Chapter 2

2.1

(a)

$$x(n) = \left\{\ldots 0, \frac{1}{3}, \frac{2}{3}, \underset{\uparrow}{1}, 1, 1, 1, 0, \ldots\right\}$$

Refer to fig 2.1-1.

(b) After folding s(n) we have

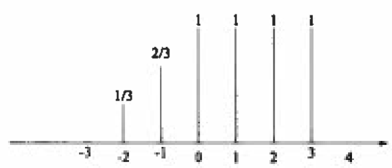

Figure 2.1-1:

$$x(-n) = \left\{\ldots 0, 1, 1, 1, \underset{\uparrow}{1}, \frac{2}{3}, \frac{1}{3}, 0, \ldots\right\}.$$

After delaying the folded signal by 4 samples, we have

$$x(-n+4) = \left\{\ldots 0, \underset{\uparrow}{0}, 1, 1, 1, 1, \frac{2}{3}, \frac{1}{3}, 0, \ldots\right\}.$$

On the other hand, if we delay x(n) by 4 samples we have

$$x(n-4) = \left\{\ldots \underset{\uparrow}{0}, 0, \frac{1}{3}, \frac{2}{3}, 1, 1, 1, 1, 0, \ldots\right\}.$$

Now, if we fold $x(n-4)$ we have

$$x(-n-4) = \left\{\ldots 0, 1, 1, 1, 1, \frac{2}{3}, \frac{1}{3}, 0, \underset{\uparrow}{0}, \ldots\right\}$$

(c)

$$x(-n+4) = \left\{\ldots \underset{\uparrow}{0}, 1, 1, 1, 1, \frac{2}{3}, \frac{1}{3}, 0, \ldots\right\}$$

(d) To obtain $x(-n+k)$, first we fold $x(n)$. This yields $x(-n)$. Then, we shift $x(-n)$ by k samples to the right if $k > 0$, or k samples to the left if $k < 0$.
(e) Yes.

$$x(n) = \frac{1}{3}\delta(n-2) + \frac{2}{3}\delta(n+1) + u(n) - u(n-4)$$

2.3

(a)

$$u(n) - u(n-1) = \delta(n) = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ 0, & n > 0 \end{cases}$$

(b)

$$\sum_{k=-\infty}^{n} \delta(k) = u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

$$\sum_{k=0}^{\infty} \delta(n-k) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

2.6

(a) No, the system is time variant. Proof: If

$$\begin{aligned} x(n) \rightarrow y(n) &= x(n^2) \\ x(n-k) \rightarrow y_1(n) &= x[(n-k)^2] \\ &= x(n^2 + k^2 - 2nk) \\ &\neq y(n-k) \end{aligned}$$

(b) (1)

$$x(n) = \left\{0, \underset{\uparrow}{1}, 1, 1, 1, 0, \ldots\right\}$$

(2)

$$y(n) = x(n^2) = \left\{\ldots, 0, 1, \underset{\uparrow}{1}, 1, 0, \ldots\right\}$$

(3)

$$y(n-2) = \left\{\ldots, 0, \underset{\uparrow}{0}, 1, 1, 1, 0, \ldots\right\}$$

(4)

$$x(n-2) = \left\{\ldots, \underset{\uparrow}{0}, 0, 1, 1, 1, 1, 0, \ldots\right\}$$

(5)

$$y_2(n) = \mathcal{T}[x(n-2)] = \left\{\ldots, 0, 1, 0, \underset{\uparrow}{0}, 0, 1, 0, \ldots\right\}$$

(6)

$$y_2(n) \neq y(n-2) \Rightarrow \text{system is time variant.}$$

(c) (1)

$$x(n) = \left\{\underset{\uparrow}{1}, 1, 1, 1\right\}$$

(2)

$$y(n) = \left\{\underset{\uparrow}{1}, 0, 0, 0, 0, -1\right\}$$

(3)

$$y(n-2) = \left\{\underset{\uparrow}{0}, 0, 1, 0, 0, 0, 0, -1\right\}$$

(4)

$$x(n-2) = \left\{\underset{\uparrow}{0}, 0, 1, 1, 1, 1\right\}$$

(5)

$$y_2(n) = \left\{\underset{\uparrow}{0}, 0, 1, 0, 0, 0, 0, -1\right\}$$

(6)

$$y_2(n) = y(n-2).$$

The system is time invariant, but this example alone does not constitute a proof.

(d) (1)

$$y(n) = nx(n),$$

$$x(n) = \left\{\ldots, 0, \underset{\uparrow}{1}, 1, 1, 1, 0, \ldots\right\}$$

(2)

$$y(n) = \left\{\ldots, \underset{\uparrow}{0}, 1, 2, 3, \ldots\right\}$$

(3)

$$y(n-2) = \left\{\ldots, \underset{\uparrow}{0}, 0, 0, 1, 2, 3, \ldots\right\}$$

(4)

$$x(n-2) = \left\{\ldots, 0, \underset{\uparrow}{0}, 0, 1, 1, 1, 1, \ldots\right\}$$

(5)

$$y_2(n) = \mathcal{T}[x(n-2)] = \{\ldots, 0, 0, 2, 3, 4, 5, \ldots\}$$

(6)

$$y_2(n) \neq y(n-2) \Rightarrow \text{the system is time variant.}$$

2.7

(a) Static, nonlinear, time invariant, causal, stable.
(b) Dynamic, linear, time invariant, noncausal and unstable. The latter is easily proved. For the bounded input $x(k) = u(k)$, the output becomes

$$y(n) = \sum_{k=-\infty}^{n+1} u(k) = \begin{cases} 0, & n < -1 \\ n+2, & n \geq -1 \end{cases}$$

since $y(n) \to \infty$ as $n \to \infty$, the system is unstable.
(c) Static, linear, timevariant, causal, stable.
(d) Dynamic, linear, time invariant, noncausal, stable.
(e) Static, nonlinear, time invariant, causal, stable.
(f) Static, nonlinear, time invariant, causal, stable.
(g) Static, nonlinear, time invariant, causal, stable.
(h) Static, linear, time invariant, causal, stable.
(i) Dynamic, linear, time variant, noncausal, unstable. Note that the bounded input $x(n) = u(n)$ produces an unbounded output.
(j) Dynamic, linear, time variant, noncausal, stable.
(k) Static, nonlinear, time invariant, causal, stable.
(l) Dynamic, linear, time invariant, noncausal, stable.
(m) Static, nonlinear, time invariant, causal, stable.
(n) Static, linear, time invariant, causal, stable.

2.8

(a) True. If

$$v_1(n) = \mathcal{T}_1[x_1(n)] \text{ and}$$
$$v_2(n) = \mathcal{T}_1[x_2(n)],$$

then

$$\alpha_1 x_1(n) + \alpha_2 x_2(n)$$

yields

$$\alpha_1 v_1(n) + \alpha_2 v_2(n)$$

by the linearity property of $\mathcal{T}_1$. Similarly, if

$$y_1(n) = \mathcal{T}_2[v_1(n)] \text{ and}$$
$$y_2(n) = \mathcal{T}_2[v_2(n)],$$

then

$$\beta_1 v_1(n) + \beta_2 v_2(n) \to y(n) = \beta_1 y_1(n) + \beta_2 y_2(n)$$

by the linearity property of $\mathcal{T}_2$. Since

$$v_1(n) = \mathcal{T}_1[x_1(n)] \text{ and}$$

$$v_2(n) = \mathcal{T}_2[x_2(n)],$$

it follows that

$$A_1x_1(n) + A_2x_2(n)$$

yields the output

$$A_1\mathcal{T}[x_1(n)] + A_2\mathcal{T}[x_2(n)],$$

where $\mathcal{T} = \mathcal{T}_1\mathcal{T}_2$. Hence $\mathcal{T}$ is linear.
(b) True. For $\mathcal{T}_1$, if

$$x(n) \rightarrow v(n) \text{ and}$$

$$x(n-k) \rightarrow v(n-k),$$

For $\mathcal{T}_2$, if

$$v(n) \rightarrow y(n)$$

$$\text{and} v(n-k) \rightarrow y(n-k).$$

Hence, For $\mathcal{T}_1\mathcal{T}_2$, if

$$x(n) \rightarrow y(n) \text{ and}$$

$$x(n-k) \rightarrow y(n-k)$$

Therefore, $\mathcal{T} = \mathcal{T}_1\mathcal{T}_2$ is time invariant.
(c) True. $\mathcal{T}_1$ is causal $\Rightarrow v(n)$ depends only on x(k) for $k \leq n$. $\mathcal{T}_2$ is causal $\Rightarrow y(n)$ depends only on $v(k)$ for $k \leq n$. Therefore, $y(n)$ depends only on $x(k)$ for $k \leq n$. Hence, $\mathcal{T}$ is causal.
(d) True. Combine (a) and (b).
(e) True. This follows from $h_1(n) * h_2(n) = h_2(n) * h_1(n)$
(f) False. For example, consider

$$\mathcal{T}_1 : y(n) = nx(n) \text{ and}$$

$$\mathcal{T}_2 : y(n) = nx(n+1).$$

Then,

$$\begin{aligned}
\mathcal{T}_2[\mathcal{T}_1[\delta(n)]] &= \mathcal{T}_2(0) = 0. \\
\mathcal{T}_1[\mathcal{T}_2[\delta(n)]] &= \mathcal{T}_1[\delta(n+1)] \\
&= -\delta(n+1) \\
&\neq 0.
\end{aligned}$$

(g) False. For example, consider

$$\mathcal{T}_1 : y(n) = x(n) + b \text{ and}$$

$$\mathcal{T}_2 : y(n) = x(n) - b, \text{ where } b \neq 0.$$

Then,

$$\mathcal{T}[x(n)] = \mathcal{T}_2[\mathcal{T}_1[x(n)]] = \mathcal{T}_2[x(n) + b] = x(n).$$

Hence $\mathcal{T}$ is linear.
(h) True.

$$\mathcal{T}_1 \text{ is stable } \Rightarrow v(n) \text{ is bounded if } x(n) \text{ is bounded.}$$

$$\mathcal{T}_2 \text{ is stable } \Rightarrow y(n) \text{ is bounded if } v(n) \text{ is bounded .}$$

Hence, y(n) is bounded if x(n) is bounded $\Rightarrow \mathcal{T} = \mathcal{T}_1\mathcal{T}_2$ is stable.
(i) Inverse of (c). $\mathcal{T}_1$ and for $\mathcal{T}_2$ are noncausal $\Rightarrow \mathcal{T}$ is noncausal. Example:

$$\begin{aligned}\mathcal{T}_1 : y(n) &= x(n+1) \text{ and}\\ \mathcal{T}_2 : y(n) &= x(n-2)\\ \Rightarrow \mathcal{T} : y(n) &= x(n-1),\end{aligned}$$

which is causal. Hence, the inverse of (c) is false.
Inverse of (h): $\mathcal{T}_1$ and/or $\mathcal{T}_2$ is unstable, implies $\mathcal{T}$ is unstable. Example:

$$\mathcal{T}_1 : y(n) = e^{x(n)}, \text{ stable and } \mathcal{T}_2 : y(n) = ln[x(n)], \text{ which is unstable.}$$

But $\mathcal{T} : y(n) = x(n)$, which is stable. Hence, the inverse of (h) is false.

2.15

(a)

$$\begin{aligned}\text{For } a = 1, \sum_{n=M}^{N} a^n &= N - M + 1\\ \text{for } a \neq 1, \sum_{n=M}^{N} a^n &= a^M + a^{M+1} + \ldots + a^N\\ (1-a)\sum_{n=M}^{N} a^n &= a^M + a^{M+1} - a^{M+1} + \ldots + a^N - a^N - a^{N+1}\\ &= a^M - a^{N+1}\end{aligned}$$

(b) For $M = 0, |a| < 1$, and $N \to \infty$,

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, |a| < 1.$$

2.17

(a)

$$\begin{aligned}x(n) &= \left\{\underset{\uparrow}{1}, 1, 1, 1\right\}\\ h(n) &= \left\{\underset{\uparrow}{6}, 5, 4, 3, 2, 1\right\}\\ y(n) &= \sum_{k=0}^{n} x(k)h(n-k)\\ y(0) &= x(0)h(0) = 6,\\ y(1) &= x(0)h(1) + x(1)h(0) = 11\\ y(2) &= x(0)h(2) + x(1)h(1) + x(2)h(0) = 15\\ y(3) &= x(0)h(3) + x(1)h(2) + x(2)h(1) + x(3)h(0) = 18\\ y(4) &= x(0)h(4) + x(1)h(3) + x(2)h(2) + x(3)h(1) + x(4)h(0) = 14\\ y(5) &= x(0)h(5) + x(1)h(4) + x(2)h(3) + x(3)h(2) + x(4)h(1) + x(5)h(0) = 10\\ y(6) &= x(1)h(5) + x(2)h(4) + x(3)h(2) = 6\\ y(7) &= x(2)h(5) + x(3)h(4) = 3\\ y(8) &= x(3)h(5) = 1\\ y(n) &= 0, n \geq 9\\ y(n) &= \left\{\underset{\uparrow}{6}, 11, 15, 18, 14, 10, 6, 3, 1\right\}\end{aligned}$$

(b) By following the same procedure as in (a), we obtain

$$y(n) = \left\{6, 11, 15, \underset{\uparrow}{18}, 14, 10, 6, 3, 1\right\}$$

(c) By following the same procedure as in (a), we obtain

$$y(n) = \left\{1, \underset{\uparrow}{2}, 2, 2, 1\right\}$$

(d) By following the same procedure as in (a), we obtain

$$y(n) = \left\{\underset{\uparrow}{1}, 2, 2, 2, 1\right\}$$

2.18

(a)

$$\begin{aligned}
x(n) &= \left\{\underset{\uparrow}{0}, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\right\} \\
h(n) &= \left\{1, 1, \underset{\uparrow}{1}, 1, 1\right\} \\
y(n) &= x(n) * h(n) \\
&= \left\{\frac{1}{3}, \underset{\uparrow}{1}, 2, \frac{10}{3}, 5, \frac{20}{3}, 6, 5, \frac{11}{3}, 2\right\}
\end{aligned}$$

(b)

$$\begin{aligned}
x(n) &= \frac{1}{3}n[u(n) - u(n-7)], \\
h(n) &= u(n+2) - u(n-3) \\
y(n) &= x(n) * h(n) \\
&= \frac{1}{3}n[u(n) - u(n-7)] * [u(n+2) - u(n-3)] \\
&= \frac{1}{3}n[u(n) * u(n+2) - u(n) * u(n-3) - u(n-7) * u(n+2) + u(n-7) * u(n-3)] \\
y(n) &= \frac{1}{3}\delta(n+1) + \delta(n) + 2\delta(n-1) + \frac{10}{3}\delta(n-2) + 5\delta(n-3) + \frac{20}{3}\delta(n-4) + 6\delta(n-5) \\
&\quad +5\delta(n-6) + 5\delta(n-6) + \frac{11}{3}\delta(n-7) + \delta(n-8)
\end{aligned}$$

2.19

2.31

From 2.30, the characteristic values are $\lambda = 4, -1$. Hence

$$y_h(n) = c_1 4^n + c_2(-1)^n$$

When $x(n) = \delta(N)$, we find that

$$y(0) = 1 \text{ and}$$

$$y(1) - 3y(0) = 2 \text{ or}$$

$$y(1) = 5.$$

Hence,

$$c_1 + c_2 = 1 \text{ and } 4c_1 - c_2 = 5$$

This yields, $c_1 = \frac{6}{5}$ and $c_2 = -\frac{1}{5}$. Therefore,

$$h(n) = \left[\frac{6}{5}4^n - \frac{1}{5}(-1)^n\right]u(n)$$

2.35

(a) $h(n) = h_1(n) * [h_2(n) - h_3(n) * h_4(n)]$
(b)

$$\begin{aligned} h_3(n) * h_4(n) &= (n-1)u(n-2) \\ h_2(n) - h_3(n) * h_4(n) &= 2u(n) - \delta(n) \\ h_1(n) &= \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n-1) + \frac{1}{2}\delta(n-2) \\ \text{Hence } h(n) &= \left[\frac{1}{2}\delta(n) + \frac{1}{4}\delta(n-1) + \frac{1}{2}\delta(n-2)\right] * [2u(n) - \delta(n)] \\ &= \frac{1}{2}\delta(n) + \frac{5}{4}\delta(n-1) + 2\delta(n-2) + \frac{5}{2}u(n-3) \end{aligned}$$

(c)

$$\begin{aligned} x(n) &= \left\{1, 0, \underset{\uparrow}{0}, 3, 0, -4\right\} \\ y(n) &= \left\{\frac{1}{2}, \frac{5}{4}, \underset{\uparrow}{2}, \frac{25}{4}, \frac{13}{2}, 5, 2, 0, 0, \ldots\right\} \end{aligned}$$

2.37

$$\begin{aligned} h(n) &= [u(n) - u(n-M)]/M \\ s(n) &= \sum_{k=-\infty}^{\infty} u(k)h(n-k) \\ &= \sum_{k=0}^{n} h(n-k) = \begin{cases} \frac{n+1}{M}, & n < M \\ 1, & n \geq M \end{cases} \end{aligned}$$

2.38

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |h(n)| &= \sum_{n=0, n even}^{\infty} |a|^n \\ &= \sum_{n=0}^{\infty} |a|^{2n} \\ &= \frac{1}{1-|a|^2} \end{aligned}$$

Stable if $|a| < 1$

2.48

(a)

$$
\begin{aligned}
y(n) &= ay(n-1) + bx(n) \\
\Rightarrow h(n) &= ba^n u(n) \\
\sum_{n=0}^{\infty} h(n) &= \frac{b}{1-a} = 1 \\
\Rightarrow b &= 1-a.
\end{aligned}
$$

(b)

$$
\begin{aligned}
s(n) &= \sum_{k=0}^{n} h(n-k) \\
&= b\left[\frac{1-a^{n+1}}{1-a}\right] u(n) \\
s(\infty) &= \frac{b}{1-a} = 1 \\
\Rightarrow b &= 1-a.
\end{aligned}
$$

(c) $b = 1 - a$ in both cases.

2.49

(a)

$$
\begin{aligned}
y(n) &= 0.8y(n-1) + 2x(n) + 3x(n-1) \\
y(n) - 0.8y(n-1) &= 2x(n) + 3x(n-1)
\end{aligned}
$$

The characteristic equation is

$$
\begin{aligned}
\lambda - 0.8 &= 0 \\
\lambda &= 0.8. \\
y_h(n) &= c(0.8)^n
\end{aligned}
$$

Let us first consider the response of the sytem

$$y(n) - 0.8y(n-1) = x(n)$$

to $x(n) = \delta(n)$. Since $y(0) = 1$, it folows that $c = 1$. Then, the impulse response of the original system is

$$
\begin{aligned}
h(n) &= 2(0.8)^n u(n) + 3(0.8)^{n-1} u(n-1) \\
&= 2\delta(n) + 4.6(0.8)^{n-1} u(n-1)
\end{aligned}
$$

(b) The inverse system is characterized by the difference equation

$$x(n) = -1.5x(n-1) + \frac{1}{2}y(n) - 0.4y(n-1)$$

Refer to fig 2.49-1

2.51

(a)

$$
\begin{aligned}
y(n) &= \frac{1}{3}x(n) + \frac{1}{3}x(n-3) + y(n-1) \\
\text{for } x(n) &= \delta(n), \text{ we have} \\
h(n) &= \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \ldots\right\}
\end{aligned}
$$

(b)

$$
\begin{aligned}
y(n) &= \frac{1}{2}y(n-1) + \frac{1}{8}y(n-2) + \frac{1}{2}x(n-2) \\
\text{with } x(n) &= \delta(n), \text{ and} \\
y(-1) &= y(-2) = 0, \text{ we obtain} \\
h(n) &= \left\{0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{16}, \frac{1}{8}, \frac{11}{128}, \frac{15}{256}, \frac{41}{1024}, \ldots\right\}
\end{aligned}
$$

(c)

$$\begin{aligned}
y(n) &= 1.4y(n-1) - 0.48y(n-2) + x(n) \\
\text{with } x(n) &= \delta(n), \text{ and} \\
y(-1) &= y(-2) = 0, \text{ we obtain} \\
h(n) &= \{1, 1.4, 1.48, 1.4, 1.2496, 1.0774, 0.9086, \ldots\}
\end{aligned}$$

(d) All three systems are IIR.

(e)

$$\begin{aligned}
y(n) &= 1.4y(n-1) - 0.48y(n-2) + x(n) \\
\text{The characteristic equation is} & \\
\lambda^2 - 1.4\lambda + 0.48 &= 0 \text{ Hence} \\
\lambda &= 0.8, 0.6. \text{ and} \\
y_h(n) &= c_1(0.8)^n + c_2(0.6)^n \text{For } x(n) = \delta(n). \text{ We have,} \\
c_1 + c_2 &= 1 \text{ and} \\
0.8c_1 + 0.6c_2 &= 1.4 \\
\Rightarrow c_1 &= 4, \\
c_2 &= -3. \text{ Therefore} \\
h(n) &= [4(0.8)^n - 3(0.6)^n]\, u(n)
\end{aligned}$$

2.58

From problem 2.57,

$$h(n) = [c_1 2^n + c_2 n 2^n]\, u(n)$$

With $y(0) = 1, y(1) = 3$, we have

$$\begin{aligned}
c_1 &= 1 \\
2c_1 + 2c_2 &= 3
\end{aligned}$$

$$\begin{aligned}
\Rightarrow c_2 &= \frac{1}{2} \\
\text{Thus } h(n) &= \left[2^n + \frac{1}{2} n 2^n\right] u(n)
\end{aligned}$$