The z-Transform and Its Application to the Analysis of L TI Systems

The Direct z-Transform

The z-transform of a discrete-time signal $x(n)$ is defined as the power series

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where z is a complex variable.

For convenience, the z-transform of a signal $x(n)$ is denoted by

$$X(z) \equiv Z\{x(n)\}$$

whereas the relationship between $x(n)$ and $X(z)$ is indicated by

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

Since the z-transform is an infinite power series, it exists only for those values of z for which this series converges. The *region of convergence* (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value. Thus any time we cite a z-transform we should also indicate its ROC.

Example

Determine the z-transforms of the following *finite-duration* signals.

(a) $x_1(n) = \{1, 2, 5, 7, 0, 1\}$

(b) $x_2(n) = \{1, 2, 5, 7, 0, 1\}$

(c) $x_3(n) = \{\underset{\uparrow}{0}, 0, 1, 2, 5, 7, 0, 1\}$

(d) $x_4(n) = \{2, 4, \underset{\uparrow}{5}, 7, 0, 1\}$

(e) $x_5(n) = \delta(n)$

(f) $x_6(n) = \delta(n - k), k > 0$

(g) $x_7(n) = \delta(n + k), k > 0$

Solution. From definition (3.1.1), we have

(a) $X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$, ROC: entire z-plane except $z = 0$

(b) $X_2(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$, ROC: entire z-plane except $z = 0$ and $z = \infty$

(c) $X_3(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$, ROC: entire z-plane except $z = 0$

(d) $X_4(z) = 2z^2 + 4z + 5 + 7z^{-1} + z^{-3}$, ROC: entire z-plane except $z = 0$ and $z = \infty$

(e) $X_5(z) = 1$ [i.e., $\delta(n) \stackrel{z}{\longleftrightarrow} 1$], ROC: entire z-plane

(f) $X_6(z) = z^{-k}$ [i.e., $\delta(n - k) \stackrel{z}{\longleftrightarrow} z^{-k}$], $k > 0$, ROC: entire z-plane except $z = 0$

(g) $X_7(z) = z^k$ [i.e., $\delta(n + k) \stackrel{z}{\longleftrightarrow} z^k$], $k > 0$, ROC: entire z-plane except $z = \infty$

Example

EXAMPLE 3.1.2

Determine the z-transform of the signal

$$x(n) = (\frac{1}{2})^n u(n)$$

Solution. The signal $x(n)$ consists of an infinite number of nonzero values

$$x(n) = \{1, (\frac{1}{2}), (\frac{1}{2})^2, (\frac{1}{2})^3, \ldots, (\frac{1}{2})^n, \ldots\}$$

The z-transform of $x(n)$ is the infinite power series

$$X(z) = 1 + \frac{1}{2}z^{-1} + (\frac{1}{2})^2 z^{-2} + (\frac{1}{2})^n z^{-n} + \cdots$$

$$= \sum_{n=0}^{\infty} (\frac{1}{2})^n z^{-n} = \sum_{n=0}^{\infty} (\frac{1}{2}z^{-1})^n$$

This is an infinite geometric series. We recall that

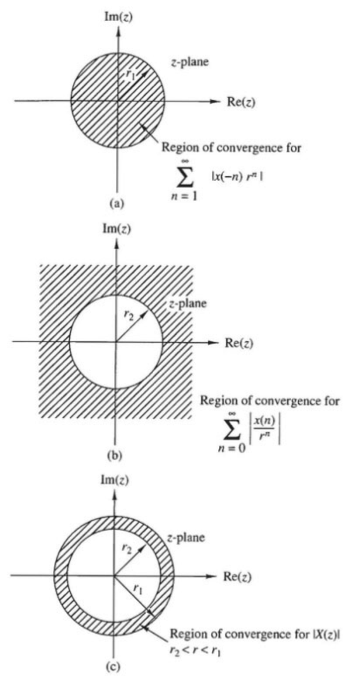
$$1 + A + A^2 + A^3 + \cdots = \frac{1}{1-A} \qquad \text{if } |A| < 1$$

Consequently, for $|\frac{1}{2}z^{-1}| < 1$, or equivalently, for $|z| > \frac{1}{2}$, $X(z)$ converges to

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \qquad \text{ROC: } |z| > \frac{1}{2}$$

We see that in this case, the z-transform provides a compact alternative representation of the signal $x(n)$.

Region of convergence for $X(z)$ and its corresponding causal and anticausal components.



Example

Determine the z-transform of the signal

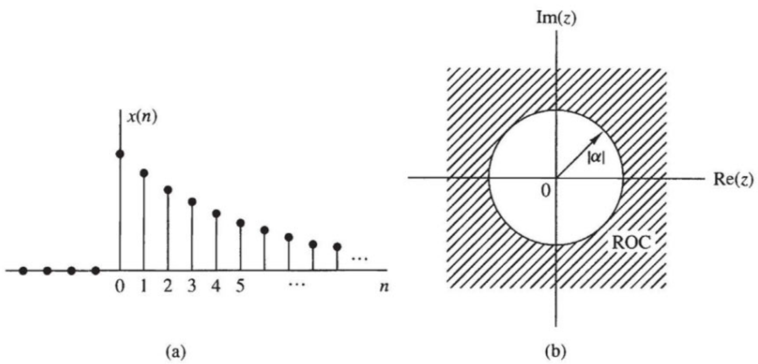
$$x(n) = \alpha^n u(n) = \begin{cases} \alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Solution. From the definition (3.1.1) we have

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

If $|\alpha z^{-1}| < 1$ or equivalently, $|z| > |\alpha|$, this power series converges to $1/(1 - \alpha z^{-1})$. Thus we have the z-transform pair

ROC



The exponential signal $x(n) = \alpha^n u(n)$ (a), and the ROC of its z-transform (b).

$$x(n) = \alpha^n u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{1 - \alpha z^{-1}}, \qquad \text{ROC: } |z| > |\alpha|$$

The ROC is the exterior of a circle having radius $|\alpha|$. Figure 3.1.2 shows a graph of the signal $x(n)$ and its corresponding ROC. Note that, in general, α need not be real.

If we set $\alpha = 1$ in (3.1.7), we obtain the z-transform of the unit step signal

$$x(n) = u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{1 - z^{-1}}, \qquad \text{ROC: } |z| > 1$$

Example

Determine the z-transform of the signal

$$x(n) = -\alpha^n u(-n-1) = \begin{cases} 0, & n \geq 0 \\ -\alpha^n, & n \leq -1 \end{cases}$$

Solution. From the definition (3.1.1) we have

$$X(z) = \sum_{n=-\infty}^{-1} (-\alpha^n) z^{-n} = -\sum_{l=1}^{\infty} (\alpha^{-1} z)^l$$

where $l = -n$. Using the formula

$$A + A^2 + A^3 + \cdots = A(1 + A + A^2 + \cdots) = \frac{A}{1-A}$$

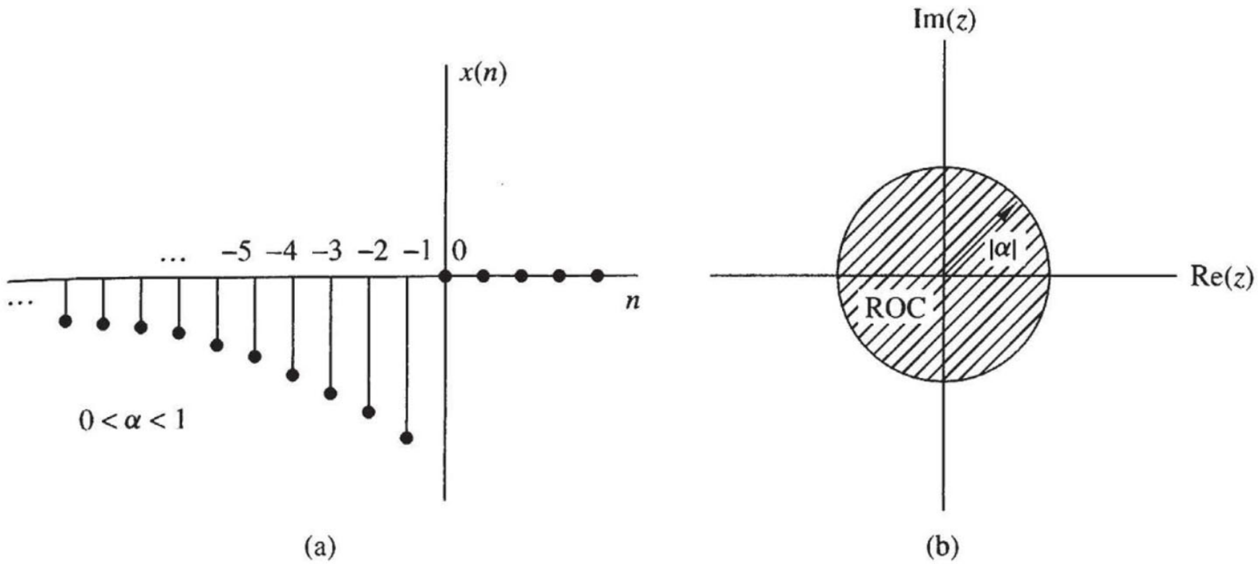
when $|A| < 1$ gives

$$X(z) = -\frac{\alpha^{-1} z}{1 - \alpha^{-1} z} = \frac{1}{1 - \alpha z^{-1}}$$

provided that $|\alpha^{-1} z| < 1$ or, equivalently, $|z| < |\alpha|$. Thus

$$x(n) = -\alpha^n u(-n-1) \xleftrightarrow{z} X(z) = -\frac{1}{1 - \alpha z^{-1}}, \qquad \text{ROC: } |z| < |\alpha|$$

The ROC is now the interior of a circle having radius $|\alpha|$. This is shown in Fig. 3.1.3.



Anticausal signal $x(n) = -\alpha^n u(-n-1)$ (a), and the ROC of its z-transform (b).

Example

Determine the z-transform of the signal

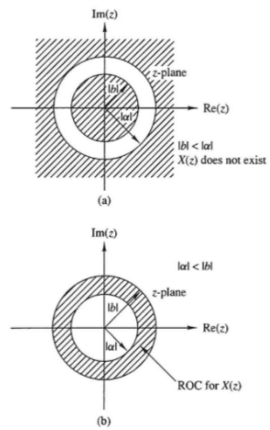
$$x(n) = \alpha^n u(n) + b^n u(-n-1)$$

Solution. From definition (3.1.1) we have

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \sum_{l=1}^{\infty} (b^{-1} z)^l$$

The first power series converges if $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$. The second power series converges if $|b^{-1} z| < 1$ or $|z| < |b|$.

In determining the convergence of $X(z)$, we consider two different cases.



ROC for z-transform in Example 3.1.5.

Case 1 $|b| < |\alpha|$: In this case the two ROC above do not overlap Consequently, we cannot find values of z for which both power series converge simultaneously. Clearly, in this case, $X(z)$ does not exist.

Case 2 $|b| > |\alpha|$: In this case there is a ring in the z-plane where both power series converge simultaneously, as shown in Fig. 3.1.4(b). Then we obtain

$$X(z) = \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-bz^{-1}}$$

$$= \frac{b-\alpha}{\alpha + b - z - \alpha b z^{-1}}$$

The ROC of $X(z)$ is $|\alpha| < |z| < |b|$.

Properties os the z-Transform

TABLE 3.2 Properties of the z-Transform

Property	Time Domain	z-Domain	ROC
Notation	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\mathrm{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$\mathrm{Im}\{x(n)\}$	$\frac{1}{2}j[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z-domain	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}\,dv$	At least, $r_{1l}r_{2l} < \|z\| < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$		

z-Transform of Basic Signals

TABLE 3.3 Some Common z-Transform Pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$(\cos\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$(\sin\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$(a^n\sin\omega_0 n)u(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

Pole-zero location

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0}{a_0} z^{-M+N} \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)}$$

$$X(z) = G z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

Example of first order system

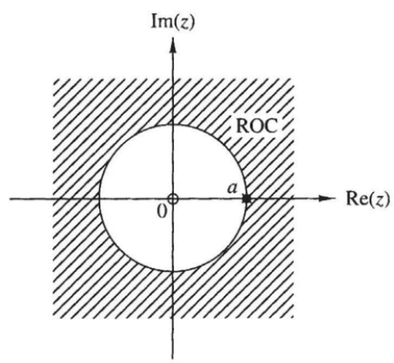
Determine the pole–zero plot for the signal

$$x(n) = a^n u(n), \qquad a > 0$$

Solution. From Table 3.3 we find that

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad \text{ROC: } |z| > a$$

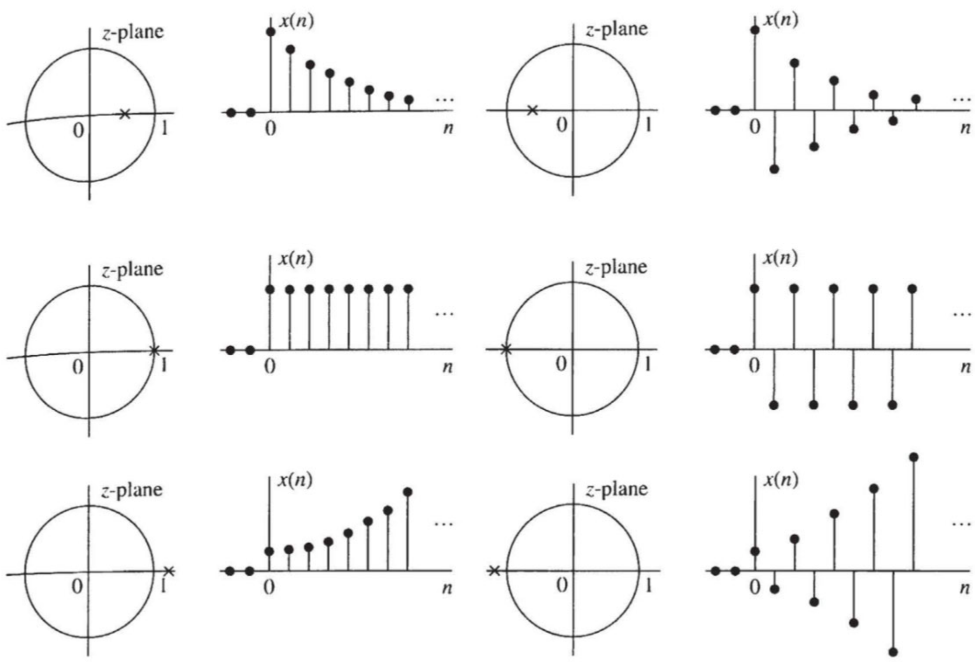
Thus $X(z)$ has one zero at $z_1 = 0$ and one pole at $p_1 = a$. The pole–zero plot is shown in Fig. 3.3.1. Note that the pole $p_1 = a$ is not included in the ROC since the z-transform does not converge at a pole.



Pole–zero plot for the causal exponential signal $x(n) = a^n u(n)$.

Time Domain Behaviour:

Time-domain behavior of a single-real-pole causal signal as a function of the location of the pole with respect to the unit circle.



System Function of LTI Systems

$$Y(z) = H(z)X(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$$

System Function derived from Difference Equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$Y(z)\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) = X(z)\left(\sum_{k=0}^{M} b_k z^{-k}\right)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

System Function of FIR System

- Let $a_k = 0 \text{ for } 1 \leq k \leq N$
- Then: $$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k}$$

There is no pole except at zero.

Example

Determine the system function and the unit sample response of the system described by the difference equation

$$y(n) = \frac{1}{2}y(n-1) + 2x(n)$$

Solution. By computing the z-transform of the difference equation, we obtain

$$Y(z) = \frac{1}{2}z^{-1}Y(z) + 2X(z)$$

Hence the system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{1}{2}z^{-1}}$$

This system has a pole at $z = \frac{1}{2}$ and a zero at the origin. Using Table 3.3 we obtain the inverse transform

$$h(n) = 2(\frac{1}{2})^n u(n)$$

This is the unit sample response of the system.

Inversion of z-Transform

1. Direct evaluation of by contour integration.

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

2. Expansion into a series of terms, in the variables z, and z^{-1}.

3. Partial-fraction expansion and table lookup.

Inverse z-Transform by Power Series Expansion

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$x(n) = c_n$ fo
division.

EXAMPLE

Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

when

(a) ROC: $|z| > 1$

(b) ROC: $|z| < 0.5$

Example

Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

when

(a) ROC: $|z| > 1$

(b) ROC: $|z| < 0.5$

Solution (a)

(a) Since the ROC is the exterior of a circle, we expect $x(n)$ to be a causal signal. Thus we seek a power series expansion in negative powers of z. By dividing the numerator of $X(z)$ by its denominator, we obtain the power series

$$X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \frac{31}{16}z^{-4} + \cdots$$

By comparing this relation with (3.1.1), we conclude that

$$x(n) = \{\underset{\uparrow}{1}, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \ldots\}$$

Solution (b)

(b) In this case the ROC is the interior of a circle. Consequently, the signal $x(n)$ is anticausal. To obtain a power series expansion in positive powers of z, we perform the long division in the following way:

$$
\begin{array}{r|l}
 & 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \cdots \\
\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & 1 \\
 & \underline{1 - 3z + 2z^2} \\
 & \quad 3z - 2z^2 \\
 & \quad \underline{3z - 9z^2 + 6z^3} \\
 & \qquad 7z^2 - 6z^3 \\
 & \qquad \underline{7z^2 - 21z^3 + 14z^4} \\
 & \qquad\quad 15z^3 - 14z^4 \\
 & \qquad\quad \underline{15z^3 - 45z^4 + 30z^5} \\
 & \qquad\qquad 31z^4 - 30z^5
\end{array}
$$

Thus

$$X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \cdots$$

In this case $x(n) = 0$ for $n \geq 0$. By comparing this result to (3.1.1), we conclude that

$$x(n) = \{\cdots 62, 30, 14, 6, 2, 0, \underset{\uparrow}{0}\}$$

Example

Determine the inverse z-transform of

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|$$

Solution. Using the power series expansion for $\log(1 + x)$, with $|x| < 1$, we have

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

Thus

$$x(n) = \begin{cases} (-1)^{n+1} \dfrac{a^n}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

Expansion of irrational functions into power series can be obtained from tables.

Inversion using partial-fraction expansion

Let $X(z)$ be a proper rational function, that is,

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

where

$$a_N \neq 0 \qquad \text{and} \qquad M < N$$

To simplify our discussion we eliminate negative powers of z by multiplying both the numerator and denominator of (3.4.12) by z^N. This results in

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

which contains only positive powers of z. Since $N > M$, the function

$$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

is also always proper.

Inversion using partial-fraction expansion

Our task in performing a partial-fraction expansion is to express This as a sum of simple fractions. We distinguish two cases.

Distinct poles. Suppose that the poles $p_1, p_2, \ldots, p_N$ are all different (distinct). Then we seek an expansion of the form

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}$$

The problem is to determine the coefficients $A_1, A_2, \ldots, A_N$. There are two ways to solve this problem, as illustrated in the following example.

Example

Determine the partial-fraction expansion of the proper function

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Solution. First we eliminate the negative powers, by multiplying both numerator and denominator by z^2. Thus

$$X(z) = \frac{z^2}{z^2 - 1.5z + 0.5}$$

The poles of $X(z)$ are $p_1 = 1$ and $p_2 = 0.5$. Consequently, the expansion is

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

A very simple method to determine A_1 and A_2 is to multiply the equation by the denominator term $(z-1)(z-0.5)$. Thus we obtain

$$z = (z - 0.5)A_1 + (z - 1)A_2$$

Solution (Continued)

Now if we set $z = p_1 = 1$ in (3.4.18), we eliminate the term involving A_2. Hence

$$1 = (1 - 0.5)A_1$$

Thus we obtain the result $A_1 = 2$. Next we return to (3.4.18) and set $z = p_2 = 0.5$, thus eliminating the term involving A_1, so we have

$$0.5 = (0.5 - 1)A_2$$

and hence $A_2 = -1$. Therefore, the result of the partial-fraction expansion is

$$\frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

General Partial-Fraction Expansion Procedure (Single Poles)

The example given above suggests that we can determine the coefficients A_1, $A_2, \ldots, A_N$, by multiplying both sides by each of the terms $(z - p_k)$, $k = 1, 2, \ldots, N$, and evaluating the resulting expressions at the corresponding pole

positions, $p_1, p_2, \ldots, p_N$. Thus we have, in general,

$$\frac{(z - p_k)X(z)}{z} = \frac{(z - p_k)A_1}{z - p_1} + \cdots + A_k + \cdots + \frac{(z - p_k)A_N}{z - p_N}$$

Consequently, with $z = p_k$, (3.4.20) yields the kth coefficient as

$$A_k = \left. \frac{(z - p_k)X(z)}{z} \right|_{z=p_k}, \qquad k = 1, 2, \ldots, N$$

Example

Determine the partial-fraction expansion of

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$

Solution. To eliminate negative powers of z we multiply both numerator and denominator by z^2. Thus

$$\frac{X(z)}{z} = \frac{z + 1}{z^2 - z + 0.5}$$

The poles of $X(z)$ are complex conjugates

$$p_1 = \frac{1}{2} + j\frac{1}{2}$$

and

$$p_2 = \frac{1}{2} - j\frac{1}{2}$$

Thus

$$\frac{X(z)}{z} = \frac{z + 1}{(z - p_1)(z - p_2)} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2}$$

we obtain

$$A_1 = \left.\frac{(z - p_1)X(z)}{z}\right|_{z=p_1} = \left.\frac{z + 1}{z - p_2}\right|_{z=p_1} = \frac{\frac{1}{2} + j\frac{1}{2} + 1}{\frac{1}{2} + j\frac{1}{2} - \frac{1}{2} + j\frac{1}{2}} = \frac{1}{2} - j\frac{3}{2}$$

$$A_2 = \left.\frac{(z - p_2)X(z)}{z}\right|_{z=p_2} = \left.\frac{z + 1}{z - p_1}\right|_{z=p_2} = \frac{\frac{1}{2} - j\frac{1}{2} + 1}{\frac{1}{2} - j\frac{1}{2} - \frac{1}{2} - j\frac{1}{2}} = \frac{1}{2} + j\frac{3}{2}$$

Multiple Order Poles (Example

Determine the partial-fraction expansion of

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2} \qquad (3.4.23)$$

Solution. First, we express (3.4.23) in terms of positive powers of z, in the form

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2}$$

$X(z)$ has a simple pole at $p_1 = -1$ and a double pole $p_2 = p_3 = 1$. In such a case the appropriate partial-fraction expansion is

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A_1}{z+1} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2} \qquad (3.4.24)$$

The problem is to determine the coefficients A_1, A_2, and A_3.

We proceed as in the case of distinct poles. To determine A_1, we multiply both sides of (3.4.24) by $(z+1)$ and evaluate the result at $z = -1$. Thus (3.4.24) becomes

$$\frac{(z+1)X(z)}{z} = A_1 + \frac{z+1}{z-1}A_2 + \frac{z+1}{(z-1)^2}A_3$$

which, when evaluated at $z = -1$, yields

$$A_1 = \left.\frac{(z+1)X(z)}{z}\right|_{z=-1} = \frac{1}{4}$$

Next, if we multiply both sides of (3.4.24) by $(z-1)^2$, we obtain

$$\frac{(z-1)^2 X(z)}{z} = \frac{(z-1)^2}{z+1}A_1 + (z-1)A_2 + A_3 \qquad (3.4.25)$$

Now, if we evaluate (3.4.25) at $z = 1$, we obtain A_3. Thus

$$A_3 = \left.\frac{(z-1)2X(z)}{z}\right|_{z=1} = \frac{1}{2}$$

The remaining coefficient A_2 can be obtained by differentiating both sides of (3.4.25) with respect to z and evaluating the result at $z = 1$. Note that it is not necessary formally to carry out the differentiation of the right-hand side of (3.4.25), since all terms except A_2 vanish when we set $z = 1$. Thus

$$A_2 = \frac{d}{dz}\left[\frac{(z-1)^2 X(z)}{z}\right]_{z=1} = \frac{3}{4} \qquad (3.4.26)$$

General Partial Fraction Expansion Procedure

The generalization of the procedure in the example above to the case of an mth-order pole $(z - p_k)^m$ is straightforward. The partial-fraction expansion must contain the terms

$$\frac{A_{1k}}{z - p_k} + \frac{A_{2k}}{(z - p_k)^2} + \cdots + \frac{A_{mk}}{(z - p_k)^m}$$

The coefficients $\{A_{ik}\}$ can be evaluated through differentiation

Causality and Stability

- A system is causal if,
  - $h(n) = 0\ for\ n < 0$
- So, an LTI system is causal if and only if the ROC of H(z) is exterior of a cirle radius $r < \infty$.

- An LTI System is BIBO stable if the unit circle lies in the region of convergence of H(z).

Causality and Stability (Example)

A linear time-invariant system is characterized by the system function

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

$$= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 3z^{-1}}$$

Specify the ROC of $H(z)$ and determine $h(n)$ for the following conditions:

(a) The system is stable.

(b) The system is causal.

(c) The system is anticausal.

Causality and Stability (Example)

Solution. The system has poles at $z = \frac{1}{2}$ and $z = 3$.

(a) Since the system is stable, its ROC must include the unit circle and hence it is $\frac{1}{2} < |z| < 3$. Consequently, $h(n)$ is noncausal and is given as

$$h(n) = (\frac{1}{2})^n u(n) - 2(3)^n u(-n-1)$$

(b) Since the system is causal, its ROC is $|z| > 3$. In this case

$$h(n) = (\frac{1}{2})^n u(n) + 2(3)^n u(n)$$

This system is unstable.

(c) If the system is anticausal, its ROC is $|z| < 0.5$. Hence

$$h(n) = -[(\frac{1}{2})^n + 2(3)^n]u(-n-1)$$

In this case the system is unstable.

One-sided z-Transform

The *one-sided* or *unilateral* z-transform of a signal $x(n)$ is defined by

$$X^+(z) \equiv \sum_{n=0}^{\infty} x(n)z^{-n}$$

We also use the notations $Z^+\{x(n)\}$ and

$$x(n) \overset{z^+}{\longleftrightarrow} X^+(z)$$

One-sided z-Transform (Examples)

Determine the z-transforms of the following *finite-duration* signals.

(a) $x_1(n) = \{1, 2, 5, 7, 0, 1\}$

(b) $x_2(n) = \{1, 2, 5, 7, 0, 1\}$

(c) $x_3(n) = \{\underset{\uparrow}{0}, 0, 1, 2, 5, 7, 0, 1\}$

(d) $x_4(n) = \{\underset{\uparrow}{2}, 4, \underset{\uparrow}{5}, 7, 0, 1\}$

(e) $x_5(n) = \delta(n)$

(f) $x_6(n) = \delta(n - k), k > 0$

(g) $x_7(n) = \delta(n + k), k > 0$

One-sided z-Transform (Examples)

Solution.

$$x_1(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\} \overset{z^+}{\longleftrightarrow} X_1^+(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

$$x_2(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\} \overset{z^+}{\longleftrightarrow} X_2^+(z) = 5 + 7z^{-1} + z^{-3}$$

$$x_3(n) = \{\underset{\uparrow}{0}, 0, 1, 2, 5, 7, 0, 1\} \overset{z^+}{\longleftrightarrow} X_3^+(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$$

$$x_4(n) = \{2, 4, \underset{\uparrow}{5}, 7, 0, 1\} \overset{z^+}{\longleftrightarrow} X_4^+(z) = 5 + 7z^{-1} + z^{-3}$$

$$x_5(n) = \delta(n) \overset{z^+}{\longleftrightarrow} X_5^+(z) = 1$$

$$x_6(n) = \delta(n - k), \qquad k > 0 \overset{z^+}{\longleftrightarrow} X_6^+(z) = z^{-k}$$

$$x_7(n) = \delta(n + k), \qquad k > 0 \overset{z^+}{\longleftrightarrow} X_7^+(z) = 0$$

One-sided z-Transform (Properties)

Shfiting Property

Case 1: Time delay If

$$x(n) \xleftrightarrow{z^+} X^+(z)$$

then

$$x(n-k) \xleftrightarrow{z^+} z^{-k}[X^+(z) + \sum_{n=1}^{k} x(-n)z^n], \qquad k > 0 \tag{3.6.2}$$

In case $x(n)$ is causal, then

$$x(n-k) \xleftrightarrow{z^+} z^{-k}X^+(z) \tag{3.6.3}$$

Proof From the definition (3.6.1) we have

$$Z^+\{x(n-k)\} = z^{-k}\left[\sum_{l=-k}^{-1} x(l)z^{-l} + \sum_{l=0}^{\infty} x(l)z^{-l}\right]$$

$$= z^{-k}\left[\sum_{l=-1}^{-k} x(l)z^{-l} + X^+(z)\right]$$

By changing the index from l to $n = -l$, the result in (3.6.2) is easily obtained.

Example

Determine the one-sided z-transform of the signals

(a) $x(n) = a^n u(n)$

(b) $x_1(n) = x(n-2)$ where $x(n) = a^n$

Solution.

(a) From (3.6.1) we easily obtain

$$X^+(z) = \frac{1}{1 - az^{-1}}$$

(b) We will apply the shifting property for $k = 2$. Indeed, we have

$$\begin{aligned} Z^+\{x(n-2)\} &= z^{-2}[X^+(z) + x(-1)z + x(-2)z^2] \\ &= z^{-2}X^+(z) + x(-1)z^{-1} + x(-2) \end{aligned}$$

Since $x(-1) = a^{-1}$, $x(-2) = a^{-2}$, we obtain

$$X_1^+(z) = \frac{z^{-2}}{1 - az^{-1}} + a^{-1}z^{-1} + a^{-2}$$

Properties (Time Advance)

Case 2: Time advance If

$$x(n) \overset{z^+}{\longleftrightarrow} X^+(z)$$

then

$$x(n+k) \overset{z^+}{\longleftrightarrow} z^k \left[X^+(z) - \sum_{n=0}^{k-1} x(n) z^{-n} \right], \qquad k > 0 \qquad (3.6.5)$$

Proof From (3.6.1) we have

$$Z^+\{x(n+k)\} = \sum_{n=0}^{\infty} x(n+k) z^{-n} = z^k \sum_{l=k}^{\infty} x(l) z^{-l}$$

where we have changed the index of summation from n to $l = n + k$. Now, from (3.6.1) we obtain

$$X^+(z) = \sum_{l=0}^{\infty} x(l) z^{-l} = \sum_{l=0}^{k-1} x(l) z^{-l} + \sum_{l=k}^{\infty} x(l) z^{-l}$$

By combining the last two relations, we easily obtain (3.6.5).

Example (Time Advance)

With $x(n)$, as given in Example 3.6.2, determine the one-sided z-transform of the signal

$$x_2(n) = x(n+2)$$

Solution. We will apply the shifting theorem for $k = 2$. From (3.6.5), with $k = 2$, we obtain

$$Z^+\{x(n+2)\} = z^2 X^+(z) - x(0)z^2 - x(1)z$$

But $x(0) = 1$, $x(1) = a$, and $X^+(z) = 1/(1 - az^{-1})$. Thus

$$Z^+\{x(n+2)\} = \frac{z^2}{1 - az^{-1}} - z^2 - az$$

Properties (Time Advance)

With $x(n)$, as given in Example 3.6.2, determine the one-sided z-transform of the signal

$$x_2(n) = x(n+2)$$

Solution. We will apply the shifting theorem for $k = 2$. From (3.6.5), with $k = 2$, we obtain

$$Z^+\{x(n+2)\} = z^2 X^+(z) - x(0)z^2 - x(1)z$$

But $x(0) = 1$, $x(1) = a$, and $X^+(z) = 1/(1 - az^{-1})$. Thus

$$Z^+\{x(n+2)\} = \frac{z^2}{1 - az^{-1}} - z^2 - az$$

Final Value Theorem

Final Value Theorem. If

$$x(n) \overset{z^+}{\longleftrightarrow} X^+(z)$$

then

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)X^+(z) \tag{3.6.6}$$

The limit in (3.6.6) exists if the ROC of $(z-1)X^+(z)$ includes the unit circle.

The proof of this theorem is left as an exercise for the reader.

This theorem is useful when we are interested in the asymptotic behavior of a signal $x(n)$ and we know its z-transform, but not the signal itself. In such cases, especially if it is complicated to invert $X^+(z)$, we can use the final value theorem to determine the limit of $x(n)$ as n goes to infinity.

Final Value Theorem (Example)

The impulse response of a relaxed linear time-invariant system is $h(n) = \alpha^n u(n)$, $|\alpha| < 1$. Determine the value of the step response of the system as $n \to \infty$.

Solution. The step response of the system is

$$y(n) = h(n) * x(n)$$

where

$$x(n) = u(n)$$

Obviously, if we excite a causal system with a causal input the output will be causal. Since $h(n)$, $x(n)$, $y(n)$ are causal signals, the one-sided and two-sided z-transforms are identical. From the convolution property (3.2.17) we know that the z-transforms of $h(n)$ and $x(n)$ must be multiplied to yield the z-transform of the output. Thus

$$Y(z) = \frac{1}{1-\alpha z^{-1}} \frac{1}{1-z^{-1}} = \frac{z^2}{(z-1)(z-\alpha)}, \qquad \text{ROC: } |z| > |\alpha|$$

Now

$$(z-1)Y(z) = \frac{z^2}{z-\alpha}, \qquad \text{ROC: } |z| < |\alpha|$$

Since $|\alpha| < 1$, the ROC of $(z-1)Y(z)$ includes the unit circle. Consequently, we can apply (3.6.6) and obtain

$$\lim_{n\to\infty} y(n) = \lim_{z\to 1} \frac{z^2}{z-\alpha} = \frac{1}{1-\alpha}$$