Lecture 7

Design of Digital Filters

Outline of this Lecture

- In this lecture, we try to use the concepts we have learned in previous lectures to design different types of digital filters. By different type, we mean what range of frequencies they pass and what frequencies they reject, i.e., are they low-pass, high-pass or band-pass, band-reject and so on. Also what is the ranges of frequencies they pass (pass-band), what range of frequencies they reject (stop-band) what is the transition band. Finally, how sharp is the transition from pass-band to stop-band and how-much ripple do we have in the pass-band and stop-band.
- First, we talk about causality, and the conditions for a system (filter) being being physically realizable (causal) and the implications causality has, e.g., the relationship between the odd and event constituents of the filter and between the real and imaginary part of the filter's spectrum (transfer function).
- Then we talk about different techniques used for the design of FIR and IIR filters.
- Finally, we will talk about the ways to transform a prototype filter into different type of filters. This simplifies an engineer's job by allowing design portability. Imagine designing a low-pass filter according to a set of criteria and then being able to change it into another low-pass filter with different frequency range, or even high-pass or band-pass filter with different frequency –domain characteristics.

Causality

- Consider an ideal low-pass filter with the cutoff frequency ω_c:

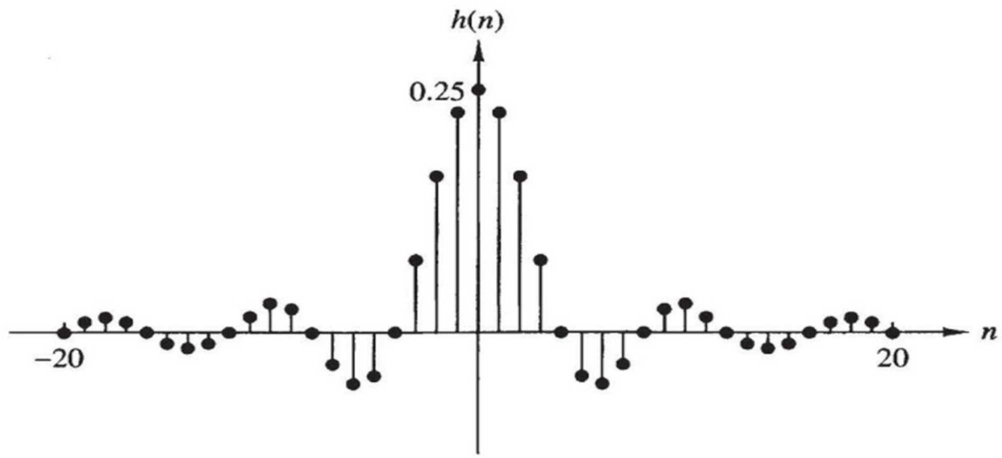
$$H(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < \omega \le \pi \end{cases}$$

- The impulse response is

$$h(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \dfrac{\omega_c}{\pi} \dfrac{\sin \omega_c n}{\omega_c n}, & n \ne 0 \end{cases}$$

Causality

- A plot of the impulse response is shown here:



It is obvious that the ideal low-pass filter is not causal.

Condition for Causality

- The question is: What is the condition that $H(\omega)$ need to satisfy in order for the filter to be causal?
- **Paley-Wiener Theorem**: If $h(n)$has finite energy and $h(n) = 0, n < 0$, then:

$$\int_{-\pi}^{\pi} |ln|H(\omega)||d\omega < \infty$$

Conversely: if $|H(\omega)|$ is square integrable and $\int_{-\pi}^{\pi} |ln|H(\omega)||d\omega < \infty$ then, it is possible to associate a phase function $\Theta(\omega)$ to the magnitude $|H(\omega)|$ to get:

$$H(\omega) = |H(\omega)|e^{j\Theta(\omega)}$$

that is the frequency response of a causal filter $h(n)$ with $h(n) = 0, n < 0$.

Note: Paley-Wiener condition implies that no filter whose $H(\omega)$ is zero over a finite band of frequency can be causal. Of course, $H(\omega)$ can be zero at some frequencies but not over a continuous band of frequencies (why?).

Implications of Causality

- We have talked about odd and even functions and the fact that any function can be written as the some of an odd and an even function:

$$h(n) = h_e(n) + h_o(n)$$

- where:

$$h_e(n) = \frac{1}{2}[h(n) + h(-n)]$$

- and,

$$h_o(n) = \frac{1}{2}[h(n) - h(-n)]$$

Now, if $h(n)$ is causal, it is possible to recover $h(n)$ from its even part $h_e(n)$ for $0 \leq n \leq \infty$ or from its odd component $h_o(n)$ for $1 \leq n \leq \infty$.

- That is:

$$h(n) = 2h_e(n)u(n) - h_e(0)\delta(n), \qquad n \geq 0$$

- and

$$h(n) = 2h_o(n)u(n) + h(0)\delta(n), \qquad n \geq 1$$

Implications of Causality

- If $h(n)$ is absolute summable, i.e., BIBO stable, then $H(\omega)$ exists and we have $H(\omega) = H_R(\omega) + jH_I(\omega)$.

In addition, if $h(n)$ is real valued and causal, the symmetry properties of the Fourier transform imply that

$$h_e(n) \overset{F}{\longleftrightarrow} H_R(\omega)$$

$$h_o(n) \overset{F}{\longleftrightarrow} H_I(\omega)$$

Since $h(n)$ is completely specified by $h_e(n)$, it follows that $H(\omega)$ is completely determined if we know $H_R(\omega)$. Alternatively, $H(\omega)$ is completely determined from $H_I(\omega)$ and $h(0)$. In short, $H_R(\omega)$ and $H_I(\omega)$ are interdependent and cannot be specified independently if the system is causal. Equivalently, the magnitude and phase responses of a causal filter are interdependent and hence cannot be specified independently.

Given $H_R(\omega)$ for a corresponding real, even, and absolutely summable sequence $h_e(n)$, we can determine $H(\omega)$.

Causality: Example

- The following example clarifies the ideas presented in the previous slides.

Consider a stable LTI system with real and even impulse response $h(n)$. Determine $H(\omega)$ if

$$H_R(\omega) = \frac{1 - a\cos\omega}{1 - 2a\cos\omega + a^2}, \qquad |a| < 1$$

Solution. The first step is to determine $h_e(n)$. This can be done by noting that

$$H_R(\omega) = H_R(z)|_{z=e^{j\omega}}$$

where

$$H_R(z) = \frac{1 - a(z + z^{-1})/2}{1 - a(z + z^{-1}) + a^2} = \frac{z - a(z^2 + 1)/2}{(z - a)(1 - az)}$$

The ROC has to be restricted by the poles at $p_1 = a$ and $p_2 = 1/a$ and should include the unit circle. Hence the ROC is $|a| < |z| < 1/|a|$. Consequently, $h_e(n)$ is a two-sided sequence, with the pole at $z = a$ contributing to the causal part and $p_2 = 1/a$ contributing to the anticausal part. By using a partial-fraction expansion, we obtain

$$h_e(n) = \tfrac{1}{2}a^{|n|} + \tfrac{1}{2}\delta(n)$$

Causality: Example

- Substituting:

$$h_e(n) = \tfrac{1}{2}a^{|n|} + \tfrac{1}{2}\delta(n)$$

- in

$$h(n) = 2h_e(n)u(n) - h_e(0)\delta(n), \qquad n \geq 0$$

- we get:

$$h(n) = a^n u(n)$$

- and

$$H(\omega) = \frac{1}{1 - ae^{-j\omega}}$$

Relationship Between $H_R(\omega)$ and $H_I(\omega)$

- Take the Fourier Transform of: $h(n) = 2h_e(n)u(n) - h_e(0)\delta(n), \qquad n \geq 0$
- to get $H(\omega)$ as twice the convolution of the Fourier Transforms of $h_e(n)$ and the unit step $u(n)$. We saw that: $h_e(n) \overset{F}{\longleftrightarrow} H_R(\omega)$
- so

$$H(\omega) = H_R(\omega) + jH_I(\omega) = \frac{1}{\pi}\int_{-\pi}^{\pi} H_R(\lambda)U(\omega - \lambda)d\lambda - h_e(0)$$

- where $U(\omega)$is the Fourier Transform of $u(n)$ given as:

$$U(\omega) = \pi\delta(\omega) + \frac{1}{1 - e^{-j\omega}} = \pi\delta(\omega) + \frac{1}{2} - j\frac{1}{2}\cot\frac{\omega}{2}, \qquad -\pi \leq \omega \leq \pi$$

- Combining the two equations, we get a relationship $H_I(\omega)$ in terms of $H_R(\omega)$:

$$H_I(\omega) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} H_R(\lambda)\cot\frac{\omega - \lambda}{2}d\lambda$$

- This is called the Hilbert Transform.

Practical versus Ideal Filters

- The facts the:
- 1) It is not possible to have infinitely sharp cutoff and,
- 2) The spectrum cannot be zero over a contiguous band of frequencies,
- rules out the possibility of implementing Ideal Filters.
- So, the challenge is to design a filter that is a close as possible to ideal subject to constraints on implementation complexity, delay, etc.
- A realizable (causal) LTI filter can be characterized using a difference equation like:

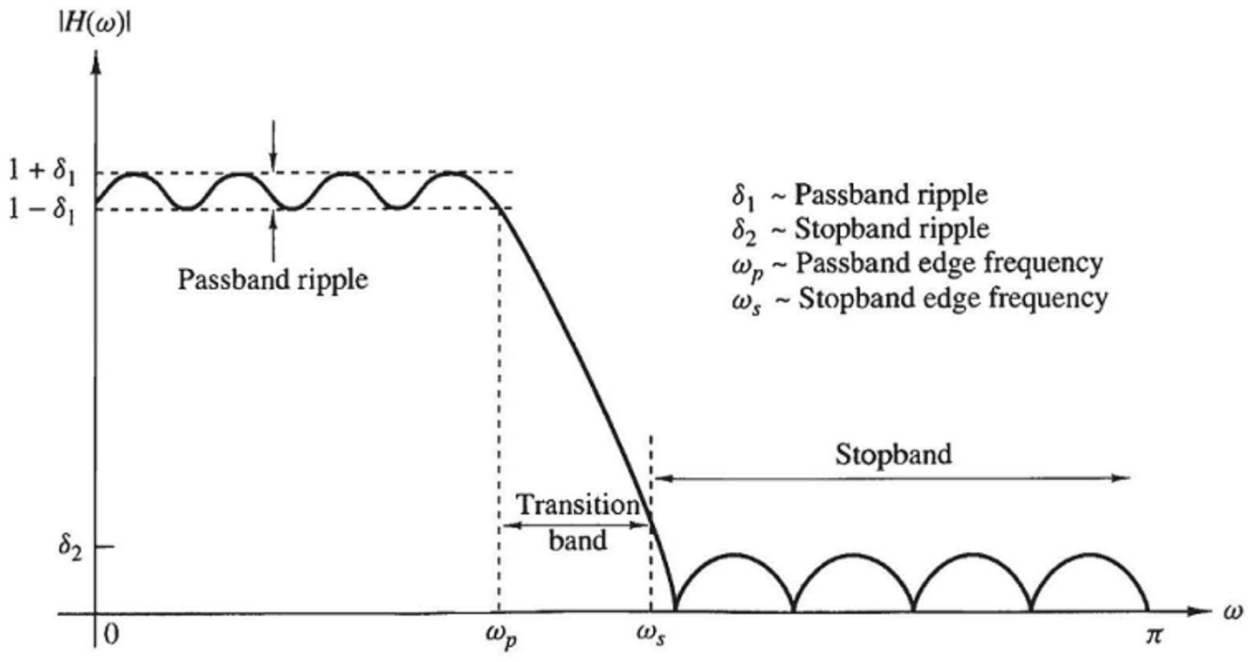
$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M-1} b_k x(n-k)$$

- This implies that the transfer function is:

$$H(\omega) = \frac{\sum_{k=0}^{M-1} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}}$$

Practical versus Ideal Filters

- The magnitude plot of such a filter typically looks like:



Practical versus Ideal Filters

- The performance of a filter like the one shown in the previous slide is characterized by:
- Ripple in the pass band: δ_1,
- Ripple in the stop-band: δ_2,
- The cutoff frequency: ω_p, and,
- The transition band: the band of frequency between where the pass-band ends and where the stop-band starts: $\omega_s - \omega_p$.
- In the next few slides, we will discuss the different techniques for design of practical filters and compare them in terms of the tradeoff between the sharpness of the spectrum, i.e., the value of $\omega_s - \omega_p$ and the ripple.

Design of FIR Filters Using Windowing

- We start with FIR filters. FIR filters are not only important as their own, but also they are important since any IIR filter, in practice can be approximated with an FIR filter with sufficient number of taps.
- The most intuitive way to implement FIR filters is using windowing.
- Assume that we are asked to design a filter with the desired spectrum:
- $H_d(\omega) = \sum_{n=0}^{\infty} h_d(n) e^{-j\omega n}$, where,

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

- Assume that we want to approximate it with an M-tap filter. The most simple thing to do is to take the values of $h_d(n)$ for values $n = 0, 1, \ldots, M-1$. That is just truncating the desired filter at point $n = M - 1$.

Design of FIR Filters Using Windowing

- This truncation is equivalent to using a rectangular window:

$$w(n) = \begin{cases} 1, & n = 0, 1, \ldots, M-1 \\ 0, & \text{otherwise} \end{cases}$$

- to the desired filter, i.e., designing an FIR filter $h(n)$ given as:

$$h(n) = h_d(n)w(n) = \begin{cases} h_d(n), & n = 0, 1, \ldots, M-1 \\ 0, & \text{otherwise} \end{cases}$$

- The window function $w(n)$ has the spectrum:

$$W(\omega) = \sum_{n=0}^{M-1} w(n)e^{-j\omega n}$$

- So, the spectrum of the FIR filter is:

$$H(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\nu)W(\omega - \nu)d\nu$$

Design of FIR Filters Using Windowing

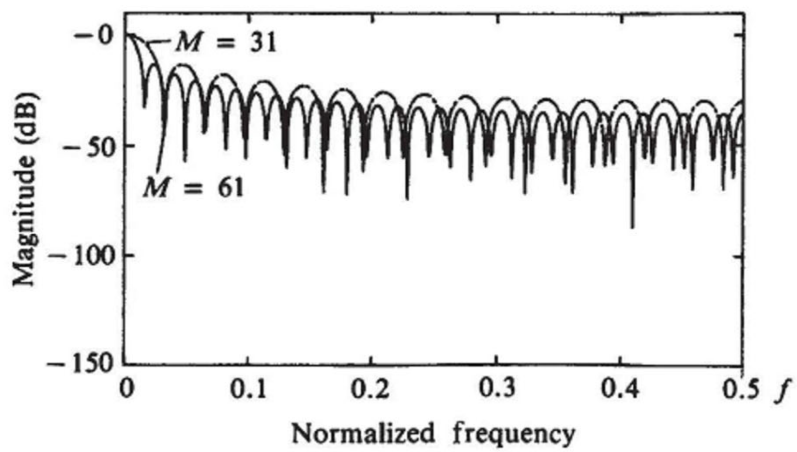
- In the case of rectangular window:

$$W(\omega) = \sum_{n=0}^{M-1} e^{-j\omega n} = \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}} = e^{-j\omega(M-1)/2} \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

- So the window has a magnitude response of:

$$|W(\omega)| = \frac{|\sin(\omega M/2)|}{|\sin(\omega/2)|}, \qquad \pi \leq \omega \leq \pi$$

Frequency response for rectangular window of lengths (a) $M = 31$, (b) $M = 61$.



Design of FIR Filters Using Windowing

- The main-lobe is reduced as M increases (it is $4\pi/M$), but the side-lobes remain unchanged.
- The rectangular window has the piece-wise linear phase:

$$\Theta(\omega) = \begin{cases} -\omega\left(\dfrac{M-1}{2}\right), & \text{when } \sin(\omega M/2) \geq 0 \\ -\omega\left(\dfrac{M-1}{2}\right) + \pi, & \text{when } \sin(\omega M/2) < 0 \end{cases}$$

- In the next slide, we present the expression for other windows, followed by some magnitude plot of some and a table comparing their characteristics.

Different filters for FIR Filter Design

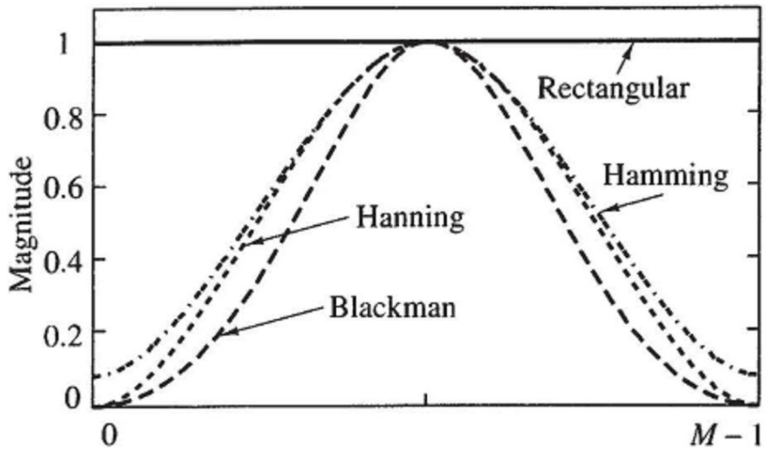
- The table shows some of the known
- window types.

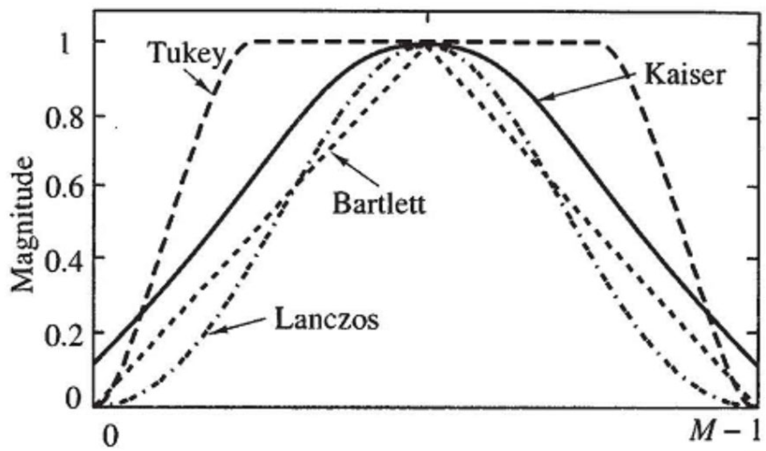
Window Functions for FIR Filter Design

Name of window	Time-domain sequence, $h(n), 0 \le n \le M-1$
Bartlett (triangular)	$1 - \dfrac{2\left\lvert n - \dfrac{M-1}{2} \right\rvert}{M-1}$
Blackman	$0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}$
Hamming	$0.54 - 0.46\cos\dfrac{2\pi n}{M-1}$
Hanning	$\dfrac{1}{2}\left(1 - \cos\dfrac{2\pi n}{M-1}\right)$
Kaiser	$\dfrac{I_0\left[\alpha\sqrt{\left(\dfrac{M-1}{2}\right)^2 - \left(n - \dfrac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\dfrac{M-1}{2}\right)\right]}$
Lanczos	$\left\{\dfrac{\sin\left[2\pi\left(n - \dfrac{M-1}{2}\right) \Big/ (M-1)\right]}{2\pi\left(n - \dfrac{M-1}{2}\right) \Big/ \left(\dfrac{M-1}{2}\right)}\right\}^L, \quad L > 0$
Tukey	$1, \left\lvert n - \dfrac{M-1}{2} \right\rvert \le \alpha\dfrac{M-1}{2}, \qquad 0 < \alpha < 1$ $\dfrac{1}{2}\left[1 + \cos\left(\dfrac{n - (1+a)(M-1)/2}{(1-\alpha)(M-1)/2}\pi\right)\right]$ $\alpha(M-1)/2 \le \left\lvert n - \dfrac{M-1}{2} \right\rvert \le \dfrac{M-1}{2}$

Different filters for FIR Filter Design

- These Figures show the shape of some of the known filters:

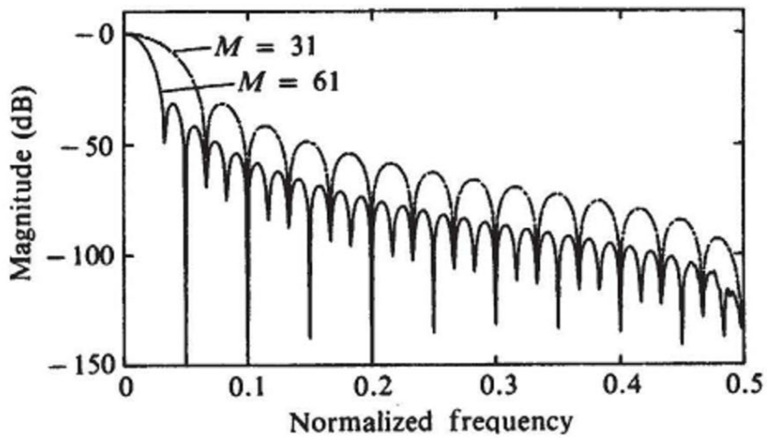




Hanning Window

- This Figure presents the magnitude response of Hanning window for M=31 and M=61:

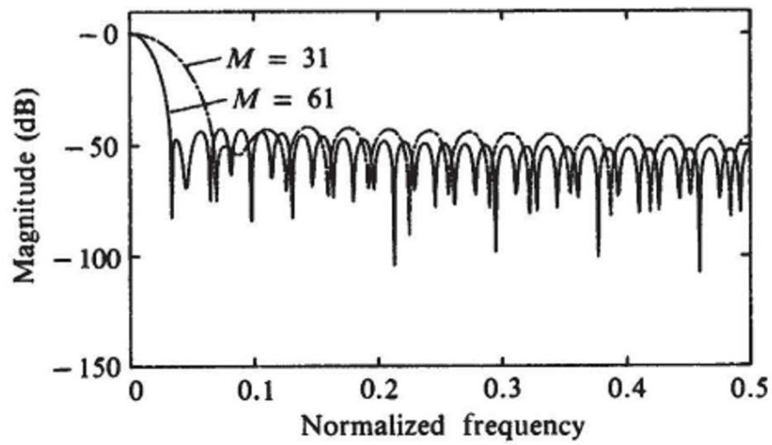
Frequency responses of Hanning window for (a) $M = 31$ and (b) $M = 61$.



- We observe that unlike the rectangular window, in this case the reduction in main-lobe (as M increases) results in attenuation of sidelobes.

Hamming Window

- This Figure presents the magnitude response of Hamming window for M=31 and M=61:



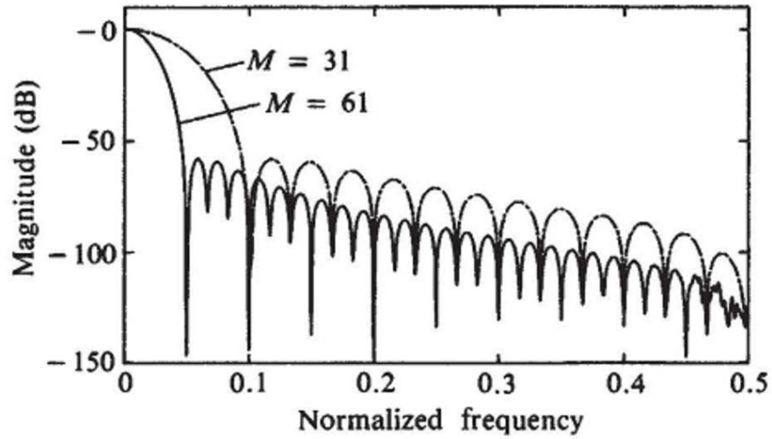
Frequency responses for Hamming window for (a) $M = 31$ and (b) $M = 61$.

- In the case of Hamming Window also the reduction of the main-lobe comes with attenuation of the sidelobes.

Blackman Window

- This Figure presents the magnitude response of the Blackman window for M=31 and M=61:

Frequency responses for Blackman window for (a) $M = 31$ and (b) $M = 61$.



- The same effect (sidelobes attenuation) is observed.

Characteristics of some Window Functions

- This table gives the frequency characteristics of some of commonly used windows in terms of the main lobe width and the sidelobe level:

Important Frequency-Domain Characteristics of Some Window Functions

Type of window	Approximate transition width of main lobe	Peak sidelobe (dB)
Rectangular	$4\pi/M$	−13
Bartlett	$8\pi/M$	−25
Hanning	$8\pi/M$	−31
Hamming	$8\pi/M$	−41
Blackman	$12\pi/M$	−57

FIR Design Using Window Functions: Example

- Assume we want to design the following low-pass filter:

$$H_d(\omega) = \begin{cases} 1e^{-j\omega(M-1)/2}, & 0 \leq |\omega| \leq \omega_c \\ 0, & \text{otherwise} \end{cases}$$

- The unit sample function for this filter is:

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega\left(n - \frac{M-1}{2}\right)} d\omega$$

$$= \frac{\sin \omega_c \left(n - \frac{M-1}{2}\right)}{\pi \left(n - \frac{M-1}{2}\right)}, \qquad n \neq \frac{M-1}{2}$$

- That is non-causal and of infinite duration.

FIR Design Using Window Functions: Example

- If we multiply $h_d(n)$ by the rectangular window, we get:

$$h(n) = \frac{\sin \omega_c \left(n - \frac{M-1}{2}\right)}{\pi \left(n - \frac{M-1}{2}\right)}, \quad 0 \leq n \leq M-1, \quad n \neq \frac{M-1}{2}$$

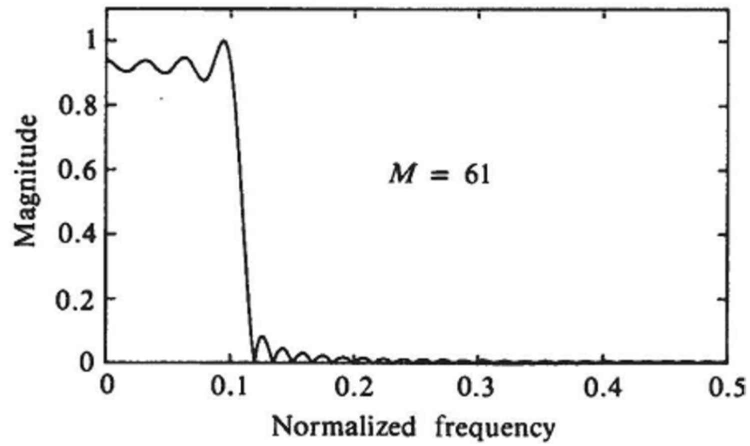
- If M is odd, the center tap will be at $n = (M-1)/2$ and will have the value:

$$h\left(\frac{M-1}{2}\right) = \frac{\omega_c}{\pi}$$

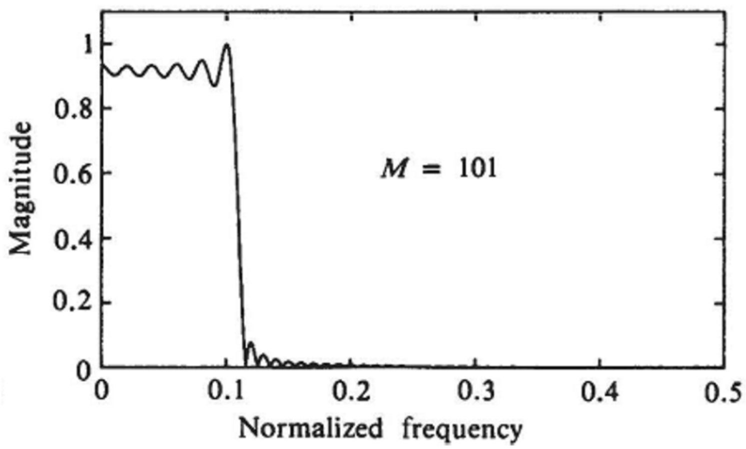
- That is non-causal and of infinite duration.

FIR Design Using Window Functions: Example

- These figures show the spectrum of the low-pass filter designed using rectangular window for M=61:

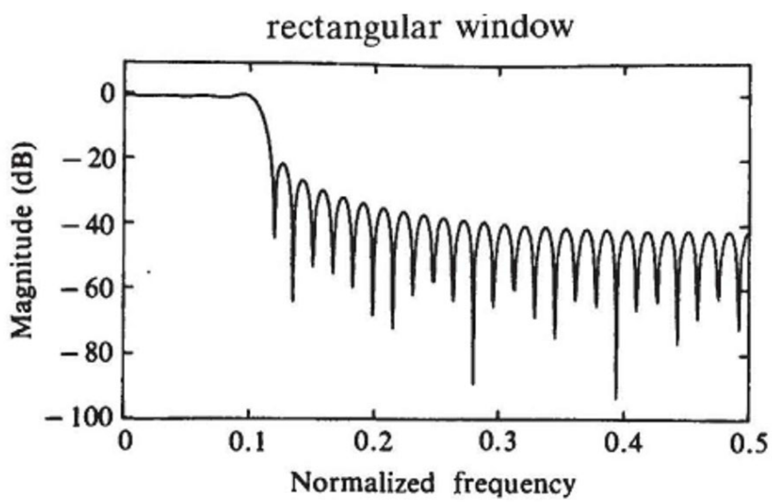


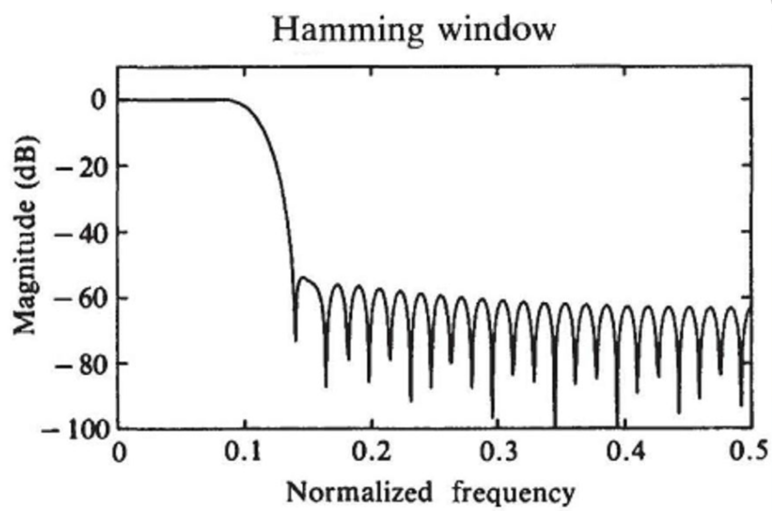
- and M=101:



FIR Design Using Window Functions: Example

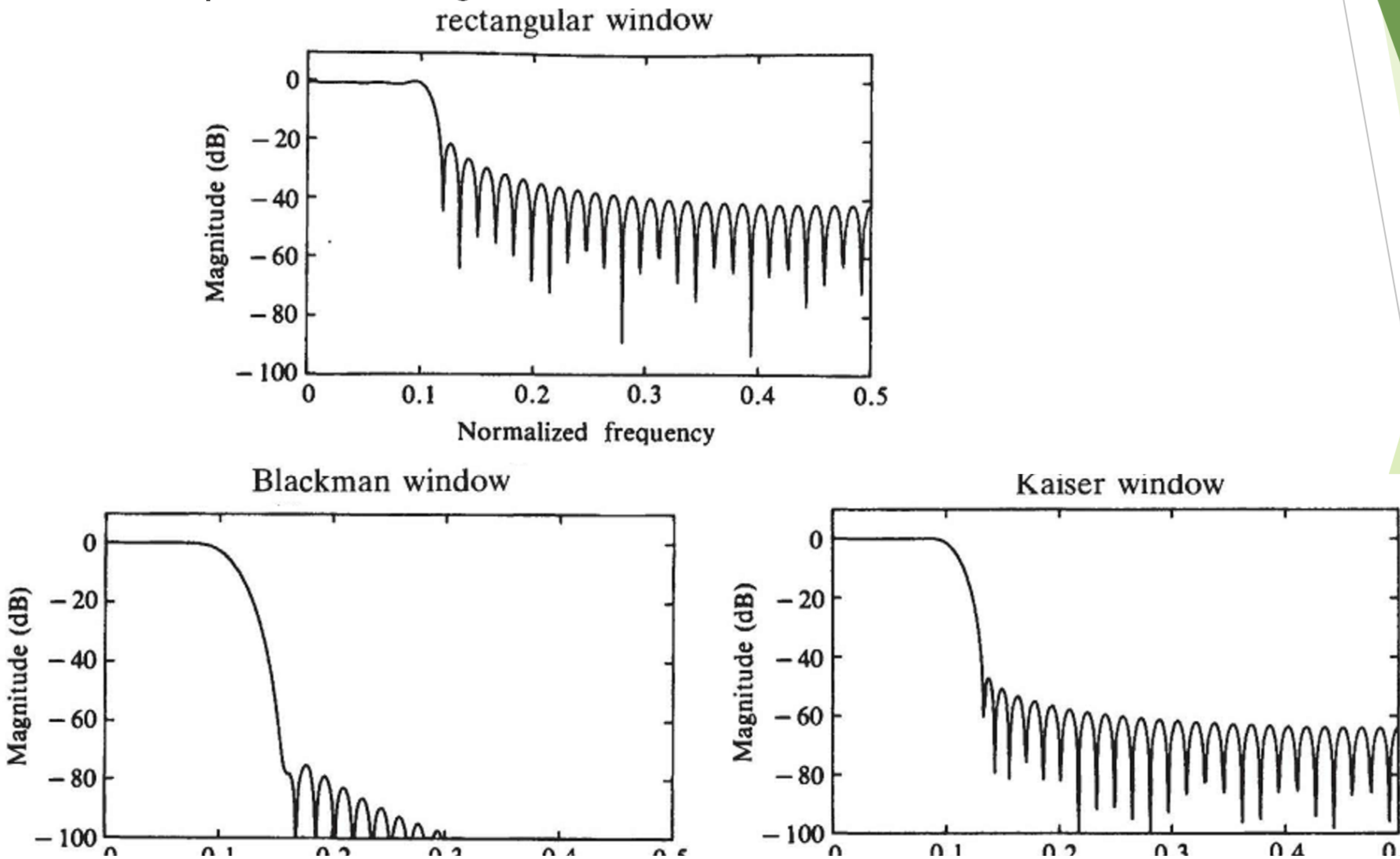
- As stated before when we use a rectangular window we truncate the sample response causing abrupt change in time-domain. This results in noticeable oscillations around the band edges referred to as the Gibbs Phenomenon.
- To avoid these fluctuations, we should avoid the abrupt change in sample response by using a window that changes gradually.
- Let's compare filter designed using Hamming vs. rectangular window:





FIR Design Using Window Functions: Example

- Here we compare the rectangular window with Blackman and Kaiser:



FIR Design: Other Techniques

- In this lecture, we have only talked about filter design using windowing.
- We can also do the design by using the frequency samples of the desired spectrum.
- Windowing and frequency-sampling methods have some limitations the most important of which is the lack of lack of control over the choice of band edges: Once you have chosen M the values of ω_p and ω_s are determined.
- Other techniques such as Chebyshev approximation technique can be used to alleviate this problem.
- Because of the time limitation, we do not discuss these other techniques. I believe that with what you have learned so far you can easily follow these topics in the text if you need it in the future.

IIR Filter Design:

- Assume that you are given an analog filter and you are asked to design a digital filter to do the same job. The filter can be represent by its system function:

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k}$$

- or by its impulse response:

$$H_a(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\, dt$$

- Or by the differential equation:

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

IIR Filter Design:

- Using any of these three characterizations results in a different technique for converting this filter to a digital filter.
- In each case, we need to use a relationship mapping from s-plane to z-plane.
- Note that an analog filter is stable if all its poles are in the Left Half Plane (LHP) and a digital filter is stable if all its poles are inside the unit circle. So, any transformation should have the property:
- 1) The $j\Omega$ axis in the s-plane should map into the unit circle in the z-plane.
- 2) The LHP of the s-plane should map into the inside of the unit circle in the z-plane.

IIR Filter Design:

- Based on the analog-domain characterization used, we have basically three techniques for analog domain to digital domain filter conversion:
- 1) IIR Filter design using the Approximation of Derivatives: using the differential equation characterization.
- 2) Impulse Invariance Technique: Using the impulse response characterization.
- 3) Bilinear Transformation: Using the system function characterization.

Filter Design by Approximation of Derivatives

- We start with: $$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$
- and approximate the derivative as: $$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(nT-T)}{T} = \frac{y(n) - y(n-1)}{T}$$
- Taking the transform, we get $s = \frac{1-z^{-1}}{T}$.
- For the second order derivative, we have:

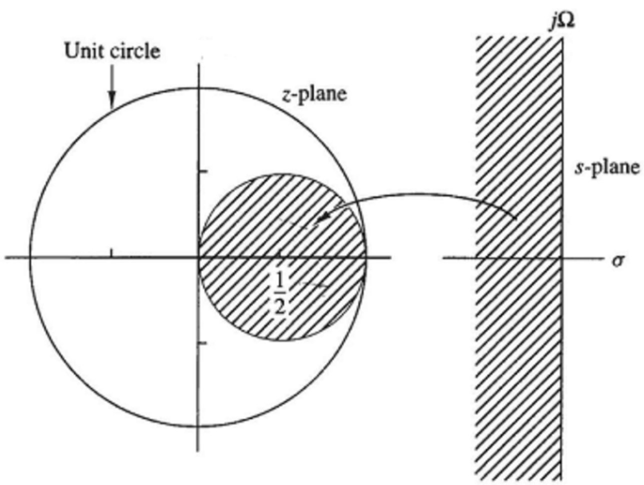
$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \frac{d}{dt}\left[\frac{dy(t)}{dt}\right]_{t=nT}$$
$$= \frac{[y(nT) - y(nT-T)]/T - [y(nT-T) - y(nT-2T)]/T}{T}$$
$$= \frac{y(n) - 2y(n-1) + y(n-2)}{T^2}$$

- or, equivalently: $$s^2 = \frac{1 - 2z^{-1} + z^{-2}}{T^2} = \left(\frac{1-z^{-1}}{T}\right)^2$$
- In general:

$$s^k = \left(\frac{1-z^{-1}}{T}\right)^k$$

Filter Design by Approximation of Derivatives

- So, the digital system function is found from the analog system function as: $H(z) = H_a(s)|_{s=(1-z^{-1})/T}$
- and approximate the derivative as:
- Note that $s = \frac{1-z^{-1}}{T}$ means $z = \frac{1}{1-sT}$.
- Let $s = j\Omega$, then: $$z = \frac{1}{1 - j\Omega T} = \frac{1}{1+\Omega^2 T^2} + j\frac{\Omega T}{1+\Omega^2 T^2}$$
- Note that as Ω goes from
- $-\infty$ to ∞ the z-plane points
- Are on a circle of radius ½
- centered around $z = \frac{1}{2}$ and
- the poles confined to the
- points inside only part of
- the unit circle.



Filter Design by Approximation of Derivatives

- The poles of the digital filter being confined to only a part of the unit circle makes the technique useful only for low-pass filters and high pass filters with low resonant frequency. To overcome this issue we can use more complex approximation for the derivative, for example, use:

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{1}{T}\sum_{k=1}^{L}\alpha_k \frac{y(nT+kT)-y(nT-kT)}{T}$$

- With the resulting mapping: $s = \frac{1}{T}\sum_{k=1}^{L}\alpha_k(z^k - z^{-k})$
- When $z = e^{j\omega}$, we get $s = j\frac{2}{T}\sum_{k=1}^{L}\alpha_k \sin\omega k$ which is purely imaginary and gives us:

$$\Omega = \frac{2}{T}\sum_{k=1}^{L}\alpha_k \sin\omega k$$

- By proper choice of $\{a_k\}$ it is possible to map the $j\Omega$ −axis into the unit circle. So, it resolves the problem of poles but still the problem of choice of the set of coefficients $\{a_k\}$ remains.

Approximation of Derivatives: Example

- Convert the analog band-pass filter: $H_a(s) = \frac{1}{(s+0.1)^2+9}$ into a digital filter using derivative approximation.
- Solution: substitute $s = \frac{1-z^{-1}}{T}$ into

$$H(z) = \frac{1}{\left(\frac{1-z^{-1}}{T}+0.1\right)^2+9}$$

$$= \frac{T^2/(1+0.2T+9.01T^2)}{1-\frac{2(1+0.1T)}{1+0.2T+9.01T^2}z^{-1}+\frac{1}{1+0.2T+9.01T^2}z^{-2}}$$

- If T is selected small enough, the poles can be near the unit circle, for example, for $T = 0.1$, we have:

$$p_{1,2} = 0.91 \pm j0.27$$

$$= 0.949e^{\pm j16.5^\circ}$$

IIR filter Design: Impulse Invariance Techniques

- In this technique, we work with the impulse response of the analog filter. The objective will be that the sample response of the digital IIR filter be equal to the samples of the analog filter's impulse response, i.e., $h(n) = h_a(nT), n = 0, 1, 2, \ldots$
- Assume that the poles of the analog filter are distinct, then we can write,

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$$

- So, we have

$$h(n) = h_a(nT) = \sum_{k=1}^{N} c_k e^{p_k T n}$$

- The system function is:

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{N} c_k e^{p_k T n} \right) z^{-n} = \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} (e^{p_k T} z^{-1})^n$$

IIR filter Design: Impulse Invariance Techniques

- Since $p_k < 1$, the inner sum converges:

$$\sum_{n=0}^{\infty}(e^{p_k T} z^{-1})^n = \frac{1}{1 - e^{p_k T} z^{-1}}$$

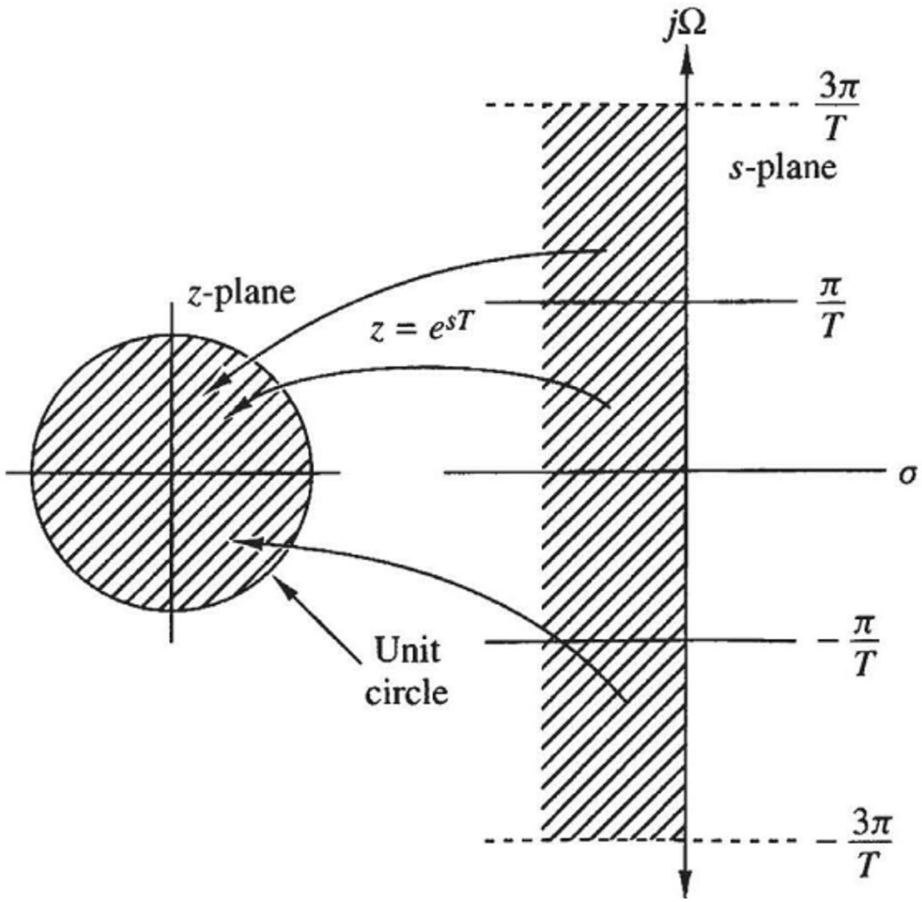
- and we have:

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

- This means that the digital filter has poles at $z_k = e^{p_k T}, \quad k = 1, 2, \ldots, N$
- This implies that the transformation from s-domain to z-domain is given by $z = e^{sT}$

IIR filters: Impulse Invariance Techniques

- The transformation $z = e^{sT}$ maps the LHP of the s-domain to the unit circle in the z-domain:
-



Impulse Invariance Techniques: Example

- Convert the analog filter with system function:

$$H_a(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9}$$

- into a digital IIR Filter.
- Solution: We see that the analog filter has a zero at $s = -0.1$ and complex conjugate poles at $p_k = -0.1 \pm j3$.
- So, we can write $H_a(s)$ as:

$$H(s) = \frac{\frac{1}{2}}{s + 0.1 - j3} + \frac{\frac{1}{2}}{s + 0.1 + j3}$$

- So,

$$H(z) = \frac{\frac{1}{2}}{1 - e^{-0.1T} e^{j3T} z^{-1}} + \frac{\frac{1}{2}}{1 - e^{-0.1T} e^{-j3T} z^{-1}}$$

- Combining the two conjugate poles:
-

$$H(z) = \frac{1 - (e^{-0.1T} \cos 3T) z^{-1}}{1 - (2e^{-0.1T} \cos 3T) z^{-1} + e^{-0.2T} z^{-1}}$$

Impulse Invariance Techniques: Aliasing

- Note that by letting $h(n) = h_a(nT), n = 0, 1, 2, \dots$ we are sampling $h_a(t)$. In the frequency domain this results in the spectrum of the analog filter $H_a(\Omega)$ repeat every $F_s = \frac{1}{T}$ Hz.
- This means that in digital domain, in terms of normalized frequency,

$$H(f) = F_s \sum_{k=-\infty}^{\infty} H_a[(f-k)F_s]$$

- or equivalently,

$$H(\omega) = F_s \sum_{k=-\infty}^{\infty} H_a[(\omega - 2\pi k)F_s]$$

- Or, in terms of the actual frequency in Hz.:

$$H(\Omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(\Omega - \frac{2\pi k}{T}\right)$$

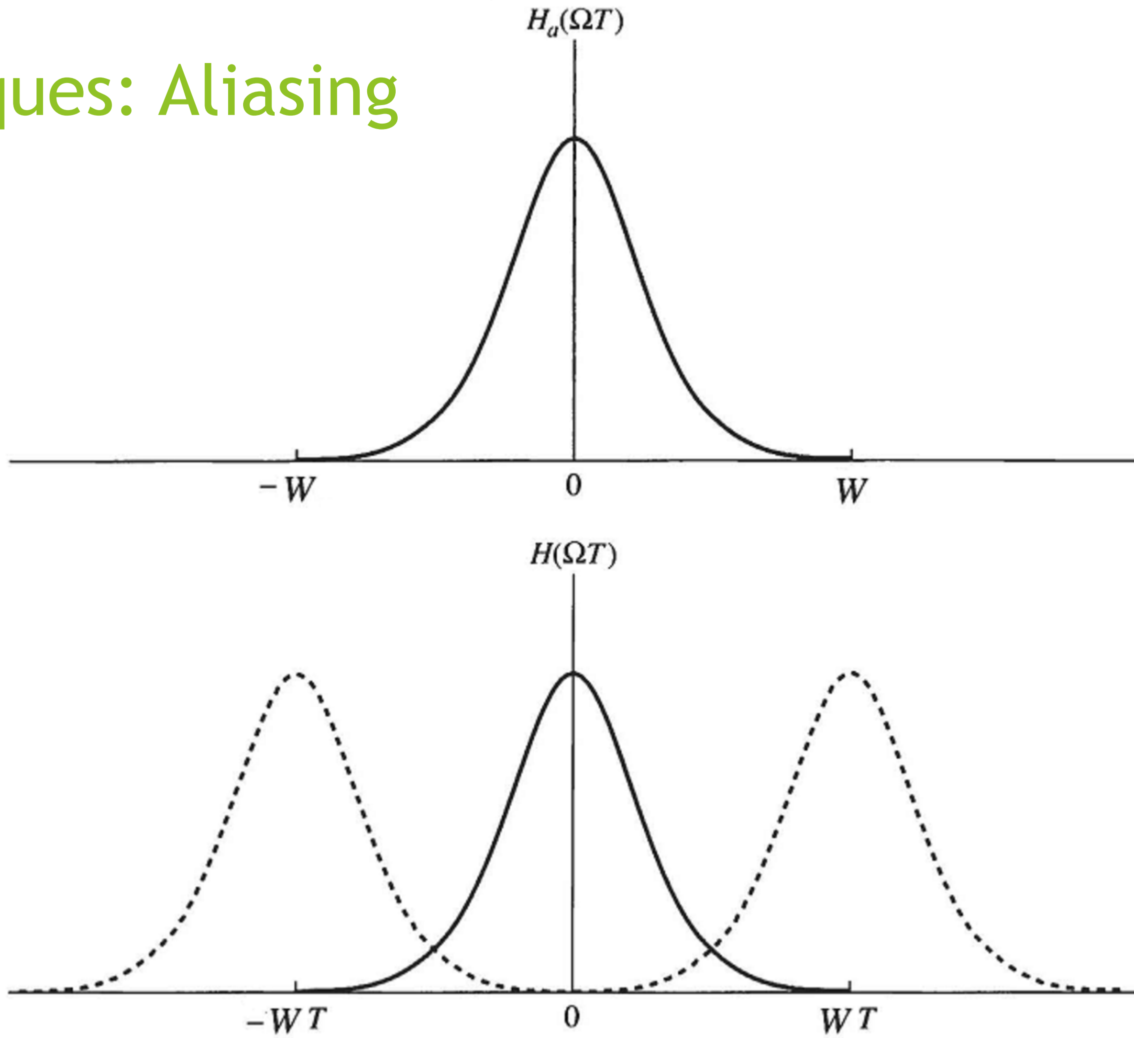
Impulse Invariance Techniques: Aliasing

- The formula:

$$H(\Omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(\Omega - \frac{2\pi k}{T}\right)$$

- is shown here.
- We observe that in order to avoid

Aliasing, we need to sufficiently large T, or, equivalently large number of samples.



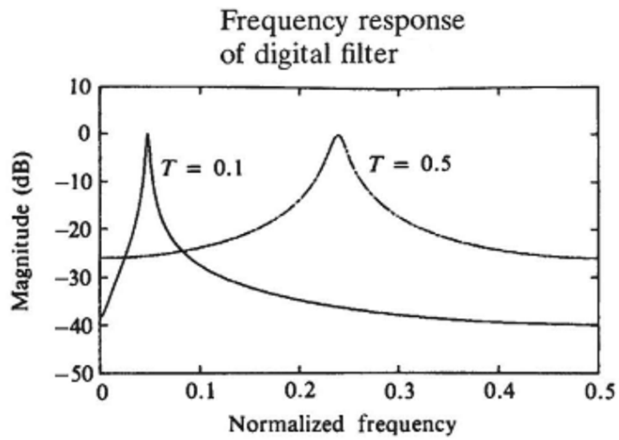
Frequency response $H_a(\Omega)$ of the analog filter and frequency re spons of the corresponding digital filter with aliasing

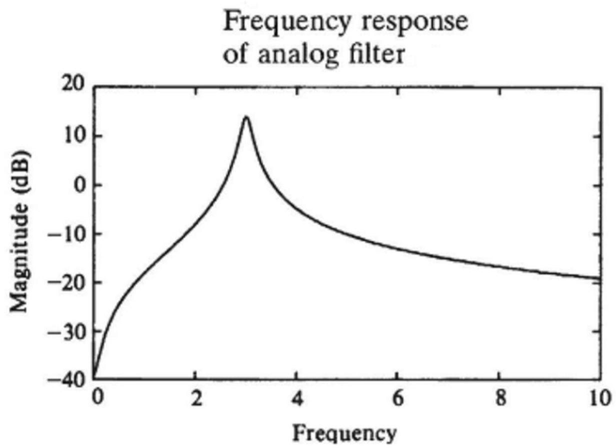
Impulse Invariance Techniques: Aliasing

- As an example, take the digital filter we designed in the above example:

$$H(z) = \frac{1 - (e^{-0.1T}\cos 3T)z^{-1}}{1 - (2e^{-0.1T}\cos 3T)z^{-1} + e^{-0.2T}z^{-1}}$$

- Here is the magnitude of the frequency response for T=0.1 and T=0.5 together with the original analog filter. We see that the aliasing is more noticeable for T=0.5. So, this techniques is not suitable for high-pass filter design.





IIR filter Design: Bilinear Transformation

- We saw that the derivative approximation and impulse invariance techniques are not suitable for high-pass filter design.
- Bilinear Transformation technique overcomes this problem by mapping the LHP of the s-plane into the inside of the unit circle in the z-plane.
- In the following slides, we describe this technique for a single pole IIR filter. The procedure, however, generalizes to the case of analog filters with multiple poles.
- Consider a single pole linear filter with the system function:
  - $H(s) = \frac{b}{s+a}$.
- The corresponding differential equation is:
  - $\frac{dy(t)}{dt} + ay(t) = bx(t)$.

IIR filter Design: Bilinear Transformation

- Let's integrate the derivative of $y'^{(t)} = \frac{dy(t)}{dt}$ to get:
  - $y(t) = \int_{t_0}^{t} y'(\tau)d\tau + y(t_0)$
- Approximating the above integral using the trapezoidal formula at: $t = nT$ and $t_0 = nT - T$, we get:
  - $y(nT) = \frac{T}{2}[y'(nT) + y'(nT - T)] + y(nT - T)$.
- $y'(nT)$ is found by evaluating the differential equation at $t = nT$:
  - $y'(nT) = -ay(nT) + x(nT)$.
- Substituting this in the expression for $y(nT)$ and using $y(n) = y(nT)$ and $x(n) = x(nT)$, we have the difference equation:
- $\left(1 + \frac{aT}{2}\right) y(n) - \left(1 - \frac{aT}{2}\right) y(n-1) = \frac{bT}{2}[x(n) + x(n-1)$.

IIR filter Design: Bilinear Transformation

- Taking the z-transform of the above difference equation:
  - $\left(1+\frac{aT}{2}\right)Y(z)-\left(1-\frac{aT}{2}\right)z^{-1}Y(z)=\frac{bT}{2}(1+z^{-1})X(z).$
- So, the system function in the z-domain is:

$$H(z)=\frac{Y(x)}{X(z)}=\frac{(\frac{bT}{2})(1+z^{-1})}{1+\frac{aT}{2}-(1-aT/2)z^{-1}}$$

or,

$$H(z)=\frac{b}{\frac{T}{2}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}$$

So, the mapping from s-plane to z-plane is through:

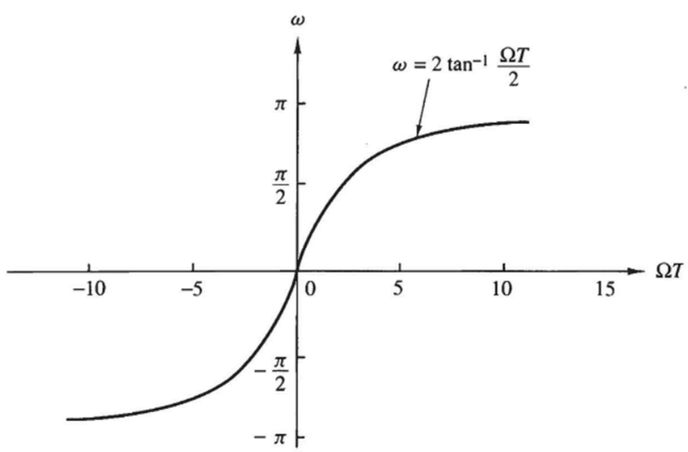
$$s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right).$$

IIR filter Design: Bilinear Transformation

- To see how the bilinear transform maps the points in the s-plane to points in the z-plane, let:
  - $z = re^{j\omega}$
- and
  - $s = \sigma + j\Omega$
- Then,
  - $s = \frac{2}{T}\frac{z-1}{z+1} = \frac{2}{T}\frac{re^{j\omega}-1}{re^{j\omega}+1} = \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right)$
- Therefore,
  - $\sigma = \frac{2}{T}\frac{r^2-1}{1+r^2+2r\cos\omega}$
- and
  - $\Omega = \frac{2}{T}\frac{2r\sin\omega}{1+r^2+2r\cos\omega}.$
- Note that $r < 1 \Rightarrow r^2 - 1 < 0 \Rightarrow \sigma < 0$ and similarly, $r > 0 \Rightarrow \sigma > 0$. So, the bilinear transformation maps the Left Hand Side of the s-plane to inside of the unit circle and the RHP to outside of the unit circle. When $r = 1$, i.e., on the unit circle, we get $\sigma = 0$. Do the $j\Omega$ axis is mapped to the unit circle.

IIR filter Design: Bilinear Transformation

- To summarize, $\sigma = \frac{2}{T}\frac{r^2-1}{1+r^2+2rcos\,\omega}$ and $\Omega = \frac{2}{T}\frac{2rsin\,\omega}{1+r^2+2rcos\,\omega}$.
- So when $r = 1$, we get $\sigma = 0$ and:
  - $\Omega = \frac{2}{T}\frac{\sin\omega}{1+\cos\omega} = \frac{2}{T}\tan\frac{\omega}{2}$,
- or, $\omega = 2tan^{-1}\frac{\Omega T}{2}$. The relationship between the frequencies in s-domain (Ω) and z-domain (ω) can be plotted as:



- Not that the mapping is highly nonlinear resulting in the so called frequency warping.

IIR filter Design: Bilinear Transformation Example

- Convert the analog filter with the system function:
  - $H_a(s) = \frac{s+0.1}{(s+0.1)^2+16}$
- into a digital IIR filter having resonant frequency $\omega_r = \pi/2$ using bilinear transformation.
- **Solution:** Note that the poles of the analog filter are at $p_{1,2} = -0.1 \pm j4$, i.e., it has the resonant frequency $\Omega_r = 4$. Using the relationship $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$, we get $4 = \frac{2}{T}\tan\frac{\pi}{4}$. So, $T = \frac{1}{2}$. The transformation will be: $s = 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ resulting in: $H(z) = \frac{0.125+0.0061z^{-1}-0.1189z^{-2}}{1+0.0006z^{-1}+0.9512z^{-2}}$.
- Ignoring the very small term in the denominator, we get the approximation:
- $H(z) = \frac{0.125+0.0061z^{-1}-0.1189z^{-2}}{1+0.9512z^{-2}}$.
- The filter has poles at $p_{1,2} = 0.987e^{\pm j\pi/2}$ and zeros at $z_1 = -1$ and $z_2 = 0.95$.

Commonly Used Analog Filters

- There are several types of analog filters each with its characteristics and properties. They are briefly explained in the text and there is a lot of information in the literature about them that you may refer to in case in your future work you are tasked with the design of a filter with certain properties and constraints.
- These filters include: Butterworth Filters, Chebyshev filters, Elliptic Filters and Bessel Filters.
- These filters are mainly characterized by the location of their poles determined by a different polynomial in each case.
- We only talk briefly about the Butterworth filters. You are encouraged to go over other types of filters discussed at the end of Chapter 10 of Proakis and Manolakis textbook.
- An important type of filter is the raised-cosine filters you have seen in your digital communications course. Due to the importance of raised-cosine filters in today's communication circuits I encourage you to read about them. Information about these filters can be found in any digtal communications textbook. You may also refer to Wikipedia (https://en.wikipedia.org/wiki/Raised-cosine_filter).

Butterworth Filters

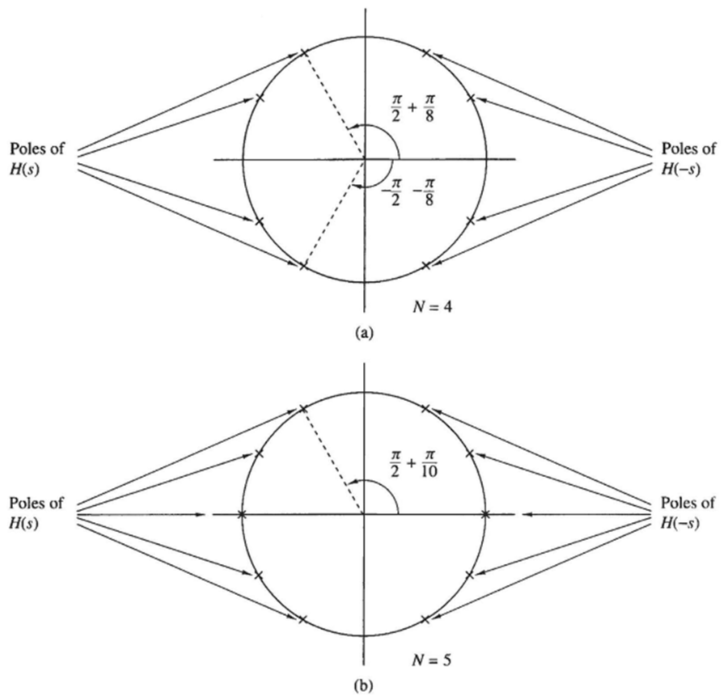
- A Butterworth filter of order N is an all-pole filter with its poles uniformly distributed around a circle. It can be characterized by its magnitude-squared frequency response:
  - $|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$
- where $N$is the order of the filter and Ω_c is the cut-off frequency or -3 dB frequency of the filter. This is the frequency where the output power of the filter is ½ of its maximum at the origin, i.e., 3 dB attenuation.
- Since $H(s)H(-s)$ evaluated at $s = j\Omega$ is equal to $|H(\Omega)|^2$, we have:
  - $H(s)H(-s) = \frac{1}{1+\left(-s^2/\Omega_c^2\right)^N}$.
- So, the poles of $H(s)H(-s)$ are equally spaced on a circle of radius Ω_c.
- Therefore,
  - $\frac{-s^2}{\Omega_2^2}$=$(-1)^{1/N}$= $e^{j(2k+1)\pi/N}, \quad k = 0, 1, \dots, N-1.$

Butterworth Filters

From the above, we find the filter poles as:

$$s_k = \Omega_c e^{j\pi/2} e^{j(2k+1)\pi/2N}, \qquad k = 0, 1, \dots, N-1.$$

The position of poles for N=4 and N=5 is shown here:



(a)

(b)

Butterworth Filters

Denoting the frequency at the edge of the passband as Ω_p, we have,

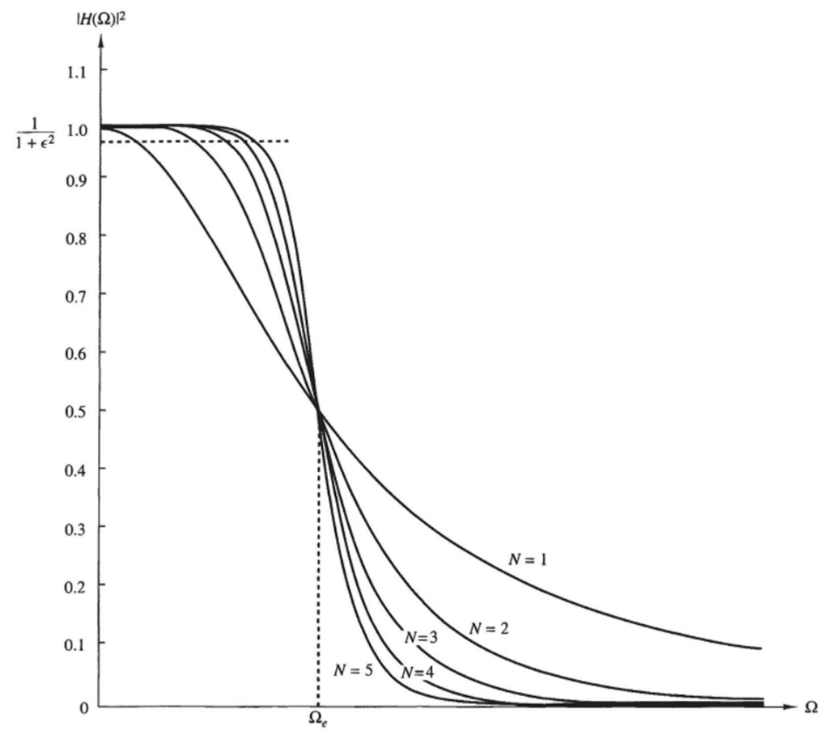
$$|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} = \frac{1}{1+\varepsilon^2(\Omega/\Omega_p)^{2N}}.$$

The attenuation at the passband edge Ω_p,

i.e., $|H(\Omega_p)|^2$ is equal to $1/(1+\varepsilon^2)$.

This figure shows the frequency response of

the Butterworth filters for a few values of N:



Butterworth Filters

Using the expression for the magnitude-squared frequency response,

$$|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} = \frac{1}{1+\varepsilon^2\left(\Omega/\Omega_p\right)^{2N}}.$$

we can compute the attenuation at and frequency Ω. For example to find the order of the filter such that the attenuation at the edge of the stop band (Ω_s) does not exceed δ_2, we calculate the expression at $\Omega = \Omega_s$ and equate the result to δ_2^2, i.e.,

$$\frac{1}{1+\varepsilon^2\left(\Omega_s/\Omega_p\right)^{2N}} = \delta_2^2.$$

The order of the filter is then,

$$N = \frac{log[(1/\delta_2^2) - 1]}{2log\left(\Omega_s/\Omega_p\right)}$$

So, the Butterworth filter is fully characterized by the parameters: N, δ_2, ε and the ratio Ω_s/Ω_p.

Butterworth Filters: Example

Example: Determine the order and the poles of a Butterworth filter with a -3 dB bandwidth of 500 Hz. and an attenuation of 40 dB at 1000 Hz.

Solution: The -3 dB frequency is $\Omega_c = 1000\pi$ and the stopband frequency is $\Omega_s = 2000\pi$. So,

$$N = \frac{log[10^4 - 1]}{2log(2)} = 6.64.$$

So, we have to have N=7. The poles are located at:

$$s_k = 1000\pi e^{j\left[\frac{\pi}{2} + (2k+1)\pi/14\right]}, \qquad k = 0, 1, \ldots, 6.$$

Frequency Transformations

- We have so far mostly talked about designing lowpass filters. If we need a highpass, a bandpass or a banstop filter, we can do by taking a prototype lowpass filter and change it to one these types of filters by transforming the spectrum of the prototype lowpass filter.
- Let's first consider the analog filters. Say, we have a lowpass filter with the edge of the passband frequency Ω_p. Assume that we need a lowpass filter with the edge of the passband frequency Ω'_p. It is easy to see that we can do this by substituting s by $\Omega_p/\Omega'_p s$, i.e., $s \to \frac{\Omega_p}{\Omega'_p}$. So, the system function of the lowpass filter is $H_l(s) = H_p\left[\left(\Omega_p/\Omega'_p\right)s\right]$. Where $H_p(s)$ is the system function of the prototype filter.
- Also, the transformation $s \to \frac{\Omega_p \Omega'_p}{s}$ turns the prototype lowpass filter into a highpass filter with the system function $H_h(s) = H_p\left(\frac{\Omega_p \Omega'_p}{s}\right)$.
- These transformations as well as transformations for bandpass and bandstop filters with the lower band edge frequency Ω_l and upper band edge frequency Ω_u are shown in the table in the next slide.

Frequency Transformations: Analog Filters

- This figure shows the frequency transformation for analog filters where a prototype low pass filter with band edge frequency Ω_p is transformed into other types of filters

Type of transformation	Transformation	Band edge frequencies of new filter
Lowpass	$s \longrightarrow \frac{\Omega_p}{\Omega'_p} s$	Ω'_p
Highpass	$s \longrightarrow \frac{\Omega_p \Omega'_p}{s}$	Ω'_p
Bandpass	$s \longrightarrow \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$	Ω_l, Ω_u
Bandstop	$s \longrightarrow \Omega_p \frac{s(\Omega_u - \Omega_c)}{s^2 + \Omega_u \Omega_l}$	Ω_l, Ω_u

Frequency Transformations: Digital Filters

- In the case of digital filters the transformations in z-domain turning a prototype digital lowpass filter with band edge frequency ω_p into types of filters are shown in the following table:

Type of transformation	Transformation	Parameters
Lowpass	$z^{-1} \longrightarrow \dfrac{z^{-1}-a}{1-az^{-1}}$	$\omega'_p =$ band edge frequency new filter $a = \dfrac{\sin[(\omega_p-\omega'_p)/2]}{\sin[(\omega_p+\omega'_p)/2]}$
Highpass	$z^{-1} \longrightarrow -\dfrac{z^{-1}+a}{1+az^{-1}}$	$\omega'_p =$ band edge frequency new filter $a = -\dfrac{\cos[(\omega_p+\omega'_p)/2]}{\cos[(\omega_p-\omega'_p)/2]}$
Bandpass	$z^{-1} \longrightarrow -\dfrac{z^{-2}-a_1z^{-1}+a_2}{a_2z^{-2}-a_1z^{-1}+1}$	$\omega_l =$ lower band edge frequency $\omega_u =$ upper band edge frequency $a_1 = 2\alpha K/(K+1)$ $a_2 = (K-1)/(K+1)$ $\alpha = \dfrac{\cos[(\omega_u+\omega_l)/2]}{\cos[(\omega_u-\omega_l)/2]}$ $K = \cot\dfrac{\omega_u-\omega_l}{2}\tan\dfrac{\omega_p}{2}$
Bandstop	$z^{-1} \longrightarrow \dfrac{z^{-2}-a_1z^{-1}+a_2}{a_2z^{-1}-a_1z^{-1}+1}$	$\omega_l =$ lower band edge frequency $\omega_u =$ upper band edge frequency $a_1 = 2\alpha/(K+1)$ $a_2 = (1-K)/(1+K)$ $\alpha = \dfrac{\cos[(\omega_u+\omega_l)/2]}{\cos[(\omega_u-\omega_l)/2]}$ $K = \tan\dfrac{\omega_u-\omega_l}{2}\tan\dfrac{\omega_p}{2}$

Frequency Transformations: Example

- **Example:** Convert the single pole lowpass Butterworth filter with the system function:
  - $H(z) = \frac{0.245(1+z^{-1})}{1-0.509z^{-1}}$
- into a bandpass filter with upper and lower cutoff frequencies ω_u and ω_l, respectively. The lowpass filter has 3 dB bandwidth $\omega_p = 0.2\pi$.
- Solution: Referring to the table in the previous slide, the required transformation is:

$$z^{-1} \longrightarrow -\frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$$

- where a_1 and a_2 are found from the table. Substitution of z^{-1} results in:

$$H(z) = \frac{0.245\left[1 - \dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}\right]}{1 + 0.509\left(\dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}\right)}$$

$$= \frac{0.245(1 - a_2)(1 - z^{-2})}{(1 + 0.509a_2) - 1.509a_1 z^{-1} + (a_2 + 0.509)z^{-2}}$$

Frequency Transformations: Example

Note that the resulting filter has zeros at $z = \pm 1$ and a pair of poles that depend on the choice of ω_u and ω_l.

For example, suppose that $\omega_u = 3\pi/5$ and $\omega_l = 2\pi/5$. Since $\omega_p = 0.2\pi$, we find that $K = 1$, $a_2 = 0$, and $a_1 = 0$. Then

$$H(z) = \frac{0.245(1 - z^{-2})}{1 + 0.509z^{-2}}$$

This filter has poles at $z = \pm j0.713$ and hence resonates at $\omega = \pi/2$.