# Chapter 3

## 3.1

(a)

$$\begin{aligned} X(z) &= \sum_n x(n)z^{-n} \\ &= 3z^5 + 6 + z^{-1} - 4z^{-2} \text{ ROC: } 0 < |z| < \infty \end{aligned}$$

(b)

$$\begin{aligned} X(z) &= \sum_n x(n)z^{-n} \\ &= \sum_{n=5}^{\infty} (\frac{1}{2})^n z^{-n} \\ &= \sum_{n=5}^{\infty} (\frac{1}{2z})^n \\ &= \sum_{m=0}^{\infty} (\frac{1}{2}z^{-1})^{m+5} \\ &= (\frac{z^{-1}}{2})^5 \frac{1}{1-\frac{1}{2}z^{-1}} \\ &= (\frac{1}{32}) \frac{z^{-5}}{1-\frac{1}{2}z^{-1}} \text{ ROC: } |z| > \frac{1}{2} \end{aligned}$$

## 3.2

(a)

$$\begin{aligned} X(z) &= \sum_n x(n)z^{-n} \\ &= \sum_{n=0}^{\infty} (1+n)z^{-n} \\ &= \sum_{n=0}^{\infty} z^{-n} + \sum_{n=0}^{\infty} nz^{-n} \\ \text{But } \sum_{n=0}^{\infty} z^{-n} &= \frac{1}{1-z^{-1}} \text{ ROC: } |z| > 1 \end{aligned}$$

$$\text{and } \sum_{n=0}^{\infty} nz^{-n} = \frac{z^{-1}}{(1-z^{-1})^2} \text{ ROC: } |z| > 1$$

$$\text{Therefore, } X(z) = \frac{1-z^{-1}}{(1-z^{-1})^2} + \frac{z^{-1}}{(1-z^{-1})^2} = \frac{1}{(1-z^{-1})^2}$$

(b)

$$\begin{aligned}
X(z) &= \sum_{n=0}^{\infty}(a^n + a^{-n})z^{-n} \\
&= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=0}^{\infty} a^{-n} z^{-n} \\
\text{But } \sum_{n=0}^{\infty} a^n z^{-n} &= \frac{1}{1-az^{-1}} \text{ ROC: } |z| > |a| \\
\text{and } \sum_{n=0}^{\infty} a^{-n} z^{-n} &= \frac{1}{(1-\frac{1}{a}z^{-1})^2} \text{ ROC: } |z| > \frac{1}{|a|} \\
\text{Hence, } X(z) &= \frac{1}{1-az^{-1}} + \frac{1}{1-\frac{1}{a}z^{-1}} \\
&= \frac{2-(a+\frac{1}{a})z^{-1}}{(1-az^{-1})(1-\frac{1}{a}z^{-1})} \text{ ROC: } |z| > \max(|a|, \frac{1}{|a|})
\end{aligned}$$

(c)

$$\begin{aligned}
X(z) &= \sum_{n=0}^{\infty}(-\frac{1}{2})^n z^{-n} \\
&= \frac{1}{1+\frac{1}{2}z^{-1}}, |z| > \frac{1}{2}
\end{aligned}$$

(d)

$$\begin{aligned}
X(z) &= \sum_{n=0}^{\infty} na^n sinw_0 n z^{-n} \\
&= \sum_{n=0}^{\infty} na^n \left[\frac{e^{jw_0 n} - e^{-jw_0 n}}{2j}\right] z^{-n} \\
&= \frac{1}{2j}\left[\frac{ae^{jw_0}z^{-1}}{(1-ae^{jw_0}z^{-1})^2} - \frac{ae^{-jw_0}z^{-1}}{(1-ae^{-jw_0}z^{-1})^2}\right] \\
&= \frac{\left[az^{-1} - (az^{-1})^3\right] sinw_0}{(1-2acosw_0 z^{-1} + a^2 z^{-2})^2}, |z| > a
\end{aligned}$$

(e)

$$\begin{aligned}
X(z) &= \sum_{n=0}^{\infty} na^n cosw_0 n z^{-n} \\
&= \sum_{n=0}^{\infty} na^n \left[\frac{e^{jw_0 n} + e^{-jw_0 n}}{2}\right] z^{-n}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\frac{ae^{jw_0}z^{-1}}{(1-ae^{jw_0}z^{-1})^2} + \frac{ae^{-jw_0}z^{-1}}{(1-ae^{-jw_0}z^{-1})^2}\right] \\
&= \frac{\left[az^{-1}+(az^{-1})^3\right] sinw_0 - 2a^2z^{-2}}{(1-2acosw_0z^{-1}+a^2z^{-2})^2}, \qquad |z| > a
\end{aligned}
$$

(f)

$$
\begin{aligned}
X(z) &= A\sum_{n=0}^{\infty} r^n cos(w_0n+\phi)z^{-n} \\
&= A\sum_{n=0}^{\infty} r^n \left[\frac{e^{jw_0n}e^{j\phi}+e^{-jw_0n}e^{-j\phi}}{2}\right] z^{-n} \\
&= \frac{A}{2}\left[\frac{e^{j\phi}}{1-re^{jw_0}z^{-1}} + \frac{e^{-j\phi}}{1-re^{-jw_0}z^{-1}}\right] \\
&= A\left[\frac{cos\phi - rcos(w_0-\phi)z^{-1}}{1-2rcosw_0z^{-1}+r^2z^{-2}}\right], \qquad |z| > r
\end{aligned}
$$

(g)

$$
\begin{aligned}
X(z) &= \sum_{n=1}^{\infty}\frac{1}{2}(n^2+n)(\frac{1}{3})^{n-1}z^{-n} \\
\text{But} \sum_{n=1}^{\infty} n(\frac{1}{3})^{n-1}z^{-1} &= \frac{(\frac{1}{3})3z^{-1}}{(1-\frac{1}{3}z^{-1})^2} = \frac{z^{-1}}{(1-\frac{1}{3}z^{-1})^2} \\
\sum_{n=1}^{\infty} n^2(\frac{1}{3})^{n-1}z^{-n} &= \frac{z^{-1}+\frac{1}{3}z^{-2}}{(1-\frac{1}{3}z^{-1})^3} \\
\text{Therefore, } X(z) &= \frac{1}{2}\left[\frac{z^{-1}}{(1-\frac{1}{3}z^{-1})^2} + \frac{z^{-1}+\frac{1}{3}z^{-2}}{(1-\frac{1}{3}z^{-1})^3}\right] \\
&= \frac{z^{-1}}{(1-\frac{1}{3}z^{-1})^3}, \qquad |z| > \frac{1}{3}
\end{aligned}
$$

(h)

$$
\begin{aligned}
X(z) &= \sum_{n=0}^{\infty}(\frac{1}{2})^n z^{-n} - \sum_{n=10}^{\infty}(\frac{1}{2})^n z^{-n} \\
&= \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{(\frac{1}{2})^{10}z^{-10}}{1-\frac{1}{2}z^{-1}} \\
&= \frac{1-(\frac{1}{2}z^{-1})^{10}}{1-\frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}
\end{aligned}
$$

The pole-zero patterns are as follows:
(a) Double pole at $z = 1$ and a zero at $z = 0$.
(b) Poles at $z = a$ and $z = \frac{1}{a}$. Zeros at $z = 0$ and $z = \frac{1}{2}(a + \frac{1}{a})$.
(c) Pole at $z = -\frac{1}{2}$ and zero at $z = 0$.
(d) Double poles at $z = ae^{jw_0}$ and $z = ae^{-jw_0}$ and zeros at $z = 0$, $z = \pm a$.
(e) Double poles at $z = ae^{jw_0}$ and $z = ae^{-jw_0}$ and zeros are obtained by solving the quadratic

$$acosw_0z^2 - 2a^2z + a^3cosw_0 = 0.$$

(f) Poles at $z = re^{jw_0}$ and $z = ae^{-jw_0}$ and zeros at $z = 0$, and $z = rcos(w_0 - \phi)/cos\phi$.
(g) Triple pole at $z = \frac{1}{3}$ and zeros at $z = 0$ and $z = \frac{1}{3}$. Hence there is a pole-zero cancellation so

that in reality there is only a double pole at $z = \frac{1}{3}$ and a zero at $z = 0$.
(h) $X(z)$ has a pole of order 9 at $z = 0$. For nine zeros which we find from the roots of

$$\begin{aligned} 1 - (\frac{1}{2}z^{-1})^{10} &= 0 \\ \text{or, equivalently, } (\frac{1}{2})^{10} - z^{10} &= 0 \\ \text{Hence, } z_n &= \frac{1}{2}e^{\frac{j2\pi n}{10}}, n = 1, 2, \ldots, k. \end{aligned}$$

Note the pole-zero cancellation at $z = \frac{1}{2}$.

## 3.3

(a)

$$\begin{aligned} X_1(z) &= \sum_{n=0}^{\infty}(\frac{1}{3})^n z^{-n} + \sum_{n=-\infty}^{0}(\frac{1}{2})^n z^{-n} - 1 \\ &= \frac{1}{1 - \frac{1}{3}z^{-1}} + \sum_{n=0}^{\infty}(\frac{1}{2})^n z^n - 1 \\ &= \frac{1}{1 - \frac{1}{3}z^{-1}} + \frac{1}{1 - \frac{1}{2}z} - 1, \\ &= \frac{\frac{5}{6}}{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{2}z)} \end{aligned}$$

The ROC is $\frac{1}{3} < |z| < 2$.
(b)

$$\begin{aligned} X_2(z) &= \sum_{n=0}^{\infty}(\frac{1}{3})^n z^{-n} - \sum_{n=0}^{\infty} 2^n z^{-n} \\ &= \frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{1}{1 - 2z^{-1}}, \\ &= \frac{-\frac{5}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})} \end{aligned}$$

The ROC is $|z| > 2$.
(c)

$$\begin{aligned} X_3(z) &= \sum_{n=-\infty}^{\infty} x_1(n+4)z^{-n} \\ &= z^4 X_1(z) \\ &= \frac{\frac{5}{6}z^4}{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{2}z)} \end{aligned}$$

The ROC is $\frac{1}{3} < |z| < 2$.
(d)

$$X_4(z) = \sum_{n=-\infty}^{\infty} x_1(-n)z^{-n}$$

$$
\begin{aligned}
&= \sum_{m=-\infty}^{\infty} x_1(m)z^{m} \\
&= X_1(z^{-1}) \\
&= \frac{\frac{5}{6}}{(1-\frac{1}{3}z)(1-\frac{1}{2}z^{-1})}
\end{aligned}
$$

The ROC is $\frac{1}{2} < |z| < 3$.

## 3.6

$$
\begin{aligned}
y(n) &= \sum_{k=-\infty}^{n} x(k) \\
\Rightarrow y(n) - y(n-1) &= x(n) \\
\text{Hence,} Y(z) - Y(z)z^{-1} &= X(z) \\
Y(z) &= \frac{X(z)}{1-z^{-1}}
\end{aligned}
$$

## 3.10

$$
\begin{aligned}
x(n) &= \frac{1}{2}\left[u(n) + (-1)^n u(n)\right] \\
X^+(z) &= \frac{\left(\frac{1}{1-z^{-1}} + \frac{1}{1+z^{-1}}\right)}{2} \\
\text{From the final value theorem} & \\
x(\infty) &= \lim_{z\to 1}(z-1)X^+(z) \\
&= \lim_{z\to 1}\left(z + \frac{z(z-1)}{z+1}\right) \\
&= \frac{1}{2}
\end{aligned}
$$

## 3.14

(a)

$$
\begin{aligned}
X(z) &= \frac{1-3z^{-1}}{1+3z^{-1}+2z^{-2}} \\
&= \frac{A}{(1+z^{-1})} + \frac{B}{(1+2z^{-1})} \\
A &= 2, B = -1 \\
\text{Hence,} x(n) &= \left[2(-1)^n - (-2)^n\right]u(n)
\end{aligned}
$$

(b)

$$
\begin{aligned}
X(z) &= \frac{1}{1-z^{-1}+\frac{1}{2}z^{-2}} \\
&= \frac{A(1-\frac{1}{2}z^{-1}) + B(\frac{1}{2}z^{-1})}{1-z^{-1}+\frac{1}{2}z^{-2}}
\end{aligned}
$$

$$
\begin{aligned}
A &= 1, B = 1 \\
\text{Hence}, X(z) &= \frac{1-\frac{1}{\sqrt{2}}(cos\frac{\pi}{4})z^{-1}}{1-2\frac{1}{\sqrt{2}}(cos\frac{\pi}{4})z^{-1}+(\frac{1}{\sqrt{2}})^2z^{-2}} \\
&+\frac{\frac{1}{\sqrt{2}}(sin\frac{\pi}{4})z^{-1}}{1-2\frac{1}{\sqrt{2}}(cos\frac{\pi}{4})z^{-1}+(\frac{1}{\sqrt{2}})^2z^{-2}} \\
\text{Hence}, x(n) &= \left[(\frac{1}{\sqrt{2}})^n cos\frac{\pi}{4}n + (\frac{1}{\sqrt{2}})^n sin\frac{\pi}{4}n\right] u(n)
\end{aligned}
$$

(c)

$$
\begin{aligned}
X(z) &= \frac{z^{-6}}{1-z^{-1}} + \frac{z^{-7}}{1-z^{-1}} \\
x(n) &= u(n-6) + u(n-7)
\end{aligned}
$$

(d)

$$
\begin{aligned}
X(z) &= \frac{1}{1+z^{-2}} + 2\frac{z^{-2}}{1+z^{-2}} \\
X(z) &= 2 - \frac{1}{1+z^{-2}} \\
x(n) &= cos\frac{\pi}{2}nu(n) + 2cos\frac{\pi}{2}(n-2)u(n-2) \\
x(n) &= 2\delta(n) - cos\frac{\pi}{2}nu(n)
\end{aligned}
$$

(e)

$$
\begin{aligned}
X(z) &= \frac{1}{4}\frac{1+6z^{-1}+z^{-2}}{(1-2z^{-1}+2z^{-2})(1-\frac{1}{2}z^{-1})} \\
&= \frac{A(1-z^{-1})}{1-2z^{-1}+2z^{-2}} + \frac{Bz^{-1}}{1-2z^{-1}+2z^{-2}} + \frac{C}{1-\frac{1}{2}z^{-1}} \\
A &= -\frac{3}{5}, B = \frac{23}{10}, C = \frac{17}{20} \\
\text{Hence}, x(n) &= \left[-\frac{3}{5}(\frac{1}{\sqrt{2}})^n cos\frac{\pi}{4}n + \frac{23}{10}(\frac{1}{\sqrt{2}})^n sin\frac{\pi}{4}n + \frac{17}{20}(\frac{1}{2})^n\right] u(n)
\end{aligned}
$$

(f)

$$
\begin{aligned}
X(z) &= \frac{2-1.5z^{-1}}{1-1.5z^{-1}+0.5z^{-2}} \\
&= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-z^{-1}} \\
x(n) &= \left[(\frac{1}{2})^n + 1\right] u(n)
\end{aligned}
$$

(g)

$$
\begin{aligned}
X(z) &= \frac{1+2z^{-1}+z^{-2}}{1+4z^{-1}+4z^{-2}} \\
&= 1 - \left(\frac{2z^{-1}+3z^{-2}}{(1+2z^{-1})(1+2z^{-1})}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 1 - \frac{2z^{-1}}{1+2z^{-1}} + \frac{z^{-2}}{(1+2z^{-1})^2} \\
x(n) &= \delta(n) - 2(-2)^{n-1}u(n-1) + (n-1)(-2)^{n-1}u(n-1) \\
&= \delta(n) + (n-3)(-2)^{n-1}u(n-1)
\end{aligned}
$$

(h)

$$
\begin{aligned}
X(z) &= \frac{1}{4}\frac{(z+\frac{1}{2})(z+\frac{1}{4})}{(z-\frac{1}{2})(z-\frac{1}{\sqrt{2}e^{j\frac{\pi}{4}}})(z-\frac{1}{\sqrt{2}e^{-j\frac{\pi}{4}}})} \\
&= \frac{1}{4}\frac{(1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2})z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1}+\frac{1}{2}z^{-2})} \\
&= \frac{A(1-\frac{1}{2}z^{-1})z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}} + \frac{A(\frac{1}{2}z^{-1})z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}} + \frac{Cz^{-1}}{1-\frac{1}{2}z^{-1}} \\
A &= -\frac{1}{2}, B = \frac{7}{8}, C = \frac{3}{4} \\
\text{Hence,}x(n) &= \left[-\frac{1}{2}(\frac{1}{2})^{\frac{n-1}{2}} cos\frac{\pi}{4}(n-1) + \frac{7}{8}(\frac{1}{2})^{\frac{n-1}{2}} sin\frac{\pi}{4}(n-1) + \frac{3}{4}(\frac{1}{2})^{n-1}\right]u(n-1)
\end{aligned}
$$

(i)

$$
\begin{aligned}
X(z) &= \frac{1-\frac{1}{4}z^{-1}}{1+\frac{1}{2}z^{-1}} \\
&= \frac{1}{1+\frac{1}{2}z^{-1}} - \frac{1}{4}\frac{z^{-1}}{1+\frac{1}{2}z^{-1}} \\
x(n) &= (-\frac{1}{2})^n u(n) + \frac{1}{4}(-\frac{1}{2})^{n-1}u(n-1)
\end{aligned}
$$

(j)

$$
\begin{aligned}
X(z) &= \frac{1-az^{-1}}{z^{-1}-a} \\
&= -\frac{1}{a}\left(\frac{1-az^{-1}}{1-\frac{1}{a}z^{-1}}\right) \\
&= -\frac{1}{a}\left[\frac{1}{1-\frac{1}{a}z^{-1}} - \frac{az^{-1}}{1-\frac{1}{a}z^{-1}}\right] \\
x(n) &= -\frac{1}{a}(\frac{1}{a})^n u(n) + (\frac{1}{a})^{n-1}u(n-1) \\
&= (-\frac{1}{a})^{n+1}u(n) + (\frac{1}{a})^{n-1}u(n-1)
\end{aligned}
$$

## 3.16

(a)

$$
\begin{aligned}
x_1(n) &= \frac{1}{4}(\frac{1}{4})^{n-1}u(n-1) \\
\Rightarrow X_1(z) &= \frac{(\frac{1}{4})z^{-1}}{1-\frac{1}{4}z^{-1}}, |z| > \frac{1}{4} \\
x_2(n) &= \left[1 + (\frac{1}{2})^n\right]u(n) \\
\Rightarrow X_2(z) &= \frac{1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, |z| > 1 \\
Y(z) &= X_1(z)X_2(z) \\
&= \frac{-\frac{4}{3}}{1-\frac{1}{4}z^{-1}} + \frac{\frac{1}{3}}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} \\
y(n) &= \left[-\frac{4}{3}(\frac{1}{4})^n + \frac{1}{3} + (\frac{1}{2})^n\right]u(n)
\end{aligned}
$$

## 3.19

(a)

$$\begin{aligned}
X(z) &= log(1-2z), |z| < \frac{1}{2} \\
Y(z) &= -z\frac{dX(z)}{dz} \\
&= \frac{-1}{1-\frac{1}{2}z^{-1}}, |z| < \frac{1}{2} \\
\Rightarrow y(n) &= (\frac{1}{2})^n, n < 0 \\
\text{Then}, x(n) &= \frac{1}{n}y(n) \\
&= \frac{1}{n}(\frac{1}{2})^n u(-n-1)
\end{aligned}$$

(b)

$$\begin{aligned}
X(z) &= log(1-\frac{1}{2}z^{-1}), |z| > \frac{1}{2} \\
Y(z) &= -z\frac{dX(z)}{dz}
\end{aligned}$$

$$\begin{aligned}
&= \frac{-\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}, |z| > \frac{1}{2} \\
\text{Hence}, y(n) &= -\frac{1}{2}(\frac{1}{2})^{n-1}u(n-1) \\
x(n) &= \frac{1}{n}y(n) \\
&= -\frac{1}{n}(\frac{1}{2})^n u(n-1)
\end{aligned}$$

## 3.25

(a)

$$\begin{aligned}
X(z) &= \frac{1}{1-1.5z^{-1}+0.5z^{-2}} \\
&= \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}} \\
\text{For } |z| < 0.5, x(n) &= [(0.5)^n - 2]u(-n-1)
\end{aligned}$$

$$\begin{aligned}
\text{For } |z| > 1, x(n) &= [2-(0.5)^n]u(n) \\
\text{For } 0.5 < |z| < 1, x(n) &= -(0.5)^n u(n) - 2u(-n-1)
\end{aligned}$$

(b)

$$\begin{aligned}
X(z) &= \frac{1}{(1-0.5z^{-1})^2} \\
&= \left[\frac{0.5z^{-1}}{(1-0.5z^{-1})^2}\right] 2z \\
\text{For } |z| > 0.5, x(n) &= 2(n+1)(0.5)^{n+1}u(n+1) \\
&= (n+1)(0.5)^n u(n) \\
\text{For } |z| < 0.5, x(n) &= -2(n+1)(0.5)^{n+1}u(-n-2) \\
&= -(n+1)(0.5)^n u(-n-1)
\end{aligned}$$

## 3.34

$$\begin{aligned}
H(z) &= \sum_{n=-1}^{-\infty} 3^n z^{-n} + \sum_{n=0}^{\infty} (\frac{2}{5})^n z^{-n} \\
&= \frac{-1}{1-3z^{-1}} + \frac{1}{1-\frac{2}{5}z^{-1}}, \text{ ROC: } \frac{2}{5} < |z| < 3 \\
X(z) &= \frac{1}{1-z^{-1}} \\
Y(z) &= H(z)X(z) \\
&= \frac{-\frac{13}{5}z^{-1}}{(1-z^{-1})(1-3z^{-1})(1-\frac{2}{5}z^{-1})}, \text{ ROC: } 1 < |z| < 2 \\
&= \frac{\frac{13}{6}}{1-z^{-1}} - \frac{\frac{3}{2}}{1-3z^{-1}} - \frac{\frac{2}{3}}{1-\frac{2}{5}z^{-1}}
\end{aligned}$$

Therefore,

$$y(n) = \frac{3}{2}3^n u(-n-1) + \left[\frac{13}{6} - \frac{2}{3}(\frac{2}{5})^n\right] u(n)$$

## 3.36

$$\begin{aligned}
H(z) &= \frac{1-2z^{-1}+2z^{-2}-z^{-3}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})(1-\frac{1}{5}z^{-1})}, \quad \frac{1}{2} < |z| < 1 \\
&= \frac{1-z^{-1}+z^{-2}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{5}z^{-1})}, \quad \frac{1}{2} < |z| < 1 \\
\text{(a) } Z_{1,2} &= \frac{1 \pm j\sqrt{3}}{2}, \ p_1 = \frac{1}{2}, \ p_2 = \frac{1}{5} \\
\text{(b) } H(z) &= 1 + \left[\frac{\frac{5}{2}}{1-\frac{1}{2}z^{-1}} + \frac{-2.8}{1-\frac{1}{5}z^{-1}}\right] z^{-1} \\
h(n) &= \delta(n) + \left[5(\frac{1}{2})^n - 14(\frac{1}{5})^n\right] u(n)
\end{aligned}$$

## 3.37

$$\begin{aligned}
y(n) &= 0.7y(n-1) - 0.12y(n-2) + x(n-1) + x(n-2) \\
Y(z) &= \frac{z^{-1}+z^{-2}}{1-0.7z^{-1}+0.12z^{-2}} X(z) \\
x(n) &= nu(n) \\
X(z) &= \frac{z^{-1}}{(1-z^{-1})^2} \\
Y(z) &= \frac{z^{-2}+z^{-3}}{(1-z^{-1})^2(1-\frac{3}{10}z^{-1})(1-\frac{2}{50}z^{-2})}
\end{aligned}$$

$\Rightarrow$ System is stable

$$Y(z) = \frac{4.76z^{-1}}{(1-z^{-1})^2} + \frac{-12.36}{(1-z^{-1})} + \frac{-26.5}{(1-\frac{3}{10}z^{-1})} + \frac{38.9}{(1-\frac{2}{5}z^{-1})}$$

$$y(n) = \left[4.76n - 12.36 - 26.5(\frac{3}{10})^n + 38.9(\frac{2}{5})^n\right] u(n)$$

## 3.38

(a)

$$\begin{aligned}
y(n) &= \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n) \\
Y(z) &= \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} X(z) \\
\text{Impulse Response: } X(z) &= 1 \\
Y(z) &= \frac{2}{1 - \frac{1}{2}z^{-1}} - \frac{1}{1 - \frac{1}{4}z^{-1}} \\
\Rightarrow h(n) &= \left[2(\frac{1}{2})^n - (\frac{1}{4})^n\right] u(n)
\end{aligned}$$

Since the poles of $H(z)$ are inside the unit circle, the system is stable (poles at $z = \frac{1}{2}, \frac{1}{4}$).

$$\begin{aligned}
\text{Step Response: } X(z) &= \frac{1}{1 - z^{-1}} \\
Y(z) &= \frac{\frac{8}{3}}{1 - z^{-1}} - \frac{2}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1 - \frac{1}{4}z^{-1}} \\
y(n) &= \left[\frac{8}{3} - 2(\frac{1}{2})^n + \frac{1}{3}(\frac{1}{4})^n\right] u(n)
\end{aligned}$$

(b)

$$\begin{aligned}
y(n) &= y(n-1) - \frac{1}{2}y(n-2) + x(n) + x(n-1) \\
Y(z) &= \frac{1 + z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} X(z)
\end{aligned}$$

$H(z)$ has zeros at $z = 0, 1$, and poles at $z = \frac{1 \pm j}{2}$. Hence, the system is stable.

$$\begin{aligned}
\text{Impulse Response: } X(z) &= 1 \\
Y(z) &= \frac{1 - (\sqrt{2})^{-1} cos\frac{\pi}{4} z^{-1}}{1 - 2(\sqrt{2})^{-1} cos\frac{\pi}{4} z^{-1} + (\frac{1}{\sqrt{2}})^2 z^{-2}} + \frac{\frac{3}{2}z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} \\
\Rightarrow y(n) = h(n) &= (\frac{1}{\sqrt{2}})^n \left[cos\frac{\pi}{4}n + sin\frac{\pi}{4}n\right] u(n) \\
\text{Step Response: } X(z) &= \frac{1}{1 - z^{-1}} \\
Y(z) &= \frac{1 + z^{-1}}{(1 - z^{-1})(1 - z^{-1} + \frac{1}{2}z^{-2}} \\
&= \frac{-(1 - \frac{1}{2}z^{-1})}{1 - z^{-1} + \frac{1}{2}z^{-2}} + \frac{\frac{1}{2}z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} + \frac{2}{1 - z^{-1}} \\
y(n) &= (\frac{1}{\sqrt{2}})^n \left[sin\frac{\pi}{4}n - cos\frac{\pi}{4}n\right] u(n) + 2u(n)
\end{aligned}$$

(c)

$$H(z) = \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$$

$$\Rightarrow h(n) = n^2 u(n)$$

Triple pole on the unit circle $\Rightarrow$ the system is unstable.

$$\begin{aligned}
\text{Step Response: } X(z) &= \frac{1}{1-z^{-1}} \\
Y(z) &= \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^4} \\
&= \frac{1}{3}\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4} + \frac{1}{2}\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} + \frac{1}{6}\frac{z^{-1}}{(1-z^{-1})^2} \\
y(n) &= (\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n)u(n) \\
&= \frac{1}{6}n(n+1)(2n+1)u(n)
\end{aligned}$$

(d)

$$\begin{aligned}
y(n) &= 0.6y(n-1) - 0.08y(n-2) + x(n) \\
Y(z) &= \frac{1}{1-0.6z^{-1}+0.08z^{-2}}X(z) \\
\text{Impulse Response: } X(z) &= 1 \\
H(z) &= \frac{1}{(1-\frac{1}{5}z^{-1})(1-\frac{2}{5}z^{-1})}
\end{aligned}$$

$\Rightarrow$ zeros at $z = 0$, poles at $p_1 = \frac{1}{2}, p_2 = \frac{2}{5}$ system is stable.

$$\begin{aligned}
H(z) &= \frac{-1}{1-\frac{1}{5}z^{-1}} + \frac{2}{1-\frac{2}{5}z^{-1}} \\
\Rightarrow h(n) &= \left[2(\frac{2}{5})^n - (\frac{1}{5})^n\right] u(n) \\
\text{Step Response: } X(z) &= \frac{1}{1-z^{-1}} \\
Y(z) &= \frac{1}{(1-\frac{1}{5}z^{-1})(1-\frac{2}{5}z^{-1})(1-z^{-1})} \\
Y(z) &= \frac{\frac{25}{12}}{1-z^{-1}} + \frac{\frac{1}{4}}{1-\frac{1}{5}z^{-1}} + \frac{-\frac{4}{3}}{1-\frac{2}{5}z^{-1}} \\
y(n) &= \left[\frac{25}{12} + \frac{1}{4}(\frac{1}{5})^n - \frac{4}{3}(\frac{2}{5})^n\right] u(n)
\end{aligned}$$

(e)

$$\begin{aligned}
y(n) &= 0.7y(n-1) - 0.1y(n-2) + 2x(n) - x(n-2) \\
Y(z) &= \frac{2-z^{-2}}{1-0.7z^{-1}+0.1z^{-2}}X(z) \\
&= \frac{2-z^{-2}}{(1-\frac{1}{5}z^{-1})(1-\frac{1}{2}z^{-1})}X(z)
\end{aligned}$$

zeros at $z = 0, 2$, and poles at $z = \frac{1}{2}, \frac{1}{5}$. Hence, the system is stable.

$$\begin{aligned}
\text{Impulse Response: } X(z) &= 1 \\
H(z) &= 2 + \left(\frac{\frac{-5}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{46}{15}}{1-\frac{1}{5}z^{-1}}\right) z^{-1}
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow h(n) &= 2\delta(n) - \frac{5}{3}(\frac{1}{2})^{n-1}u(n-1) + \frac{46}{15}(\frac{1}{5})^{n-1}u(n-1) \\
\text{Step Response: } X(z) &= \frac{1}{1-z^{-1}} \\
Y(z) &= \frac{2-z^{-2}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})(1-\frac{1}{5}z^{-1})} \\
&= \frac{\frac{5}{2}}{1-z^{-1}} + \frac{\frac{10}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{-23}{6}}{1-\frac{1}{5}z^{-1}} \\
y(n) &= \left[\frac{5}{2} + \frac{10}{3}(\frac{1}{2})^n - \frac{23}{6}(\frac{1}{5})^n\right] u(n)
\end{aligned}
$$

## 3.43

$$
\begin{aligned}
[aY(z) + X(z)]\,z^{-2} &= Y(z) \\
Y(z) &= \frac{z^{-2}}{1-az^{-2}}X(z)
\end{aligned}
$$

Assume that $a > 0$. Then

$$
\begin{aligned}
H(z) &= -\frac{1}{a} + \frac{\frac{1}{a}}{(1-\sqrt{a}z^{-1})(1+\sqrt{a}z^{-1})} \\
&= -\frac{1}{a} + \frac{1}{2a}\frac{1}{1-\sqrt{a}z^{-1}} + \frac{1}{2a}\frac{1}{1+\sqrt{a}z^{-1}} \\
h(n) &= -\frac{1}{a}\delta(n) + \frac{1}{2a}\left[(\sqrt{a})^n + (-\sqrt{a})^n\right] u(n)
\end{aligned}
$$