Lecture 3

Sampling and Reconstruction of Signals

Sampling

- To use digital signal processing techniques on continuous-time signals, we need to them into a sequence of numbers. This is done by taking samples of the analog signal periodically (one evert T seconds):

$$x(n) = x_a(nT), \qquad -\infty < n < \infty$$

- The sampling frequency $F_s = 1/T$ has to be large enough so that the signal can be recovered from the samples x[n]. The limit is the Nyquist rate that is twice the highest frequency of the analog signal. That is, if the highest frequency of the signal is B Hz. the sampling rate need to be greater than (or at least equal to) 2B samples per second.

Sampling

- If the signal $x_a(t)$ is an aperiodic signal with finite energy, its spectrum is:

$$X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi Ft}\, dt$$

- And it can be recovered from its spectrum using:

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi Ft}\, dF$$

Sampling

- The spectrum of a discrete-time signal is:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

- Or equivalently,

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n}$$

Sampling

- The signal x[n] can be recovered from the spectrum:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\,d\omega$$

$$= \int_{-1/2}^{1/2} X(f)e^{j2\pi f n}\,df$$

The relationship beteen time in analog domain (t) and digital domain (n) is:

$$t = nT = \frac{n}{F_s}$$

Sampling

- In fact x[n] represents a sequence of samples of the analog signal at time nT:

$$x(n) \equiv x_a(nT) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi nF/F_s}\, dF$$

So,

$$\int_{-1/2}^{1/2} X(f) e^{j2\pi fn}\, df = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi nF/F_s}\, dF$$

The relationship between analog and digital frequencies is:

$$f = \frac{F}{F_s}$$

Sampling

- Substituting $f = \frac{F}{F_s}$ in the equality in the previous slide, we get:

$$\frac{1}{F_s}\int_{-F_s/2}^{F_s/2} X(F)e^{j2\pi nF/F_s}\,dF = \int_{-\infty}^{\infty} X_a(F)e^{j2\pi nF/F_s}\,dF$$

- The right-hand side can be written as,

$$\int_{-\infty}^{\infty} X_a(F)e^{j2\pi nF/F_s}\,dF = \sum_{k=-\infty}^{\infty}\int_{(k-1/2)F_s}^{(k+1/2)F_s} X_a(F)e^{j2\pi nF/F_s}\,dF$$

Sampling

- We observe that $X_a(F)$ in the interval $\left(k-\frac{1}{2}\right)F_s$ to $\left(k+\frac{1}{2}\right)F_s$ is identical to $X_a(F-kF_s)$ in the interval $-F_s/2$ to $F_s/2$, so:

$$\sum_{k=-\infty}^{\infty}\int_{(k-1/2)F_s}^{(k+1/2)F_s} X_a(F)e^{j2\pi nF/F_s}\,dF = \sum_{k=-\infty}^{\infty}\int_{-F_s/2}^{F_s/2} X_a(F-kF_s)e^{j2\pi nF/F_s}\,dF$$

$$= \int_{-F_s/2}^{F_s/2}\left[\sum_{k=-\infty}^{\infty} X_a(F-kF_s)\right]e^{j2\pi nF/F_s}\,d$$

We have used the fact that

$$e^{j2\pi n(F+kF_s)/F_s} = e^{j2\pi nF/F_s}$$

Sampling

- Comparing the equations in the previous two slides, we get,

$$X(F) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s)$$

- Or equivalently,

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a[(f - k)F_s]$$

- So if $F_s \geq 2B$, that is, if the sampling rate exceeds the Nyquist rate, the analog signal, then the analog signal can be recovered from the samples,

$$X(F) = F_s X_a(F), \qquad |F| \leq F_s/2$$

Reconstruction

- Given the sampled signals spectrum X(f), we get,

$$X_a(F) = \begin{cases} \frac{1}{F_s} X(F), & |F| \le F_s/2 \\ 0, & |F| > F_s/2 \end{cases}$$

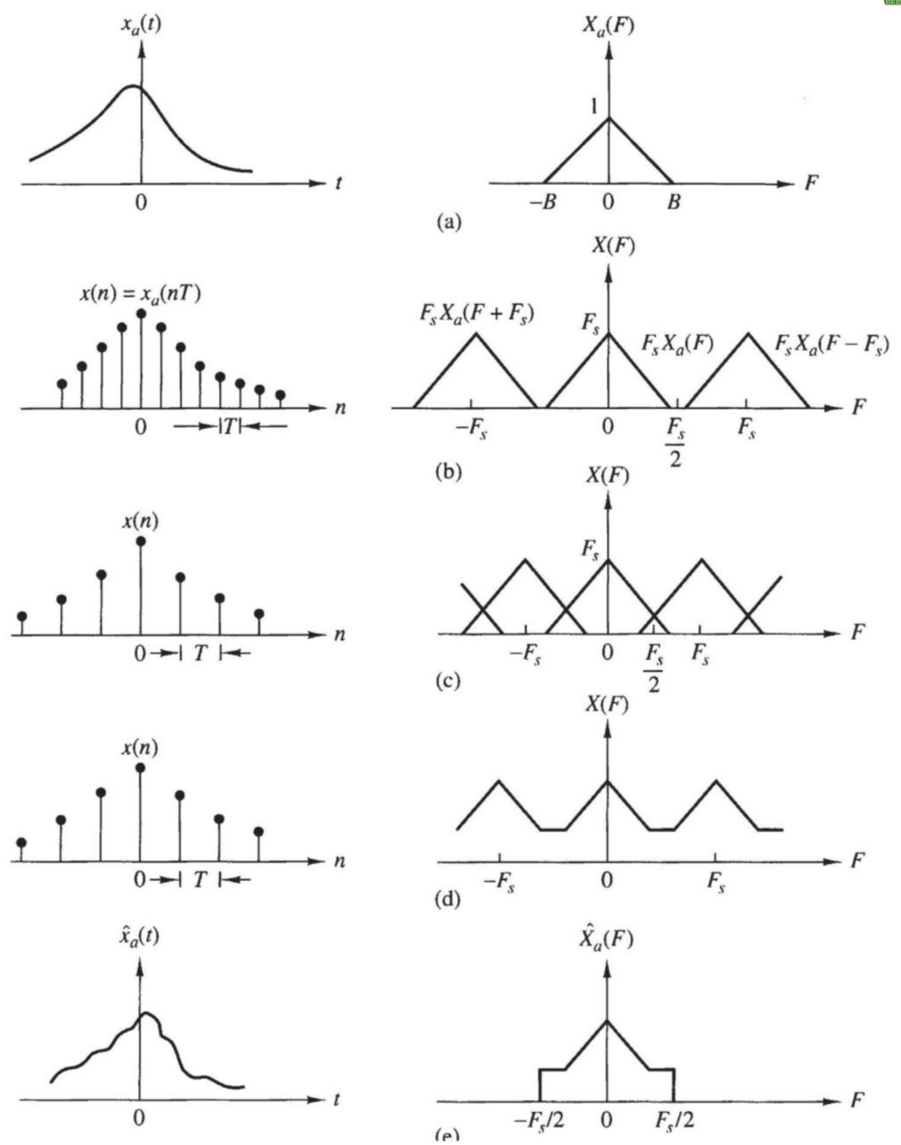
- From Fourier transform,

$$X(F) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi Fn/F_s}$$

- We have

$$x_a(t) = \int_{-F_s/2}^{F_s/2} X_a(F) e^{j2\pi Ft} \, dF$$

Reconstruction



Sampling of an analog bandlimited signal and aliasing of spectral components

Aliasing: $F_s < 2B$

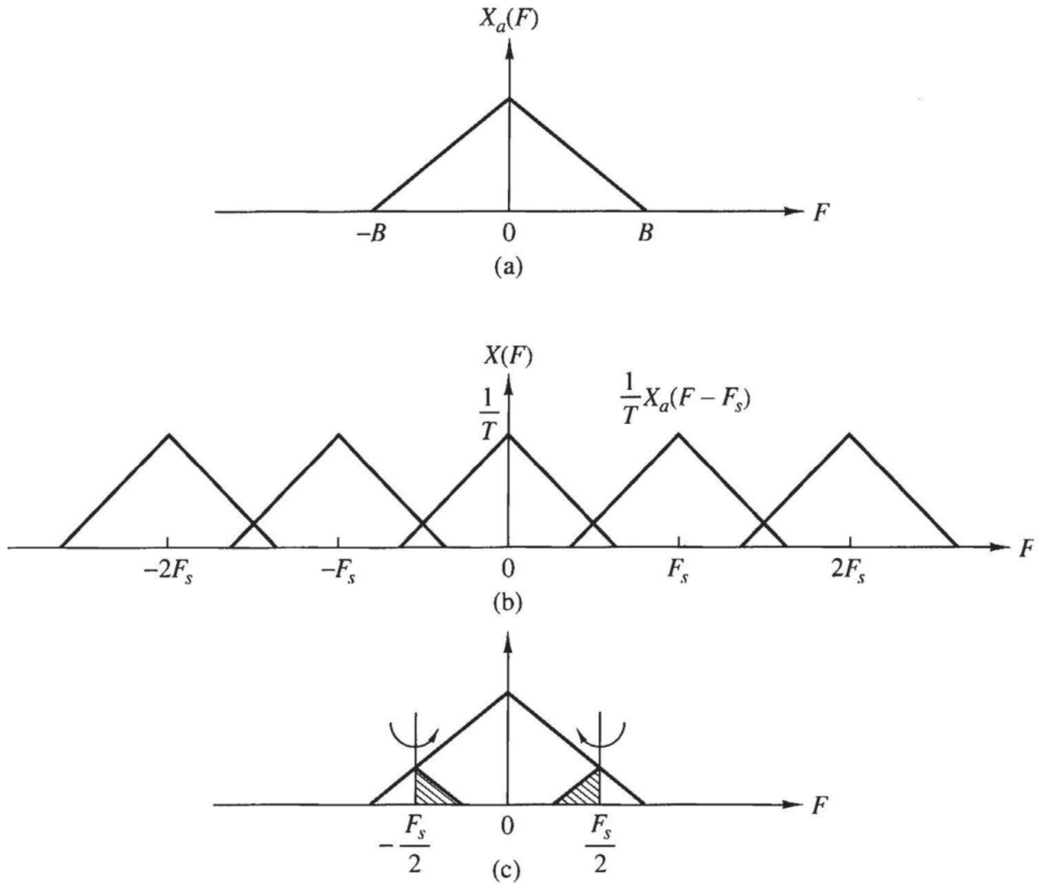


Illustration of aliasing around the folding frequency.

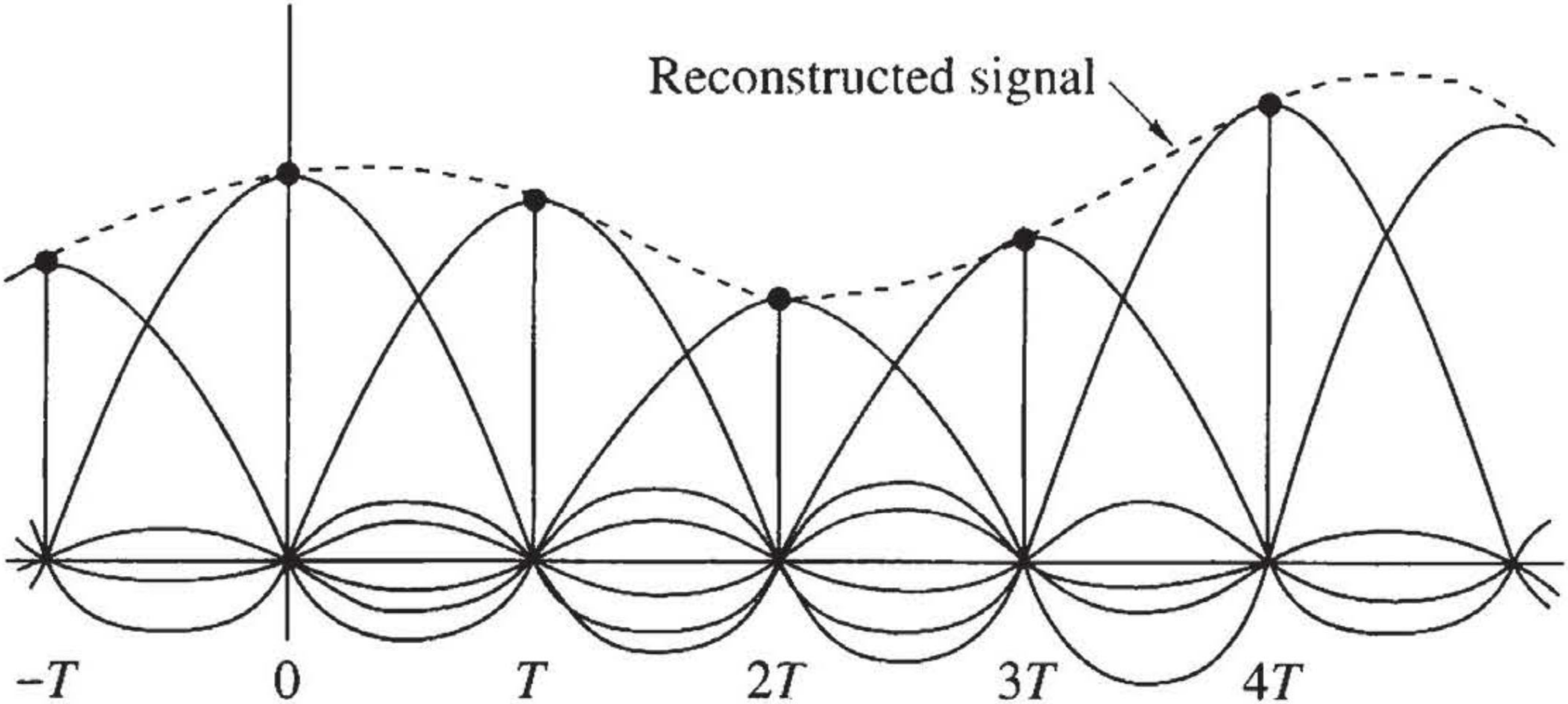
Reconstruction: $F_s \geq 2B$

$$x_a(t) = \frac{1}{F_s}\int_{-F_s/2}^{F_s/2}\left[\sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi Fn/F_s}\right]e^{j2\pi Ft}\,dF$$

$$= \frac{1}{F_s}\sum_{n=-\infty}^{\infty} x(n)\int_{-F_s/2}^{F_s/2} e^{j2\pi F(t-n/F_s)}\,dF$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT)\frac{\sin(\pi/T)(t-nT)}{(\pi/T)(t-nT)}$$

Reconstruction: $F_s \geq 2B$



Sampling (Nyquist) Theorem

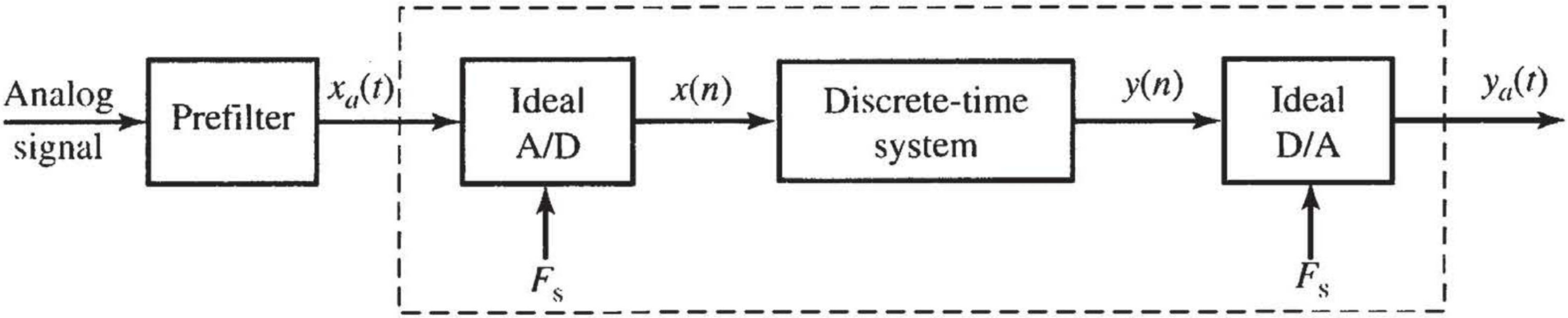
Sampling Theorem. A bandlimited continuous-time signal, with highest frequency (bandwidth) B hertz, can be uniquely recovered from its samples provided that the sampling rate $F_s \geq 2B$ samples per second.

Sampling: Example

- Speech Signals mainly contain components in frequencies less that 3400 Hz. So, 6800 samples per second is the minimum sampling rate. To make reconstruction easier (less sharp filters) 8000 samples are taken per second. That is one sample every $T = 1\ \mu s$ (micro-seconds).
- Audio contains frequencies up to 20 kHz. So minimum is 40,000 samples/sec. Usually 44.1 k samples are taken each second.
- Video: The number of samples for a video signal depends on the number of pixels (picture elements) per frame and the number of frames per second.

Discrete-Processing of Analog Signals

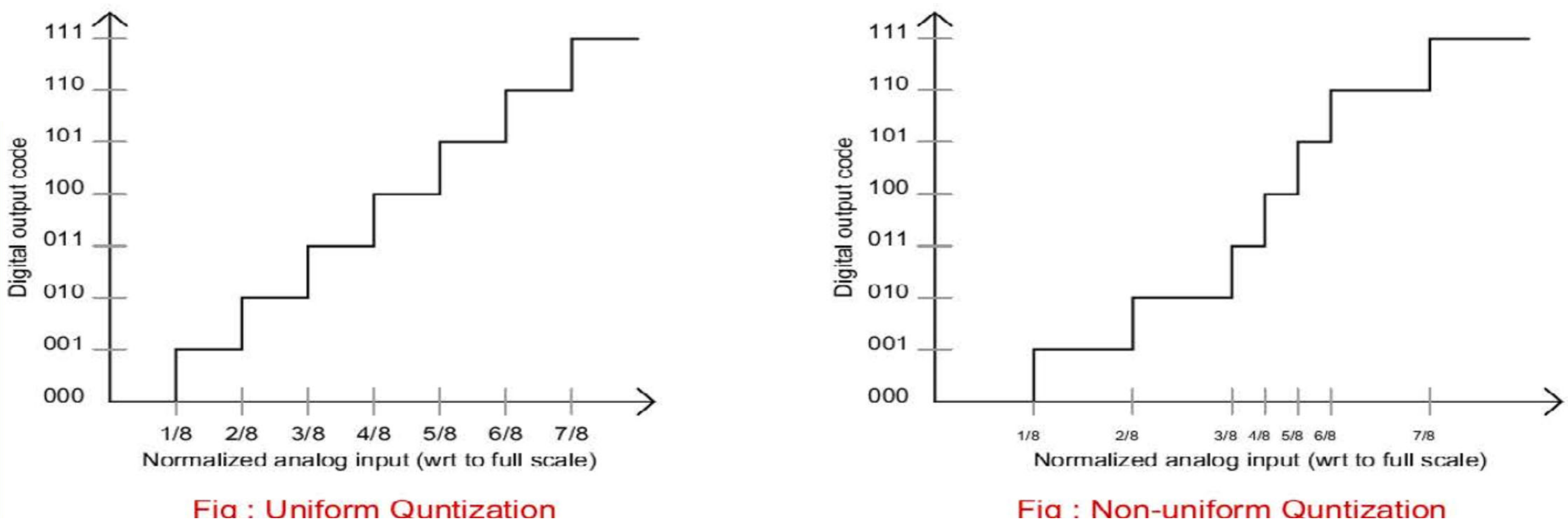
- First the analog signal is digital signal using Analog-to-Digital (A to D or A/D) converter, then it is processed by the Discrete-Time (most often Digital) system and then reconverted to Analog using D/A converter.



System for the discrete-time processing of continuous-time signals.

Quantization

After sampling a source whether audio or video, we need to convert the voltage level obtained into a finite number of values so that we can represent each sample of the audio signal or each pixel with a finite number of bits.



Quantization

An input sample x is mapped into a discrete level $\hat{x}$. So, the quantization error is $e = x - \hat{x}$. The average squared error (MSE) will be,

$$\sigma_q^2 = E[(x - \hat{x})^2]$$

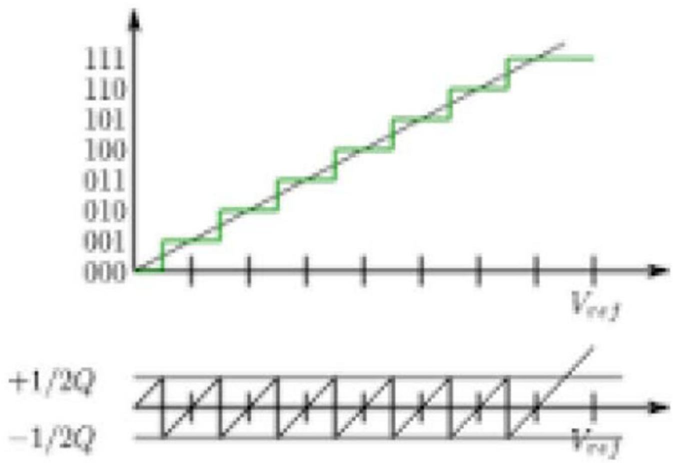
$$= \int_{t_0}^{t_L} (x - \hat{x})^2 \mathrm{p(x)dx}$$

$$= \sum_{i=1}^{L} \int_{t_{i-1}}^{t_i} (x - q_i)^2 \mathrm{p(x)dx}.$$

Where t_i, $i = 0, 1, \dots, L$ are the thresholds and q_i, $i = 1, \dots, L$ are the discrete level value. That is if $t_{i-1} < x \leq t_i$, then $\hat{x} = q_i$.

Quantization

- For a uniform source e is uniformly distributed between $-\frac{Q}{2}$ and $\frac{Q}{2}$.



Quantization Error

- So,

$$\sigma_q^2 = \frac{1}{Q}\int_{-\frac{Q}{2}}^{\frac{Q}{2}} e^2 de = \frac{Q^2}{12}.$$

Let the peak-to-peak value of the signal be 2V, i.e., $t_0 = -V$ and $t_k = +V$. Then $Q = \frac{2V}{L}$ and $\sigma_q^2 = \frac{V^2}{3L^2} = \frac{V^2}{3\times 2^{2n}}$.

Where $n = \log_2 L$ is the number of bits required for representing L levels of the ADC.

Denoting the signal power by σ_S^2, we have the signal-to-quantization-noise ratio (SQNR) in dB as:

$$SQNR = 10\log\left(\frac{3\sigma_S^2}{V^2}\right) + 6n.$$

Quantization Error : Example

- Exercise : a) Find the SQNR in dB for a sinusoidal signal with amplitude A quantized with an 8 bit uniform quantizer.

  b) Find SQNR for a Gaussian source quantized with an 8 bit uniform quantizer. The probability of overload should be less than 1%.