Lecture 5

Implementation of Linear Time-invariant Systems

Different Implementation OPtions

- **We will mainly discuss the following structures:**

- Direct Form Implementation,
- Cascade Form Implementation (Cascade Structure),
- Parallel Structure,
- Lattice Structure,

Direct Implementation

- Consider the general form of a linear time invariant system (filter). That is one with both feedback and feed forward path. The linear difference equation describing such a filter is:

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

- Using z-transform, we find the transfer function of the system as:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Structure of FIR Systems

- Assume that there is no feedback, i.e., the output y[n] does not depend on the past outputs and only depends on the input sequence:

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$

- The denominator of H(z) will be equal to one and we will have,

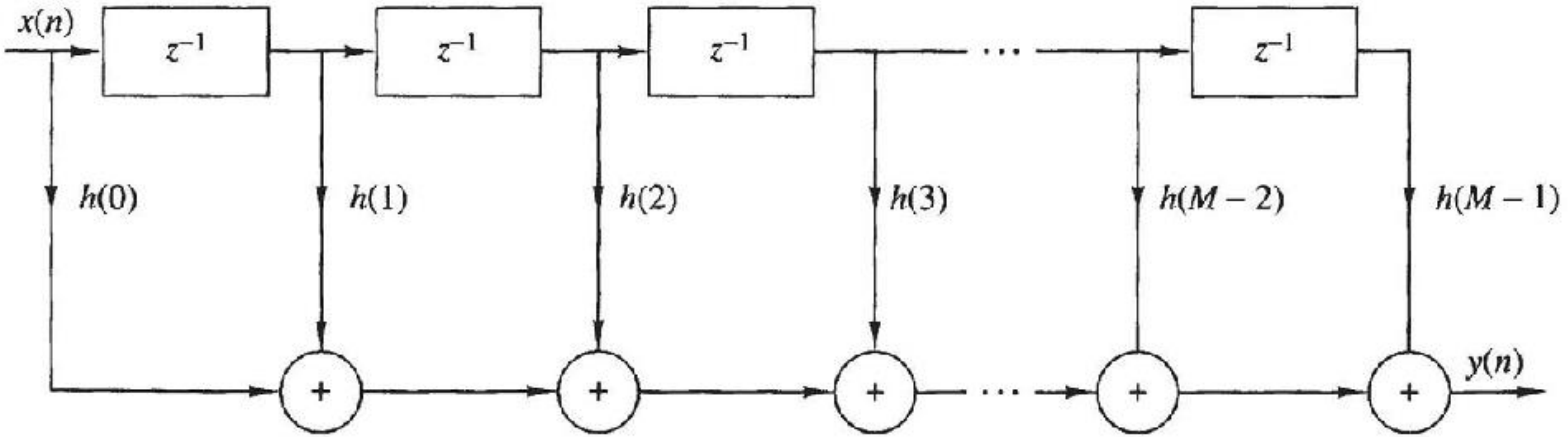
$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k}$$

- The unit sample response of the FIR system will be:

$$h(n) = \begin{cases} b_n, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

Direct Implementation of FIR Systems

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$



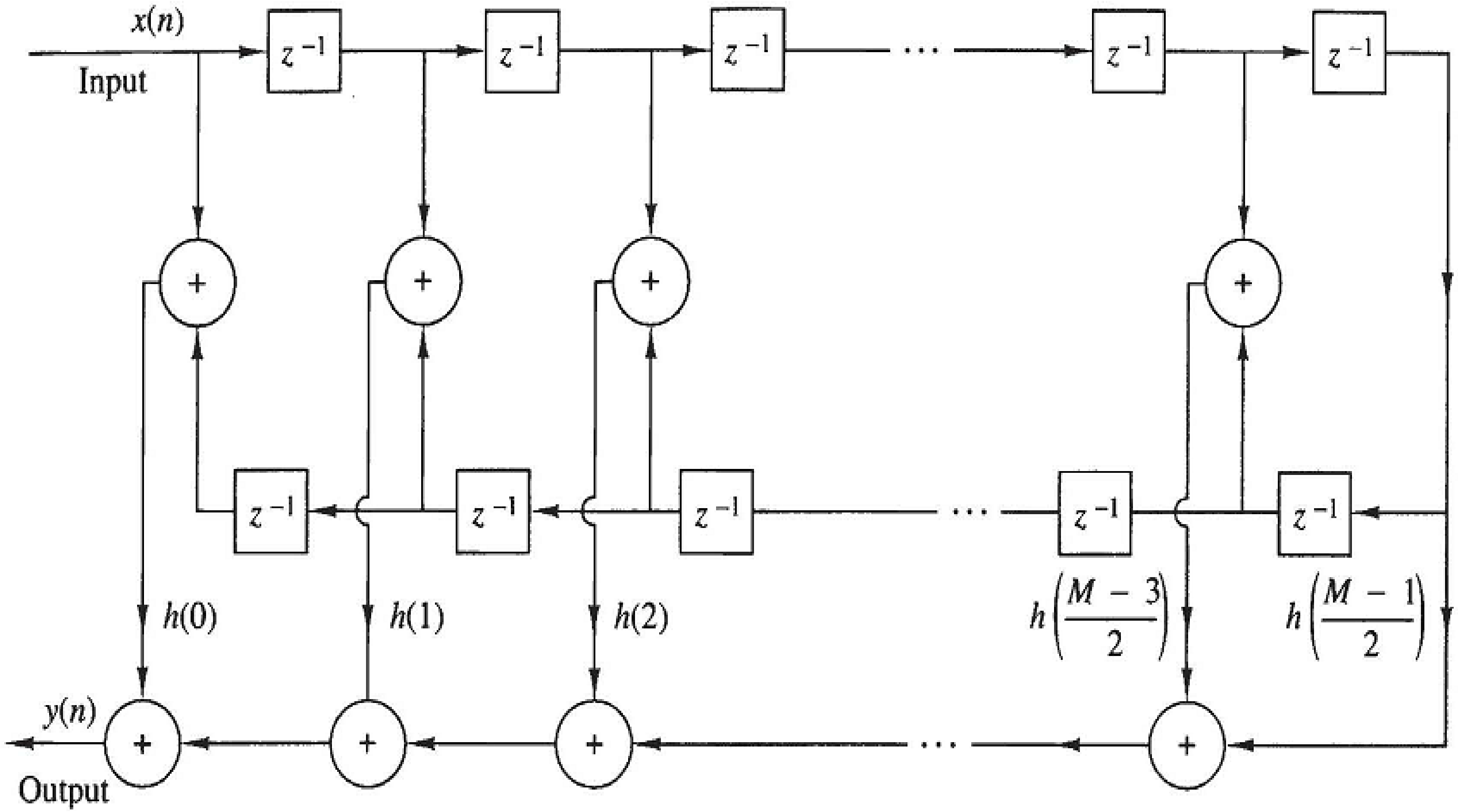
- When the FIR system has linear phase (it is symmetric):

$$h(n) = \pm h(M-1-n)$$

Linear Phase FIR Systems

- When the FIR system has linear phase (it is symmetric):

$$h(n) = \pm h(M-1-n)$$



Exercise:

1) Implement $y[n] = 3x[n] - 2x[n-1] + x[n-2]$.

2) Is the filter y[n]=x[n]-2x[n-1]-x[n-2] linear phase?

3) Implement the filter:

$$y[n] = 0.5x[n] + x[n-1] + 2x[n-2] + x[n-3] + 0.5x[n-4]$$

Cascade Implementation of FIR Systems

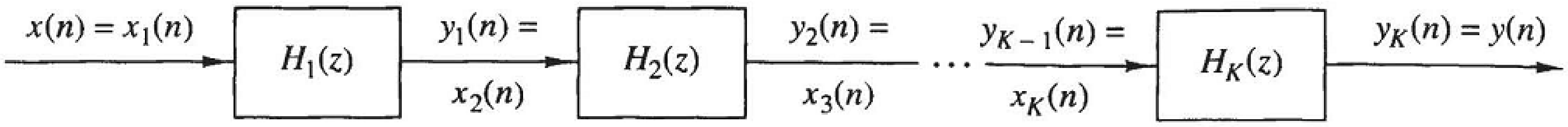
- It is easy to factor H(z) into K second order filters:

$$H(z) = \prod_{k=1}^{K} H_k(z)$$

- where,

$$H_k(z) = b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}, \qquad k = 1, 2, \ldots, K$$

- and K is the M/2 for M even and (M+1)/2 for M odd.

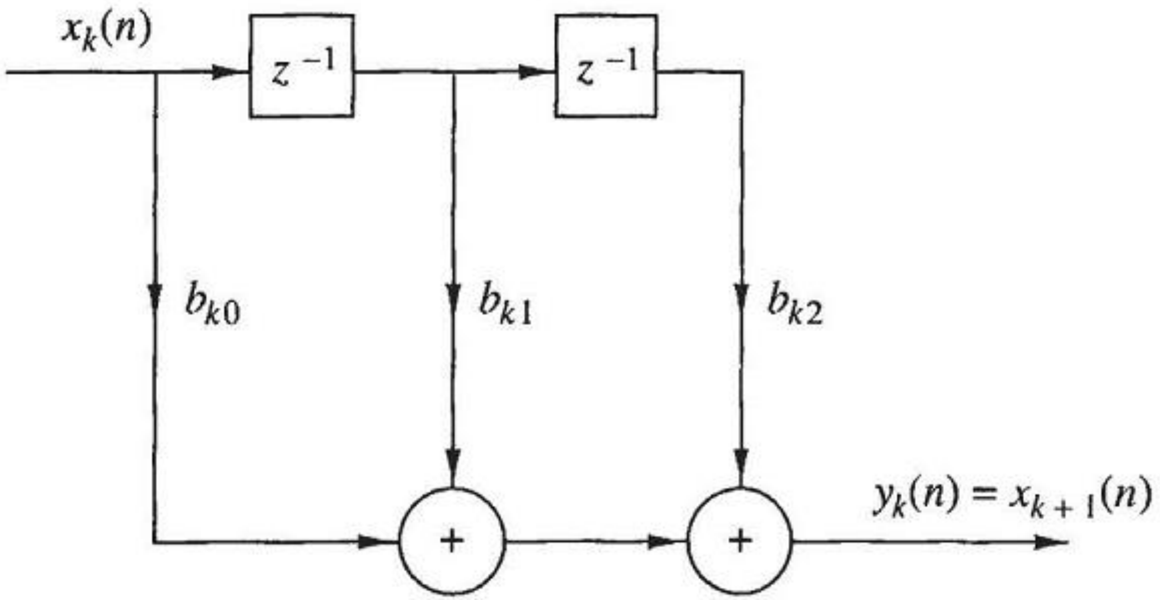


Cascade Implementation of FIR Systems

- Where each section,

$$H_k(z) = b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}, \qquad k = 1, 2, \ldots, K$$

- Is implemented as



Lattice Structure

- Lattice filter implementation is widely used in adaptive filtering. Assume that we have a filter with transfer function H(z). We can write,

$$H_m(z) = A_m(z), \qquad m = 0, 1, 2, \ldots, M-1$$

- where $A_m(z)$ is a polynomial with $A_0(z) = 1$

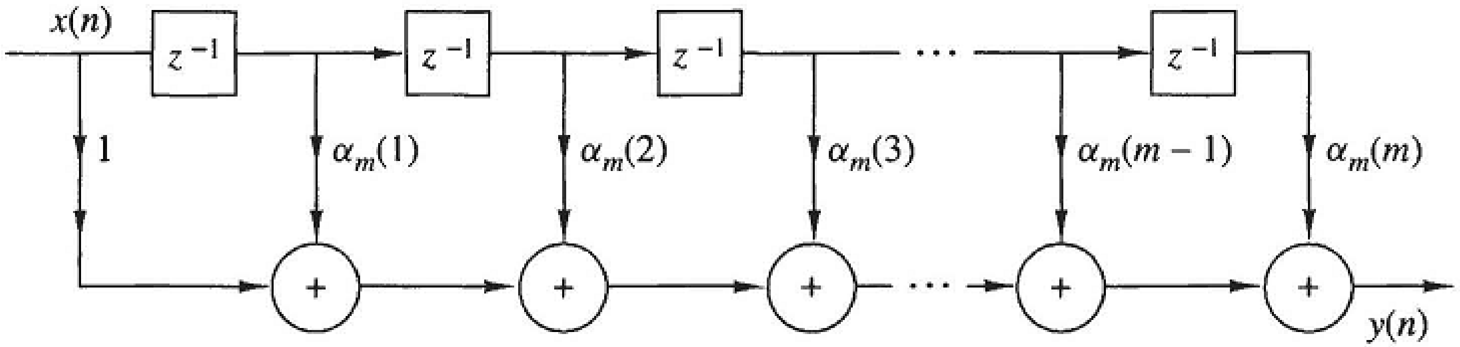
$$A_m(z) = 1 + \sum_{k=1}^{m} \alpha_m(k) z^{-k}, \qquad m \geq 1$$

- Y[n] can be written as

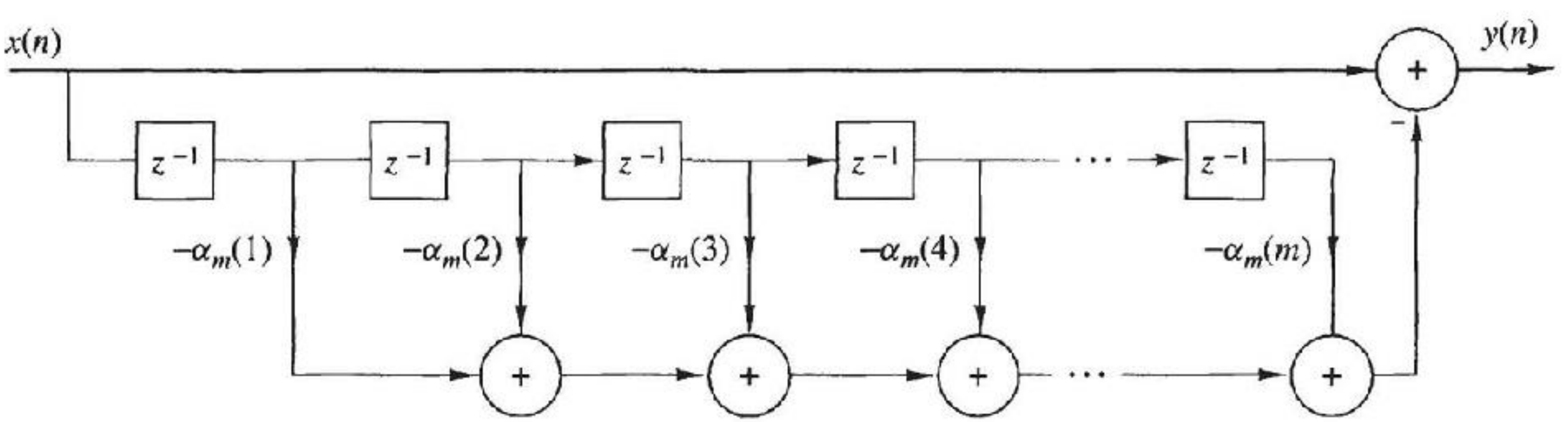
$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k) x(n-k)$$

Lattice Structure

- The direct form implementation can be expressed as



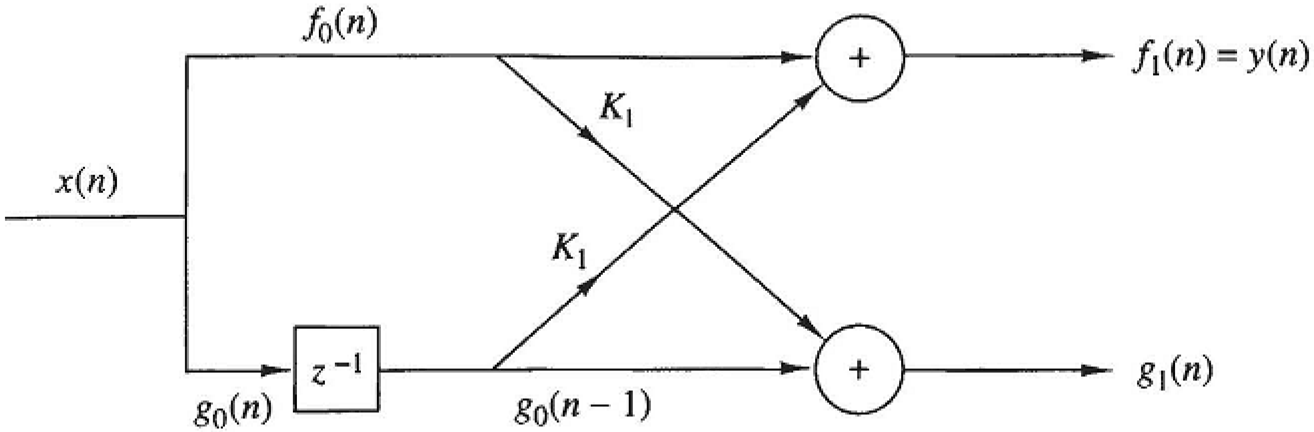
- Or



- This is called a prediction error filter.

Lattice Structure

- Let's consider a first order FIR filter, i.e., m=1: $y(n) = x(n) + \alpha_1(1)x(n-1)$
- Let the reflection coefficient $K_1 = \alpha_1(1)$. to get:



$$f_0(n) = g_0(n) = x(n)$$
$$f_1(n) = f_0(n) + K_1 g_0(n-1) = x(n) + K_1 x(n-1)$$
$$g_1(n) = K_1 f_0(n) + g_0(n-1) = K_1 x(n) + x(n-1)$$

Lattice Structure

- Now consider m=2:

$$y(n) = x(n) + \alpha_2(1)x(n-1) + \alpha_2(2)x(n-2)$$

- We cascade two lattice stages:
- The out put of the first stage is,

$$f_1(n) = x(n) + K_1 x(n-1)$$

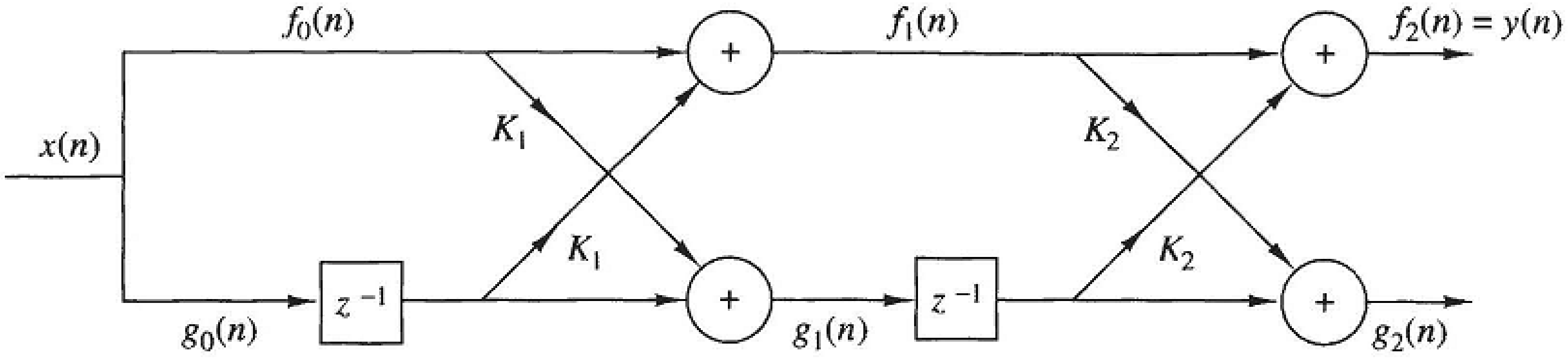
$$g_1(n) = K_1 x(n) + x(n-1)$$

- And the output of the second stage is:

$$f_2(n) = f_1(n) + K_2 g_1(n-1)$$

$$g_2(n) = K_2 f_1(n) + g_1(n-1)$$

Lattice Structure



Two-stage lattice filter.

- Let’s consider on $f_2[n]$:

$$f_2(n) = x(n) + K_1 x(n-1) + K_2[K_1 x(n-1) + x(n-2)]$$
$$= x(n) + K_1(1+K_2)x(n-1) + K_2 x(n-2)$$

- $f_2[n]$ will be y[n] if:

$$\alpha_2(2) = K_2, \qquad \alpha_2(1) = K_1(1+K_2)$$

- or, equivalently if:

$$K_2 = \alpha_2(2), \qquad K_1 = \frac{\alpha_2(1)}{1+\alpha_2(2)}$$

Lattice Structure

- In general:

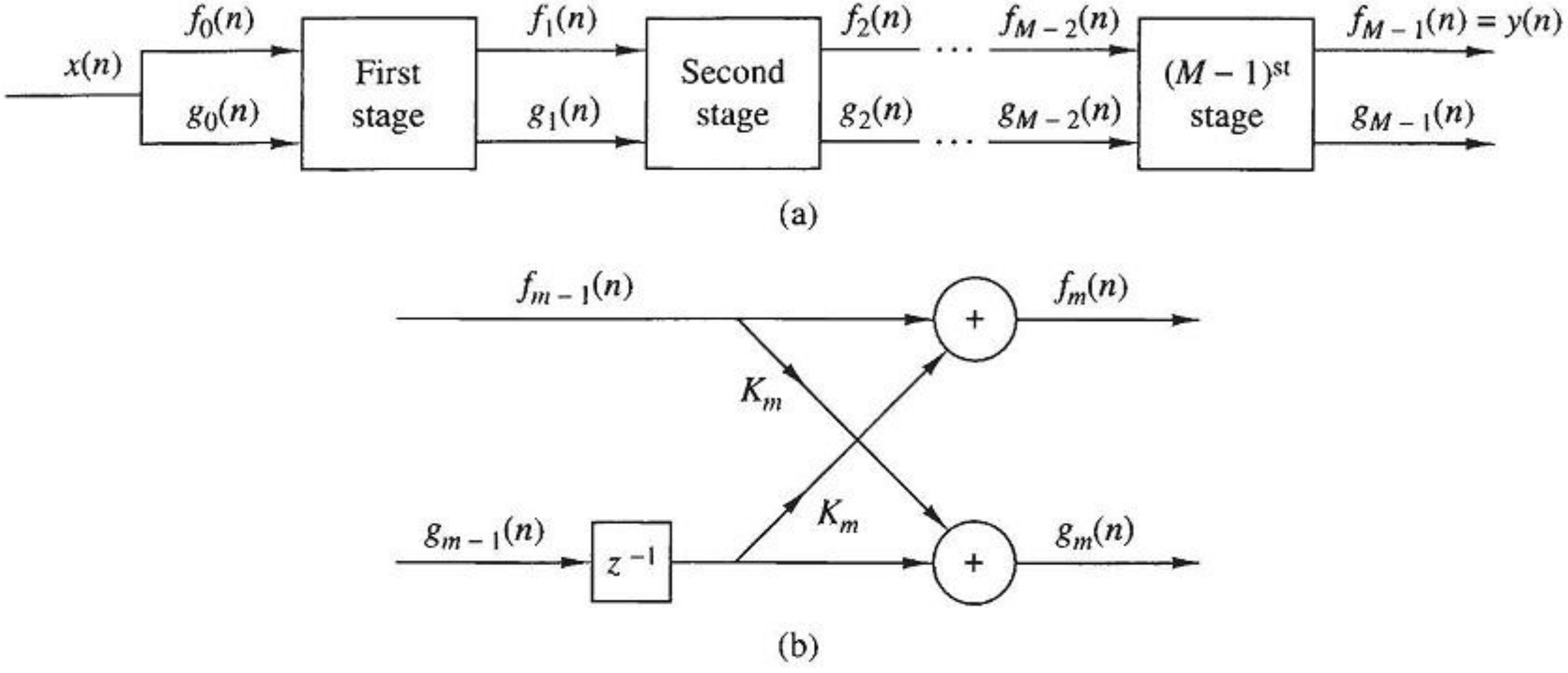
$$f_0(n) = g_0(n) = x(n)$$

$$f_m(n) = f_{m-1}(n) + K_m g_{m-1}(n-1), \qquad m = 1, 2, \ldots, M-1$$

$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1), \qquad m = 1, 2, \ldots, M-1$$

- Then:

$$y(n) = f_{M-1}(n)$$



$(M-1)$-stage lattice filter.

Conversion of reflection coefficients to filter taps

Use the following equations recursively,

$$A_0(z) = B_0(z) = 1$$

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z), \qquad m = 1, 2, \ldots, M-1$$

$$B_m(z) = z^{-m} A_m(z^{-1}), \qquad m = 1, 2, \ldots, M-1$$

EXAMPLE

Given a three-stage lattice filter with coefficients $K_1 = \frac{1}{4}$, $K_2 = \frac{1}{4}$, $K_3 = \frac{1}{3}$, determine the FIR filter coefficients for the direct-form structure.

Solution. $m = 1$. Thus we have

$$A_1(z) = A_0(z) + K_1 z^{-1} B_0(z)$$

$$= 1 + K_1 z^{-1} = 1 + \tfrac{1}{4} z^{-1}$$

Hence the coefficients of an FIR filter corresponding to the single-stage lattice are $\alpha_1(0) = 1$, $\alpha_1(1) = K_1 = \frac{1}{4}$. Since $B_m(z)$ is the reverse polynomial of $A_m(z)$, we have

$$B_1(z) = \tfrac{1}{4} + z^{-1}$$

Conversion of reflection coefficients to filter taps

Next we add the second stage to the lattice. For $m = 2$,

$$A_2(z) = A_1(z) + K_2 z^{-1} B_1(z)$$
$$= 1 + \tfrac{3}{8} z^{-1} + \tfrac{1}{2} z^{-2}$$

Hence the FIR filter parameters corresponding to the two-stage lattice are $\alpha_2(0) = 1$, $\alpha_2(1) = \frac{3}{8}$, $\alpha_2(2) = \frac{1}{2}$. Also, $B_2(z) = \frac{1}{2} + \frac{3}{8} z^{-1} + z^{-2}$

Finally, the addition of the third stage to the lattice results in the polynomial

$$A_3(z) = A_2(z) + K_3 z^{-1} B_2(z)$$
$$= 1 + \tfrac{13}{24} z^{-1} + \tfrac{5}{8} z^{-2} + \tfrac{1}{3} z^{-3}$$

Consequently, the desired direct-form FIR filter is characterized by the coefficients

$$\alpha_3(0) = 1, \qquad \alpha_3(1) = \tfrac{13}{24}, \qquad \alpha_3(2) = \tfrac{5}{8}, \qquad \alpha_3(3) = \tfrac{1}{3}$$

Conversion of FIR Taps to Lattice reflection coefficients

- We start with $A_{M-1}(z)$ and find lattice coefficients using

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z)$$
$$= A_{m-1}(z) + K_m[B_m(z) - K_m A_{m-1}(z)]$$

Or, equivalently,

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}, \qquad m = M-1, M-2, \ldots, 1$$

EXAMPLE

Determine the lattice coefficients corresponding to the FIR filter with system function

$$H(z) = A_3(z) = 1 + \tfrac{13}{24}z^{-1} + \tfrac{5}{8}z^{-2} + \tfrac{1}{3}z^{-3}$$

Solution. First we note that $K_3 = \alpha_3(3) = \frac{1}{3}$. Furthermore,

$$B_3(z) = \tfrac{1}{3} + \tfrac{5}{8}z^{-1} + \tfrac{13}{24}z^{-2} + z^{-3}$$

Conversion of FIR Taps to Lattice reflection coefficients

The step-down relationship with $m = 3$ yields

$$A_2(z) = \frac{A_3(z) - K_3 B_3(z)}{1 - K_3^2}$$

$$= 1 + \tfrac{3}{8}z^{-1} + \tfrac{1}{2}z^{-2}$$

Hence $K_2 = \alpha_2(2) = \frac{1}{2}$ and $B_2(z) = \frac{1}{2} + \frac{3}{8}z^{-1} + z^{-2}$. By repeating the step-down recursion we obtain

$$A_1(z) = \frac{A_2(z) - K_2 B_2(z)}{1 - K_2^2}$$

$$= 1 + \tfrac{1}{4}z^{-1}$$

Hence $K_1 = \alpha_1(1) = \frac{1}{4}$.

- A recursive formula for finding
- the lattice coefficients:

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - K_m \beta_m(k)}{1 - K_m^2}$$

$$= \frac{\alpha_m(k) - \alpha_m(m)\alpha_m(m-k)}{1 - \alpha_m^2(m)}, \qquad 1 \le k \le m-1$$

Structure of IIR Systems

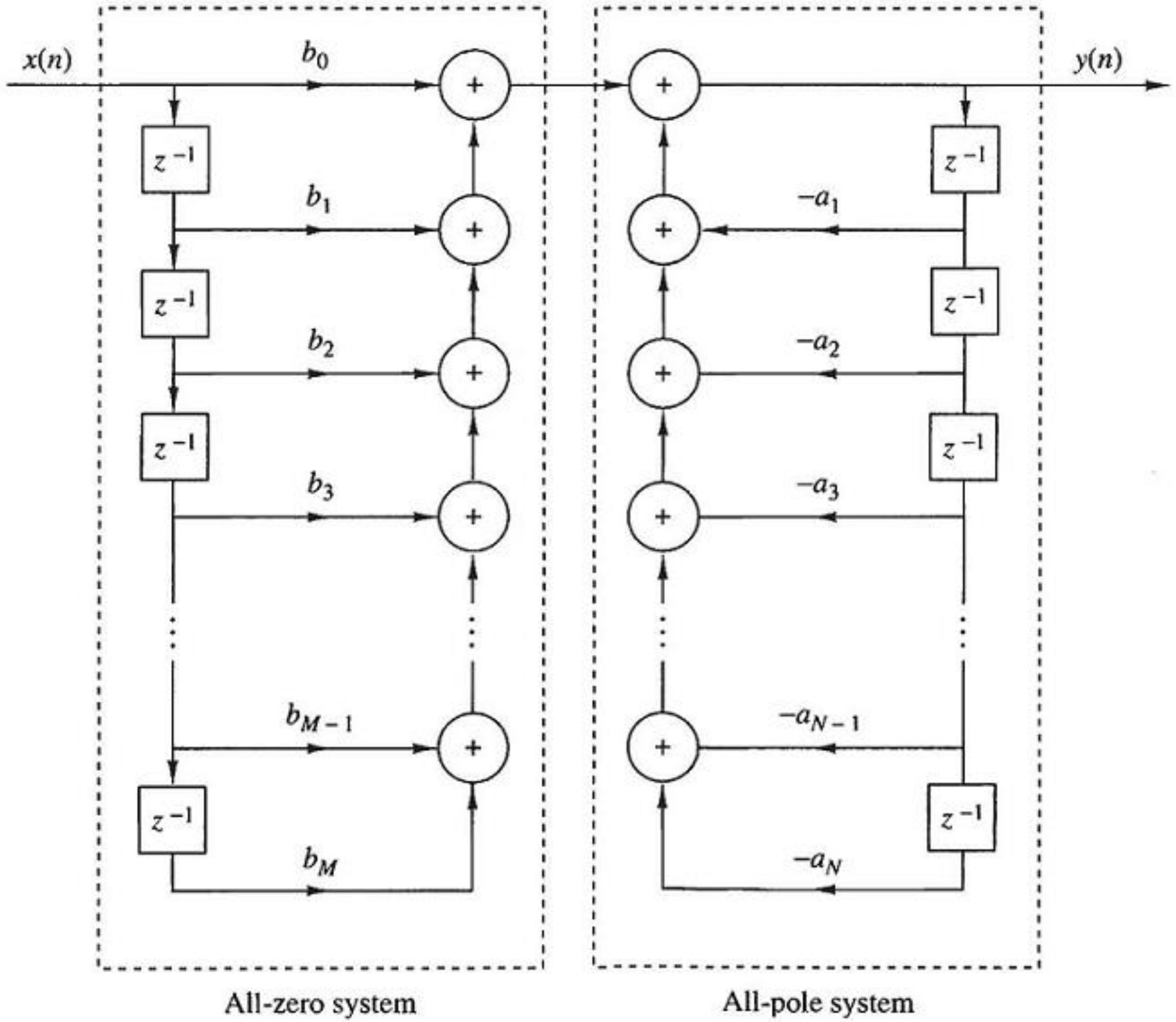
- An IIR System can be expressed as $H(z) = H_1(z)H_2(z)$
- With an all-zero part

$$H_1(z) = \sum_{k=0}^{M} b_k z^{-k}$$

- and an all-pole part

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Structure of IIR Systems: Direct form I



Direct form I realization.

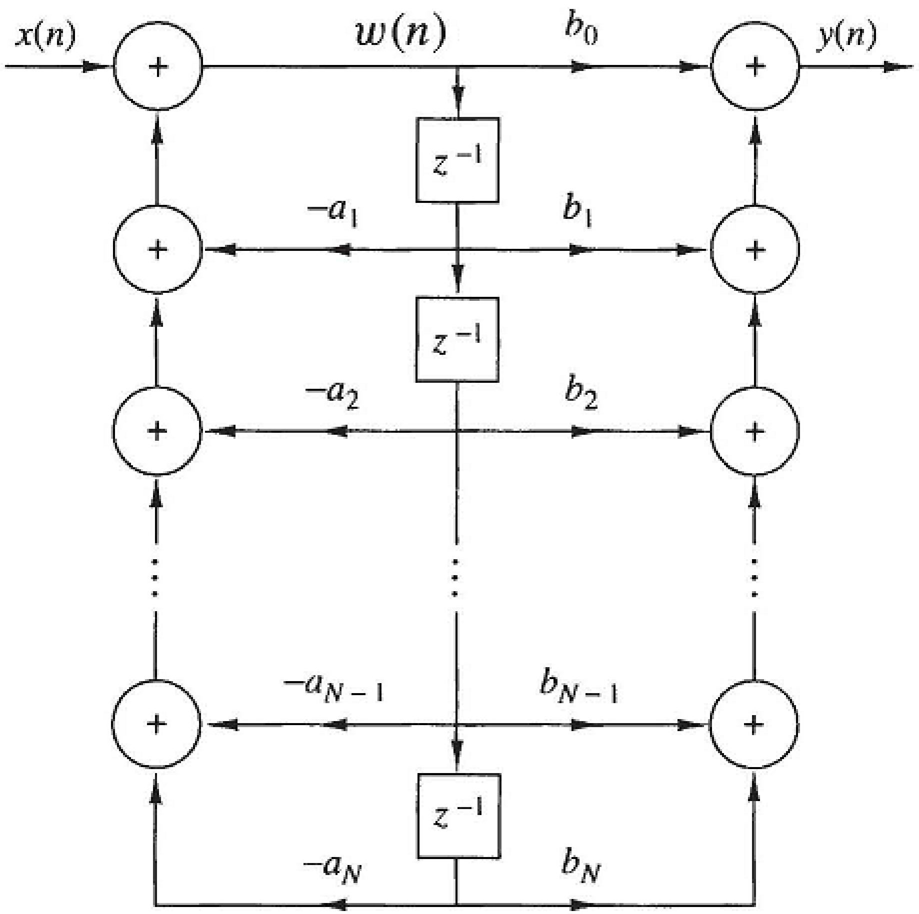
Structure of IIR Systems: Direct form II

- If we put the all-pole filter firs, we get
- w[n] is the output of $H_2(z)$ and
- the input to $H_1(z)$
- So, we have

$$w(n) = -\sum_{k=1}^{N} a_k w(n-k) + x(n)$$

and

$$y(n) = \sum_{k=0}^{M} b_k w(n-k)$$



Structure of IIR Systems: Cascade Implementation

- The transfer function H(z) can be written as:

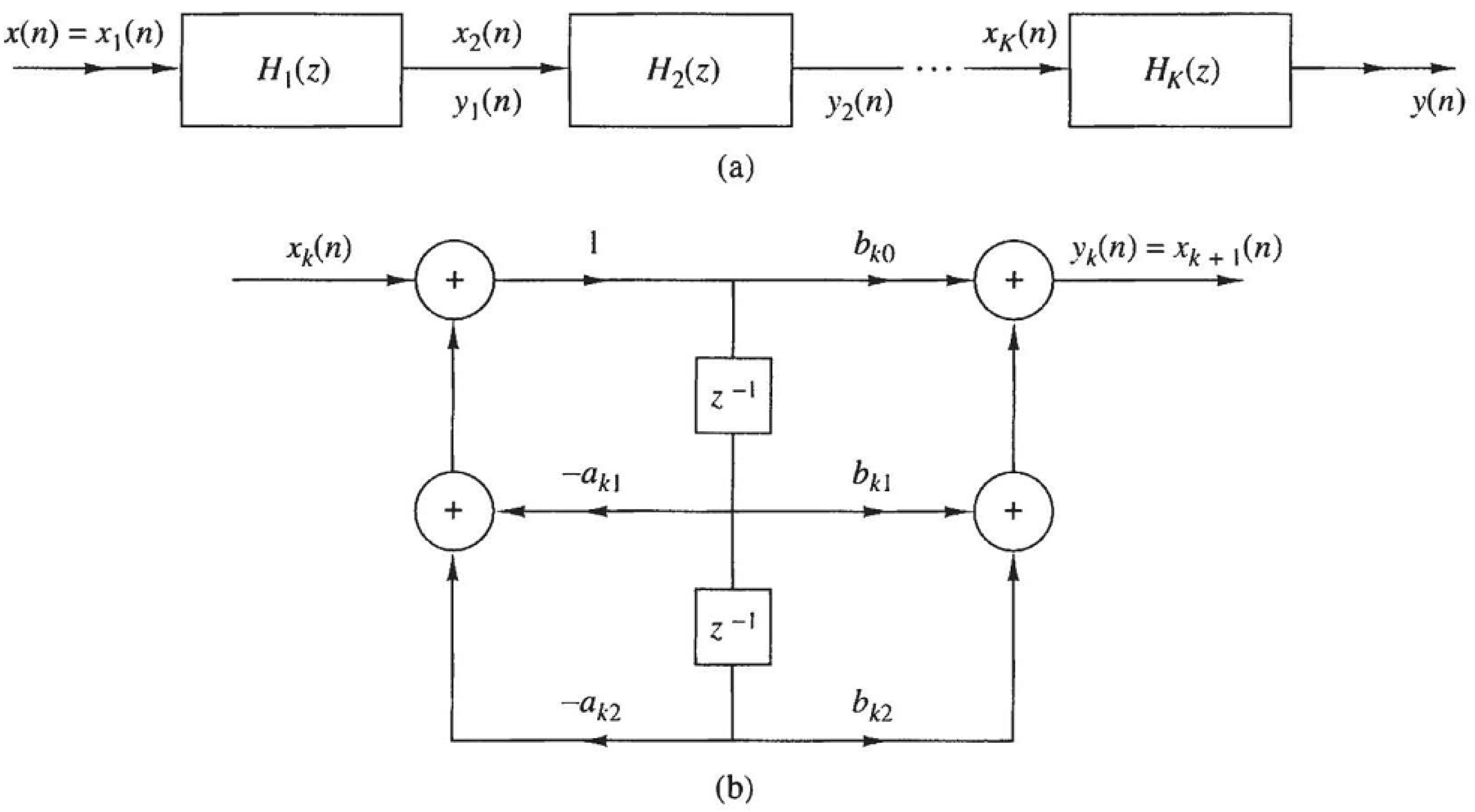
$$H(z) = \prod_{k=1}^{K} H_k(z)$$

- where K is [N/2]. That is it is N/2 for N even and (N+1)/2 for N odd.

- $H_k(z)$ has the general form of:

$$H_k(z) = \frac{b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}$$

Structure of IIR Systems: Cascade Implementation



Cascade structure of second-order systems and a realization of each second-order section.

Structure of IIR Systems: Parallel Implementation

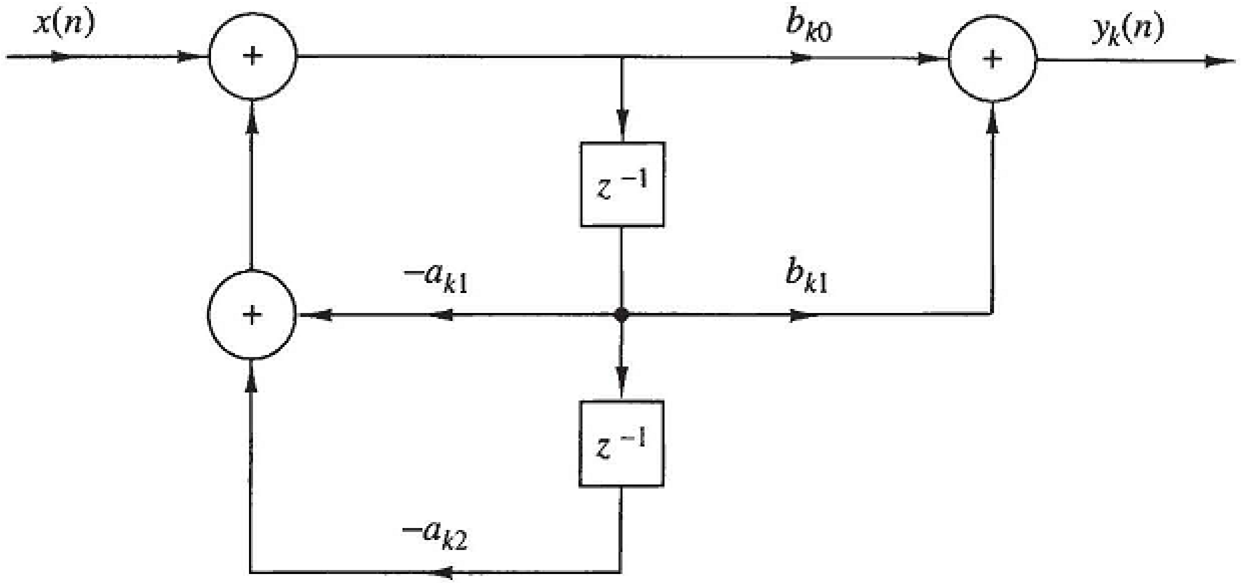
Performing partial fraction expansion, we get,

$$H(z) = C + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

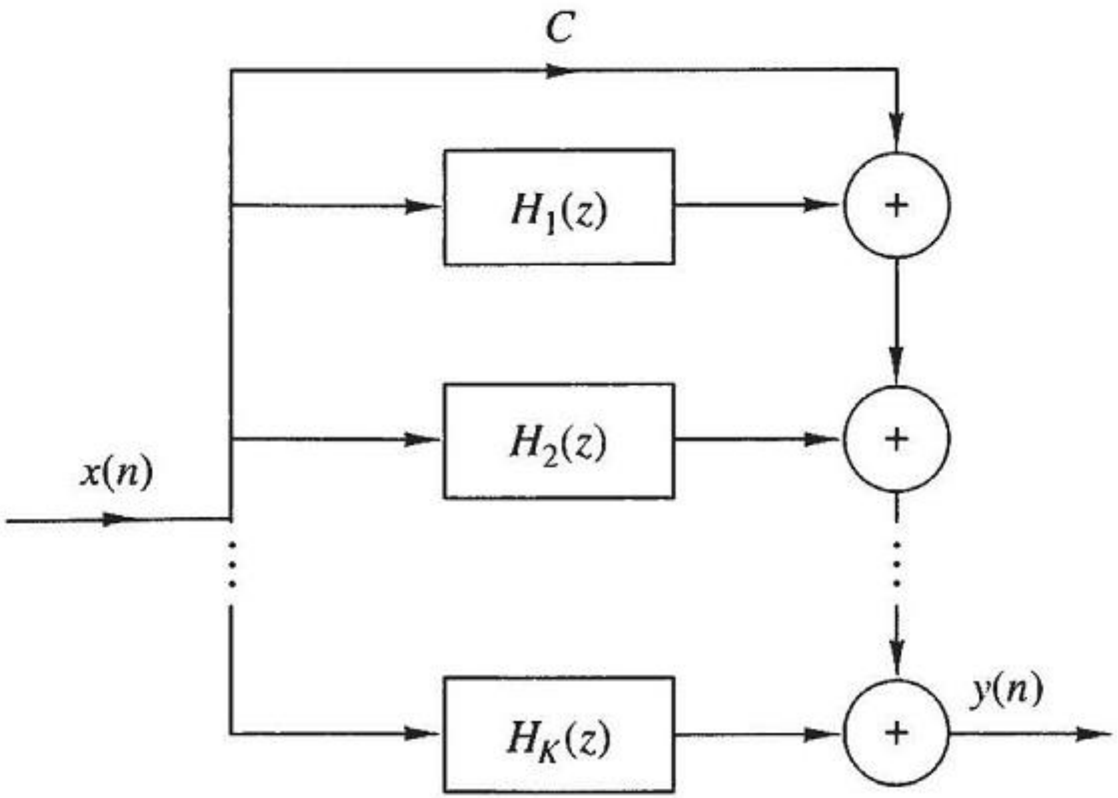
so we can implement the filter using N parallel branches each with a single pole.

Most often the poles are complex. The complex poles are pairs of conjugate poles. So, we can implement the filter with parallel second order branches:

$$H_k(z) = \frac{b_{k0} + b_{k1} z^{-1}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}}$$



Structure of IIR Systems: Parallel Implementation



$$H_k(z) = \frac{b_{k0} + b_{k1}z^{-1}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}$$

Example:

Find the cascade and parallel implementation of the filter:

$$H(z) = \frac{10(1-\frac{1}{2}z^{-1})(1-\frac{2}{3}z^{-1})(1+2z^{-1})}{(1-\frac{3}{4}z^{-1})(1-\frac{1}{8}z^{-1})[1-(\frac{1}{2}+j\frac{1}{2})z^{-1}][1-(\frac{1}{2}-j\frac{1}{2})z^{-1}]}$$

We have

$$H_1(z) = \frac{1-\frac{2}{3}z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}}$$

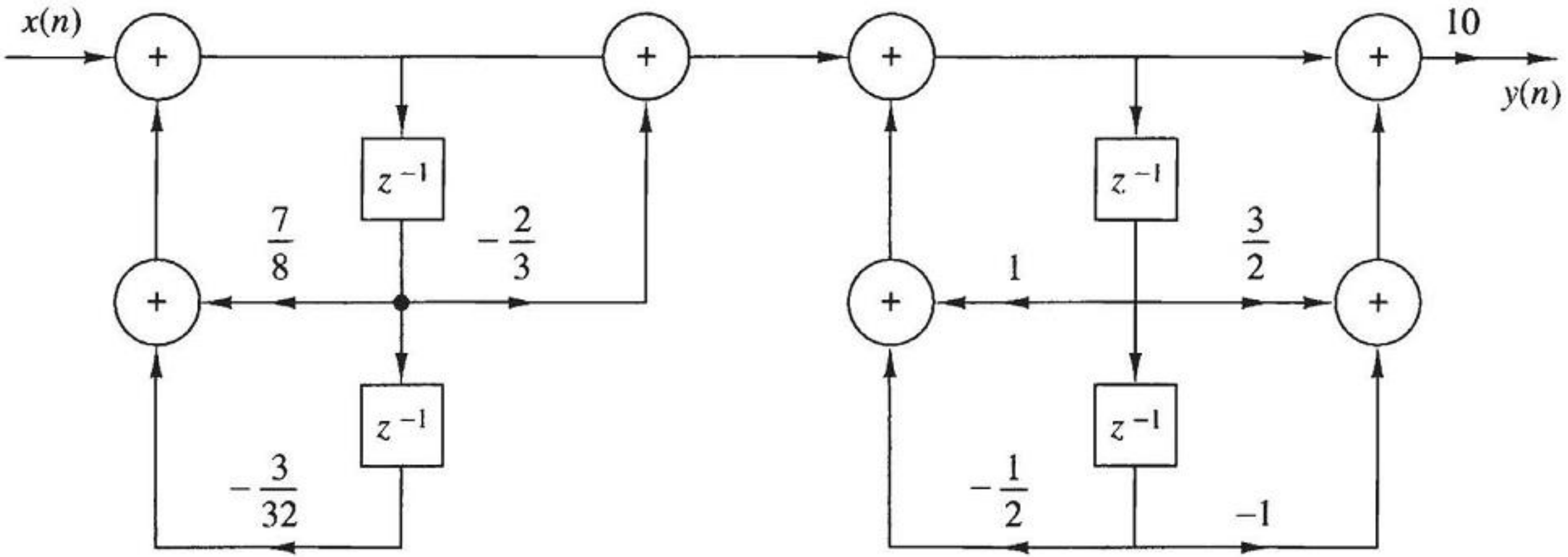
$$H_2(z) = \frac{1+\frac{3}{2}z^{-1}-z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

So

$$H(z) = 10H_1(z)H_2(z)$$

Example:

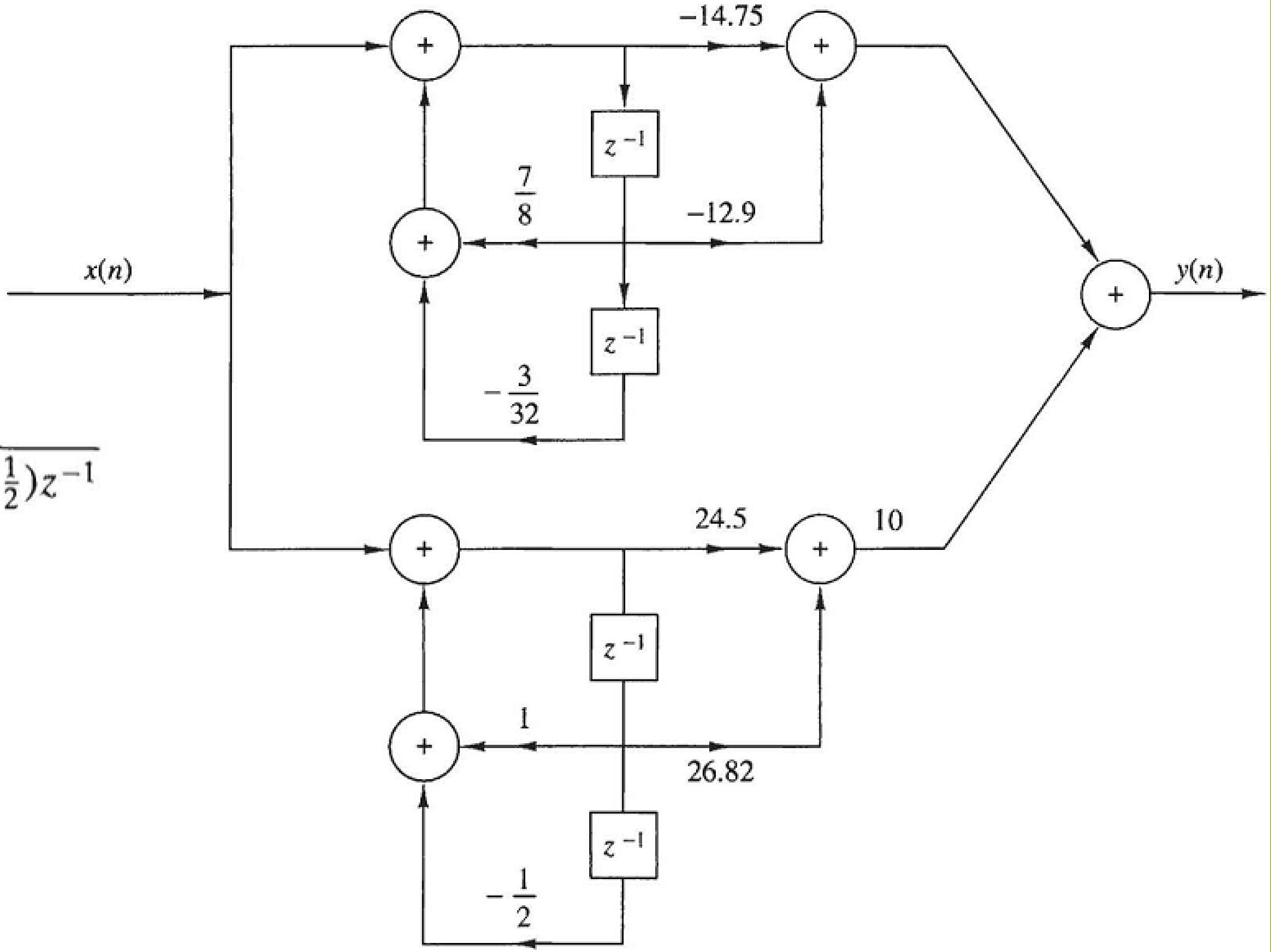
And the cascade implementation is:



Example:

For parallel implementation, we have:

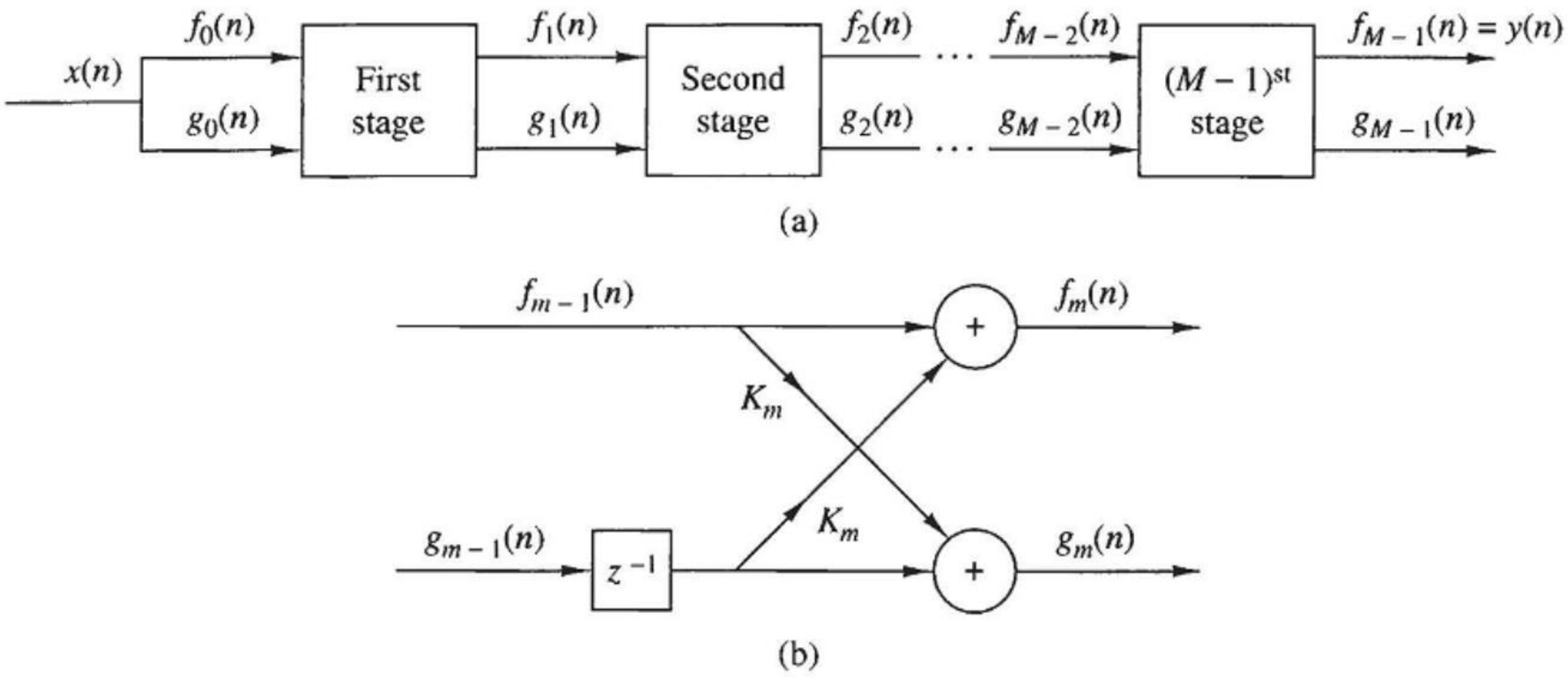
$$H(z) = \frac{A_1}{1-\frac{3}{4}z^{-1}} + \frac{A_2}{1-\frac{1}{8}z^{-1}} + \frac{A_3}{1-(\frac{1}{2}+j\frac{1}{2})z^{-1}} + \frac{A_3^*}{1-(\frac{1}{2}-j\frac{1}{2})z^{-1}}$$



Structure of IIR Systems: Lattice Implementation

We have already done the all-zero filter:

$$H_1(z) = \sum_{k=0}^{M} b_k z^{-k}$$



$(M-1)$-stage lattice filter.

Structure of IIR Systems: Lattice Implementation

Now, let's consider the all-pole filter:

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

or,

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_N(k) z^{-k}} = \frac{1}{A_N(z)}$$

We can write y[n] as:

$$y(n) = -\sum_{k=1}^{N} a_N(k) y(n-k) + x(n)$$

Structure of IIR Systems: Lattice Implementation

Then x[n] can be written as: $$x(n) = y(n) + \sum_{k=1}^{N} a_N(k) y(n-k)$$

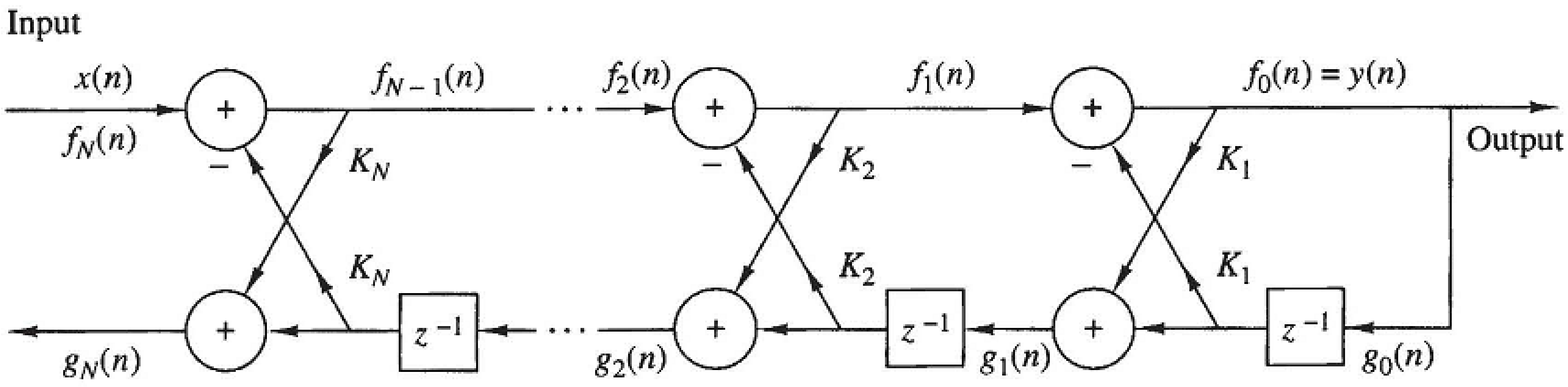
This is a lattice filter with similar to the one we saw before but the role of input and output changed. That is we have $x(n) = f_N(n)$ and $y(n) = f_0(n)$ We use:

$$f_N(n) = x(n)$$

$$f_{m-1}(n) = f_m(n) - K_m g_{m-1}(n-1), \qquad m = N, N-1, \ldots, 1$$

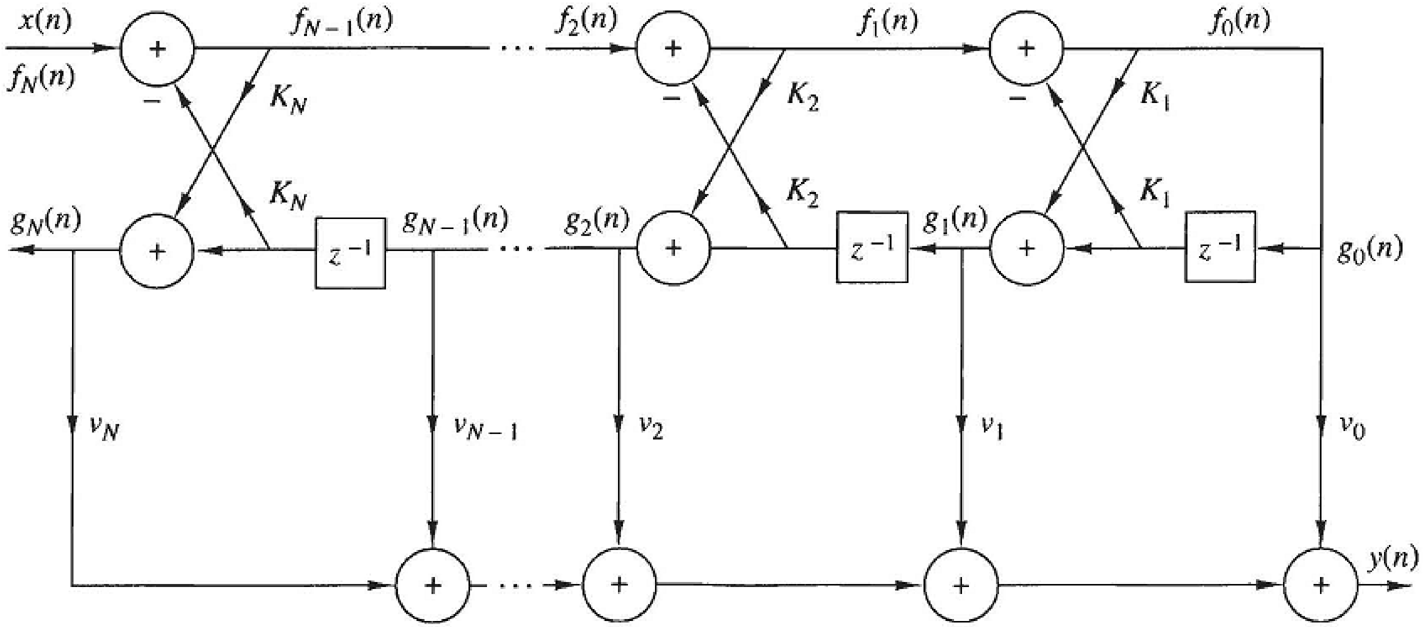
$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1), \qquad m = N, N-1, \ldots, 1$$

$$y(n) = f_0(n) = g_0(n)$$



Lattice structure for an all-pole IIR system.

Lattice Ladder Implementation of IIR Systems



Lattice-ladder structure for the realization of a pole–zero system.