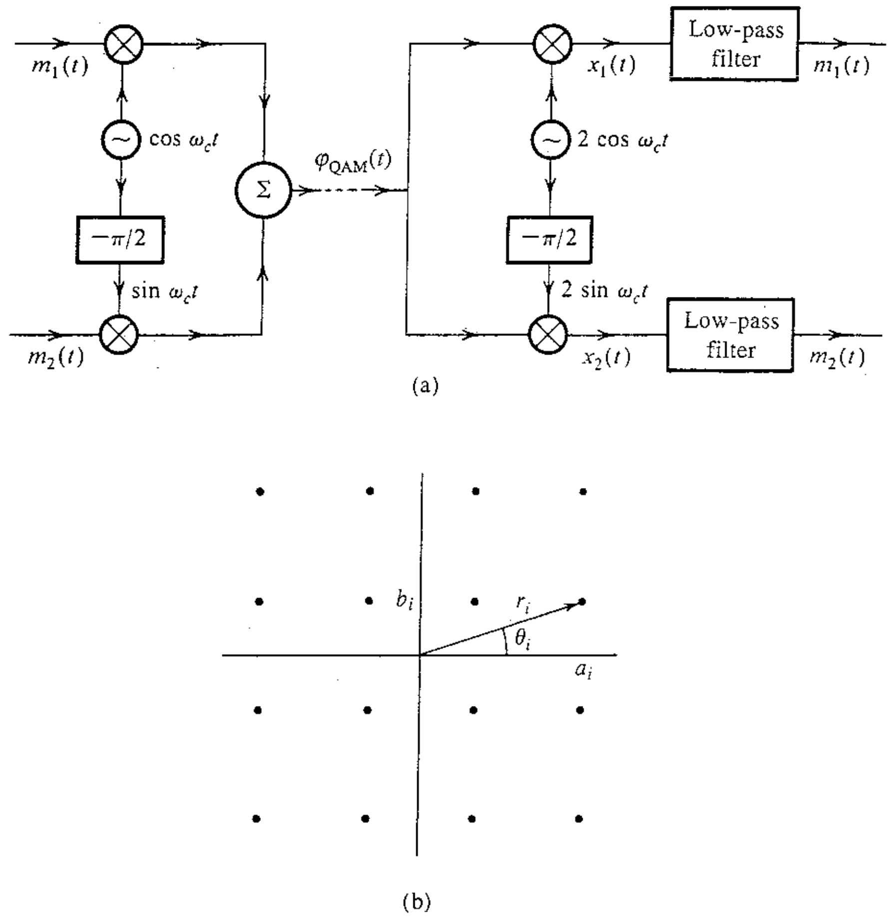


Figure 7.32 (a) QAM or quadrature multiplexing. (b) 16-point QAM ($M = 16$).

$a_i p(t)$ and $b_i p(t)$. From the knowledge of $(a_i,\ b_i)$, we can determine the four transmitted bits. Further analysis of 16-ary QAM on a noisy channel is carried out in Example 14.3.

Such a QAM scheme is used on telephone lines for data transmission. At each end of the telephone line, we need a modulator and a demodulator to transmit as well as to receive data. The two devices, **modulator** and **demodulator**, are usually packaged in one unit called a **modem**.

7.9 DIGITAL MULTIPLEXING

Several low-bit-rate signals can be multiplexed, or combined, to form one high-bit-rate signal to be transmitted over a high-frequency medium. Because the medium is time-shared by various incoming signals, this is a case of time-division multiplexing (TDM). The signals from various incoming channels, or tributaries, may be of such diverse nature as a digitized voice signal

In noncoherent detection, the incoming signal is applied to a bank of two filters tuned to ω_{c_0} and ω_{c_1}. Each filter is followed by an envelope detector (see Fig. 7.31a). The outputs of the two envelope detectors are sampled and compared. A **0** is transmitted by a pulse of frequency ω_{c_0}, and this pulse will appear at the output of the filter tuned to ω_{c_0}. Practically no signal appears at the output of the filter tuned to ω_{c_1}. Hence, the sample of the envelope detector output following the ω_{c_0} filter will be greater than the sample of the envelope detector output following the ω_{c_1} filter, and the receiver decides that a **0** was transmitted. In the case of a **1**, the situation is reversed.

FSK can also be detected coherently by generating two references of frequencies ω_{c_0} and ω_{c_1}, and demodulating the received signal by two demodulators using the two carriers and then comparing the outputs of the two demodulators, as shown in Fig. 7.31b.

From the point of view of noise immunity, coherent PSK is superior to all other schemes. PSK also requires a smaller bandwidth than FSK (see Fig. 7.29). A quantitative discussion of this topic can be found in Chapter 14.

Digital Signal Transmission Using QAM

Quadrature amplitude modulation (QAM) discussed in Chapter 4 (Fig. 4.14) can be conveniently used for digital signals as well. Figure 7.32a shows the QAM modulator and demodulator. Each of the signals $m_1(t)$ and $m_2(t)$ is a baseband binary polar pulse sequence. These signals are modulated by a carrier of the same frequency but in phase quadrature. Note that both of the modulated signals are PSK signals. For this reason, it is also known as **quadrature PSK (QPSK)**. As seen in Sec. 4.4, we can transmit and receive both of these signals on the same channel, thus doubling the transmission rate.

M-ary QAM

We can increase the transmission rate further by using M-ary QAM.* One practical case with $M = 16$ uses the following 16 pulses (16 symbols):

$$\begin{aligned} p_i(t) &= a_i p(t) \cos \omega_c t + b_i p(t) \sin \omega_c t \\ &= r_i p(t) \cos (\omega_c t - \theta_i) \qquad i = 1, 2, \dots, 16 \end{aligned}$$

where

$$r_i = \sqrt{a_i^2 + b_i^2} \qquad \text{and} \qquad \theta_i = \tan^{-1} \frac{b_i}{a_i}$$

and $p(t)$ is a properly shaped baseband pulse. The signal $p_i(t)$ can be generated using QAM by letting $m_1(t) = a_i p(t)$ and $m_2(t) = b_i p(t)$. One possible choice of r_i and θ_i for 16 pulses is shown graphically in Fig. 7.32b. The transmitted pulse $p_i(t)$ can take on 16 distinct forms and is, therefore, a 16-ary pulse. Since $M = 16$, each pulse can transmit the information of $\log_2 16 = 4$ binary digits. This can be done as follows: There are 16 possible sequences of four binary digits and there are 16 combinations (a_i, b_i) in Fig. 7.32b. Thus, every possible 4-bit sequence is transmitted by a particular (a_i, b_i) or (r_i, θ_i). Therefore, one signal pulse $r_i p(t) \cos (\omega_c t - \theta_i)$ transmits 4 bits. The bit rate is quadrupled without increasing the bandwidth. The transmission rate can be increased further by increasing the value of M.

Modulation as well as demodulation can be performed by using the system in Fig. 7.32a. The inputs are $m_1(t) = a_i p(t)$ and $m_2(t) = b_i p(t)$. The two outputs at the demodulator are

* The M-ary QAM discussed here is also called **amplitude phase keying (APK)**.

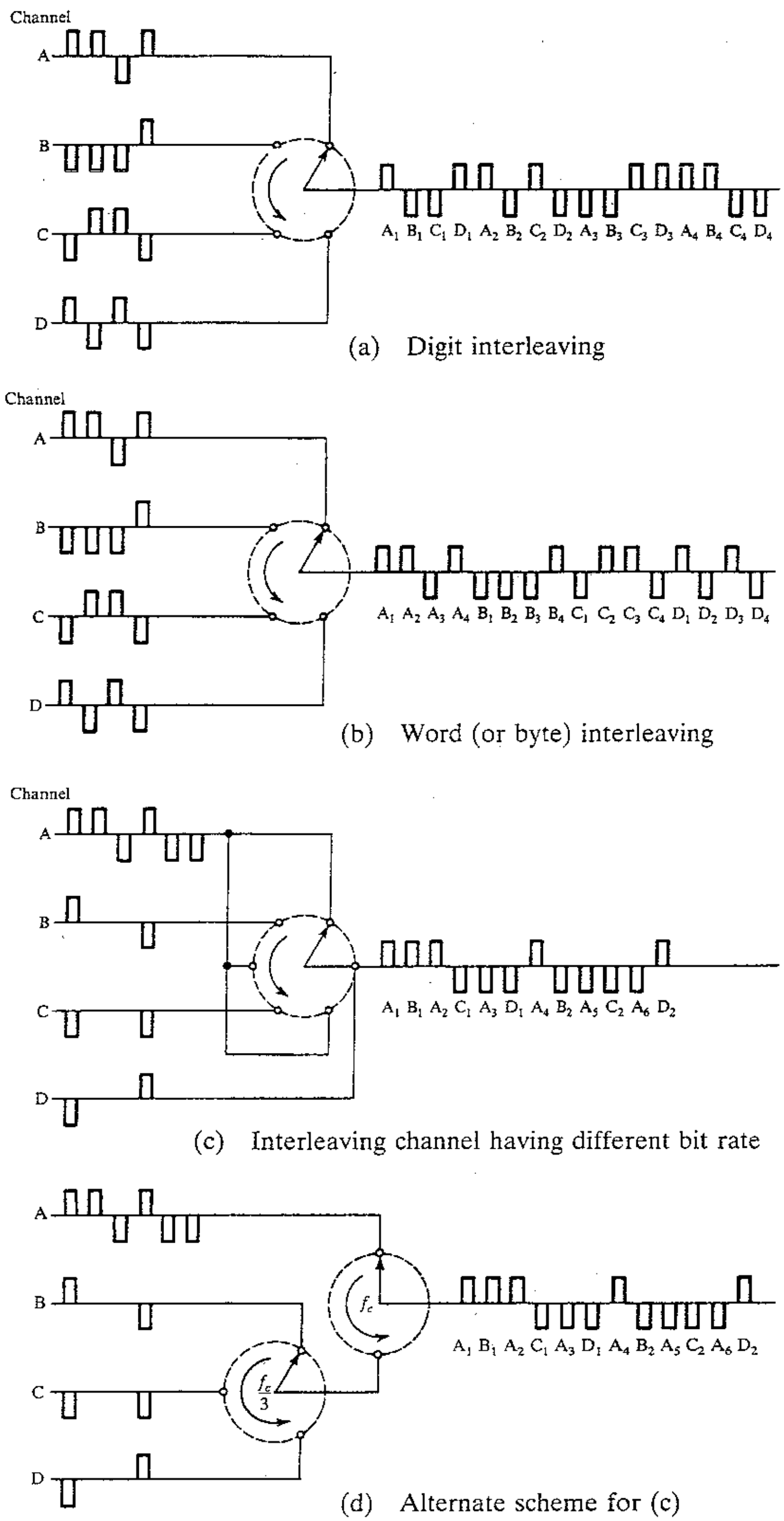


Figure 7.33 Time-division multiplexing of digital signals.

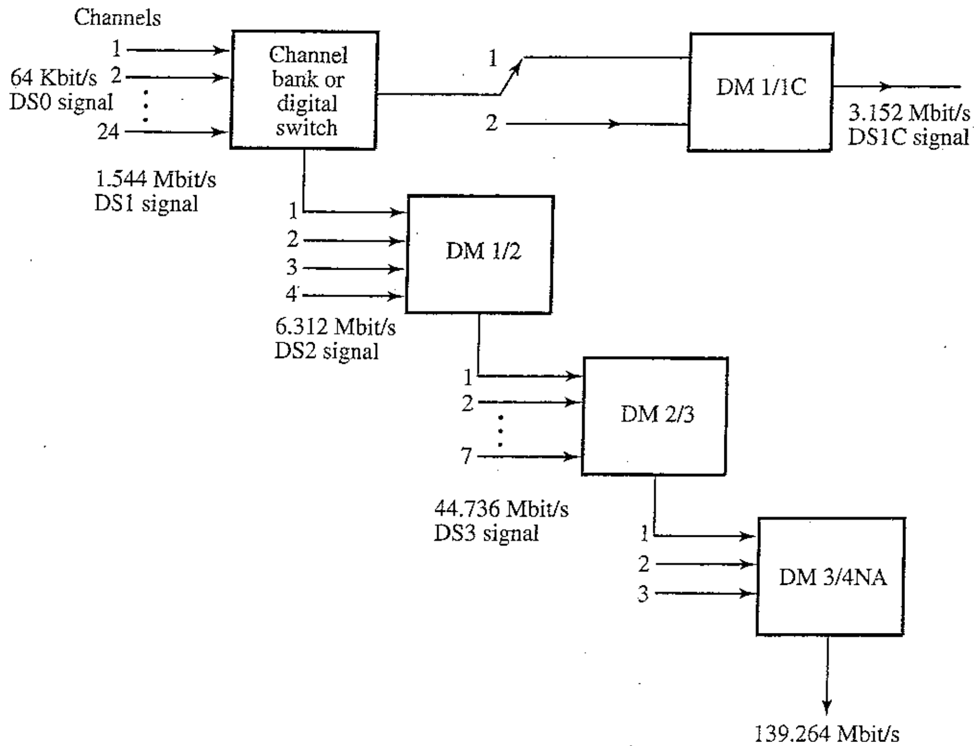


Figure 7.36 North American digital hierarchy (AT&T system).

signal at a rate of 44.736 Mbit/s. Finally, three DS3 signals are multiplexed by a DM3/4NA multiplexer to yield a DS4NA signal at a rate of 139.264 Mbit/s. There is also a lower rate multiplexing hierarchy, known as the **digital data system (DDS)**, which provides standards for multiplexing digital signals with rates as low as 2.4 kbit/s into a DS0 signal for transmission through the network.

The multiplexer DM 1/1C is useful for channeling paired cable plant but it cannot be multiplexed into higher level signals. Although DS1C is a "dead end" rate, which cannot be multiplexed to higher levels, it has proved useful for channeling interoffice cable pairs with more channels than can be carried by a comparable DS1 system. Although lightwave interoffice transmission has diminished the importance of DS1C, there are still a number of T1C lines in service in the Northern American network.

The inputs to the T1 multiplexer need not be restricted only to digitized voice channels. Any digital signal of 64 kbit/s of appropriate format can be transmitted. The case of the higher levels is similar. For example, all the incoming channels of the DM1/2 multiplexer need not be DS1 signals obtained by multiplexing 24 channels of 64 kbit/s each. Some of them may be 1.544 Mbit/s digital signals of appropriate format, and so on.

In Europe and the rest of the world, another hierarchy, recommended by the CCITT (Consultative Committee on International Telephony and Telegraphy) as an international standard, has been adopted. This hierarchy, based on the lowest level PCM international standard of 2.048 Mbit/s (30 channels), is shown in Fig. 7.37.

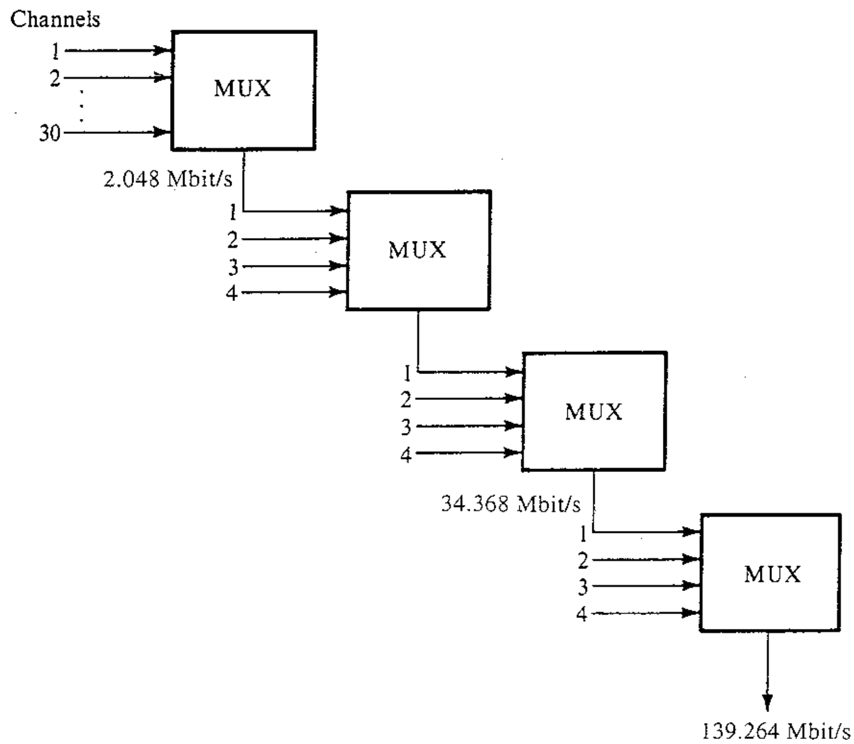


Figure 7.37 Digital hierarchy, CCITT recommendation.

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