

ELEC 6831 / ELEC 462

Digital Transmission Systems

Summer 2011

X Lecture 1: May 2011

- Course outline

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- Review of Probability and random processes.

Relative Frequency versus the axiomatic definition of probability.

• Sample Space  $S = \{A_1, A_2, \dots, \dots\}$ :

a set of events

•  $P[A]$ : A mapping from <sup>subsets of</sup>  $S$  to  $[0, 1]$  such that:

i)  $0 \leq P[A] \leq 1 \quad \forall A \in S$

ii)  $P[S] = 1$

iii)  $P[A \cup B] = P[A] + P[B]$

if  $A \cap B = \emptyset$

If  $A \cap B \neq \emptyset$  then

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

Conditional Probability

$$P[B|A] = \frac{P[A \cap B]}{P[A]}$$

or

$$P[A \cap B] = P[A]P[B|A]$$

similarly

$$P[A \cap B] = P[B]P[A|B]$$

So,

$$P[A]P[B|A] = P[B]P[A|B]$$

or

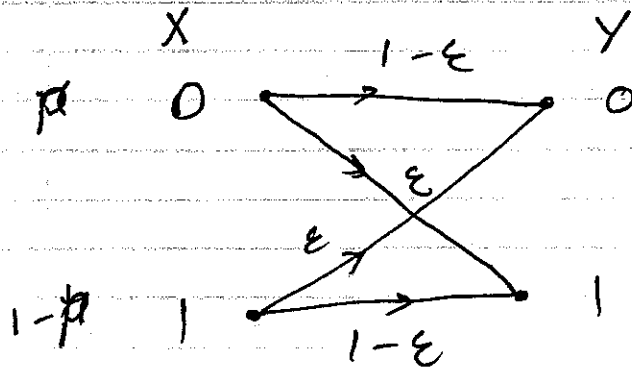
$$P[A|B] = \frac{P[A]P[B|A]}{P[B]}$$

when  $P[A|B] = P[A]$ , we have:

$$P[A \cap B] = P[A]P[B]$$

and say A and B are independent.

Binary Symmetric Channel (BSC):



$$P[Y=0] = p(1-\epsilon) + (1-p)\epsilon$$

$$P[Y=1] = p\epsilon + (1-p)(1-\epsilon)$$

So,

$$P[X=0|Y=0] = \frac{P[X=0]P[Y=0|X=0]}{P[Y=0]}$$

$$= \frac{p(1-\epsilon)}{p(1-\epsilon) + (1-p)\epsilon}$$

$$P[X=1|Y=1] = \frac{P[X=1]P[Y=1|X=1]}{P[Y=1]}$$

$$= \frac{(1-p)(1-\epsilon)}{p\epsilon + (1-p)(1-\epsilon)}$$

## Random Variables

A random variable  $X$  or  $X(s)$  is a function from  $S$  to  $\mathbb{R}$ , i.e., it maps each  $s_i \in S$  to a real value  $X(s_k) = x$ .

## Cumulative Distribution Function (CDF)

$$F_X(x) = P[X \leq x]$$

it is clear that

I)  $0 \leq F_X(x) \leq 1$  since it is a probability

II)  $F_X(-\infty) = 0$

$$F_X(\infty) = 1$$

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III)  $P[x_1 \leq X \leq x_2] = F_X(x_2) - F_X(x_1)$

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## Probability Density Function (pdf)

$$f_X(x) = \frac{d}{dx} F_X(x)$$

So,

$$F_X(x) = \int_{-\infty}^x f_X(x') dx'$$

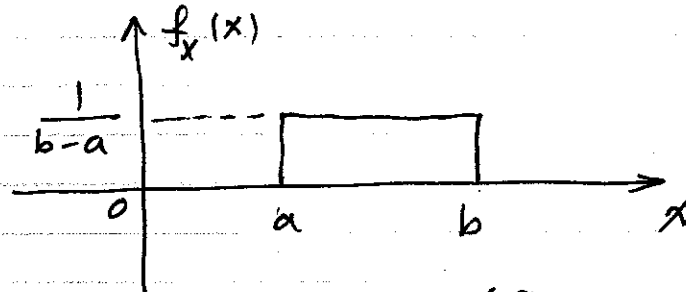
therefore,

$$P[X \leq x_i] = \int_{-\infty}^{x_i} f_X(x) dx$$

and,

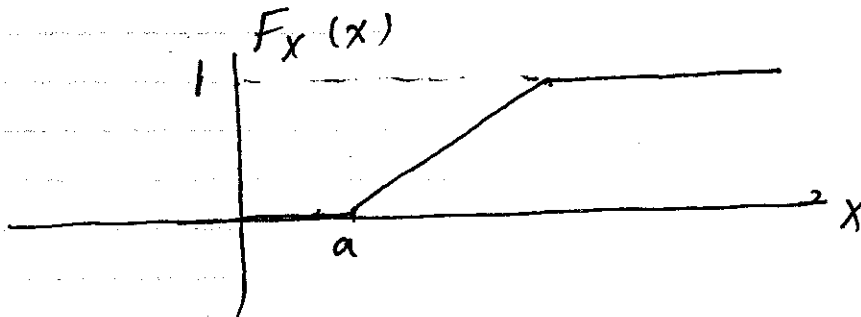
$$\begin{aligned} P[x_1 \leq X \leq x_2] &= F_X(x_2) - F_X(x_1) \\ &= \int_{x_1}^{x_2} f_X(x) dx \end{aligned}$$

Uniform Distribution:



$$f_X(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & a < x \leq b \\ 0 & x > b \end{cases}$$

So,



$$F_X(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & x > b \end{cases}$$

Note that

$$\int_{-\infty}^{\infty} f_X(x) dx = 1$$

Multiple random variables:

$$F_{X,Y}(x,y) = P[X \leq x, Y \leq y]$$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(\xi, \eta) d\xi d\eta = 1$$

and

$$F_X(x) = \int_{-\infty}^{\infty} \int_{-\infty}^x f_{X,Y}(\xi, \eta) d\xi d\eta$$

and

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, \eta) d\eta$$

Conditional probability density function

$$f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

Mean: (average)

$$\mu_x = E[X] = \int_{-\infty}^{\infty} x f_x(x) dx$$

Similarly if

$Y = g(X)$  is a function of  $X$

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f_x(x) dx$$

Example

$$Y = \cos(X)$$

where

$$f_x(x) = \begin{cases} \frac{1}{2\pi} & -\pi < x < \pi \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y] = \int_{-\pi}^{\pi} \cos(x) \frac{1}{2\pi} dx = \frac{1}{2\pi} \sin(x) \Big|_{-\pi}^{\pi} = 0$$

Moments

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_x(x) dx$$

is called the  $n$ -th moment of the r.v.  $X$ .

For  $n=1$ , we get the mean.

$$\text{For } n=2, \text{ we have } E[X^2] = \int_{-\infty}^{\infty} x^2 f_x(x) dx$$

## Central moments

$$E[(X - \mu_x)^n] = \int_{-\infty}^{\infty} (x - \mu_x)^n f_x(x) dx$$

for  $n=2$ , we have

$$E[(X - \mu_x)^2] = \int_{-\infty}^{\infty} (x - \mu_x)^2 f_x(x) dx$$

this is called the variance.

$$\text{Var}(X) = \sigma_x^2 = E[(X - \mu_x)^2]$$

$\sigma_x$  (the square-root of the variance) is called the standard deviation.

For  $n=1$  the central moment is equal to zero, i.e., subtracting the mean results in zero-mean (as expected!!).

Note that:

$$\begin{aligned} \sigma_x^2 = E[(X - \mu_x)^2] &= E[X^2] - E[2\mu_x X] \\ &\quad + E[\mu_x^2] \end{aligned}$$

$$= E[X^2] - 2\mu_x E[X] + \mu_x^2$$

$$= E[X^2] - \mu_x^2$$



Chebyshev's inequality

$$P[|X - \mu_x| \geq \epsilon] < \frac{\sigma_x^2}{\epsilon^2}$$

is an indication of the importance of mean and variance in limiting the region where we expect to observe the result of an experiment.

Covariance

$$\text{Cov}[X, Y] = E[(X - E(X))(Y - E(Y))]$$

$$= E[XY] - \mu_x \mu_y$$

The correlation coefficient

$$\rho = \frac{\text{Cov}[X, Y]}{\sigma_x \sigma_y}$$

if  $\text{Cov}[X, Y] = 0$  we say  $X$  and  $Y$  are uncorrelated.

if  $E[XY] = 0$  we say  $X$  and  $Y$  are orthogonal.

## Random Processes

A random process is an assignment to a function (a time function or signal) to each outcome of a sample space.

$$X(t, s) \quad -T \leq t \leq T$$

Mean

$$\mu_x(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x) dx$$

if  $f_{X(t_1)}(x) = f_{X(t_2)}(x)$

then

$$\mu_x(t) = \mu_x \quad \text{for all } t$$

and we say that  $X(t)$  is first order stationary.

A second order stationary random process has all  $f_{X(t_1), X(t_2)}(x_1, x_2)$  <sup>that</sup> only depend on  $t_2 - t_1$ .

A random process is stationary (in the strict sense) if all its statistics are time independent (shift independent).

Weak sense (or wide sense) stationary  
(WSS)

$$\text{if } \mu_X(t) = \mu_X \quad \forall t$$

and

$$R_X(t_1, t_2) = \iint_{-\infty}^{\infty} x_1 x_2 f_{X(t_1), X(t_2)}(x_1, x_2) dx_1 dx_2 = R_X(t_2 - t_1)$$

then  $X(t)$  is called WSS.

Properties of the autocorrelation function

For a stationary process:

$$R_X(\tau) = E[X(t+\tau)X(t)] \quad \forall t$$

1)

$$R_X(0) = E[X^2(t)]$$

2)

$$R_X(\tau) = R_X(-\tau)$$

3)

$$|R_X(\tau)| \leq R_X(0)$$

Proof:

$$E[\{X(t+\tau) \mp X(t)\}^2] \geq 0$$

$$E[X^2(t+\tau)] \mp 2E[X(t+\tau)X(t)] + E[X^2(t)] \geq 0$$

$$2R_x(0) \pm 2R_x(\tau) \geq 0$$

$$-R_x(0) \leq R_x(\tau) \leq R_x(0)$$

Example:

$$X(t) = A \cos(2\pi f_c t + \theta)$$

where phase  $\theta$  is uniformly distributed in  $[-\pi, \pi]$

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta \leq \pi \\ 0 & \text{elsewhere} \end{cases}$$

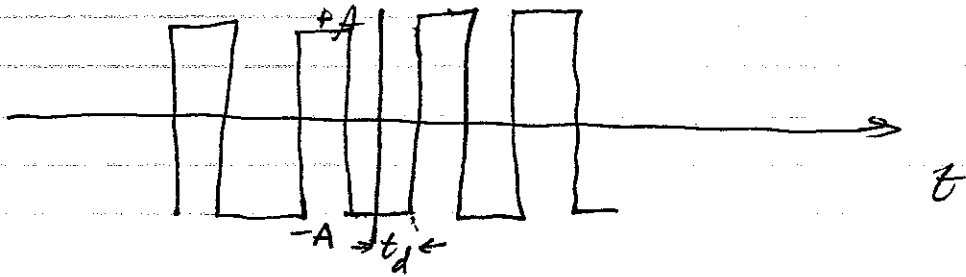
$$R_x(\tau) = E[X(t+\tau)X(t)] = E[A^2 \cos(2\pi f_c t + 2\pi f_c \tau + \theta) \cos(2\pi f_c t + \theta)]$$

$$\begin{aligned} R_x(\tau) &= \frac{A^2}{2} E[\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)] + \frac{A^2}{2} E[\cos(2\pi f_c \tau)] \\ &= \frac{A^2}{2} \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos(2\pi f_c t + 2\pi f_c \tau + 2\theta) d\theta + \frac{A^2}{2} \cos(2\pi f_c \tau) \end{aligned}$$

or

$$R_x(\tau) = \frac{A^2}{2} \cos(2\pi f_c \tau)$$

Ex: Random binary signal:



$$f_{T_d}(t_d) = \begin{cases} \frac{1}{T} & 0 \leq t_d \leq T \\ 0 & \text{elsewhere} \end{cases}$$

a) if  $|t_k - t_i| > T$  then

$$E[X(t_k)X(t_i)] = E[X(t_k)]E[X(t_i)] = 0$$

b) if  $|t_k - t_i| < T$ , then

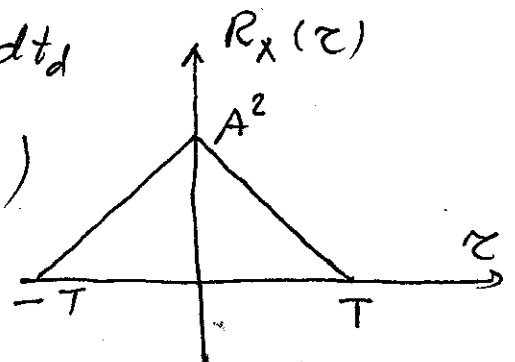
if  $|t_k - t_i| < T - t_d$  then either we have  $(-A)(-A)$  or  $(+A)(+A)$ , i.e., there is no new level. If, on the other hand  $|t_k - t_i| > T - t_d \Rightarrow$  the two levels are independent and the average is zero.

$$E[X(t_k)X(t_i)|t_d] = \begin{cases} A^2 & t_d < T - |t_k - t_i| \\ 0 & \text{otherwise.} \end{cases}$$

or

$$E[X(t_k)X(t_i)] = \int_0^{T - |t_k - t_i|} A^2 f_{T_d}(t_d) dt_d$$

$$= A^2 \left(1 - \frac{|t_k - t_i|}{T}\right)$$



## Cross-Correlation

$$R_{xy}(t, u) = E[X(t)Y(u)]$$

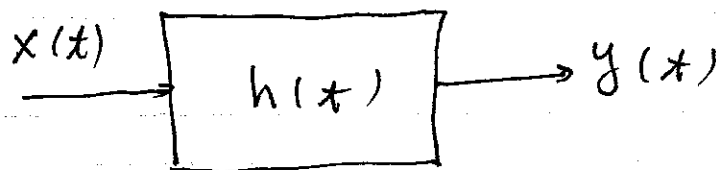
if  $X$  and  $Y$  are WSS then

$$R_{xy}(t, u) = R_{xy}(\tau)$$

where  $\tau = t - u$ .

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Transmission of a Random Process  
through a Linear filter:



$$y(t) = \int_{-\infty}^{\infty} h(\tau_1) x(t - \tau_1) d\tau_1$$

$$\begin{aligned} \mu_y(t) &= E[y(t)] = E\left[\int_{-\infty}^{\infty} h(\tau_1) x(t - \tau_1) d\tau_1\right] \\ &= \int_{-\infty}^{\infty} h(\tau_1) \mu_x(t - \tau_1) d\tau_1 \end{aligned}$$

for wide sense stationary Process  $X(t)$

$$\mu_x(t) = \mu_x$$

So

$$\mu_y(t) = \mu_x \int_{-\infty}^{\infty} h(\tau_1) d\tau_1 = \mu_x H(0) = \mu_y$$

$$R_Y(t, u) = E[Y(t)Y(u)]$$

$$R_Y(t, u) = E \left[ \int_{-\infty}^{\infty} h(\tau_1) X(t - \tau_1) d\tau_1 \int_{-\infty}^{\infty} h(\tau_2) X(u - \tau_2) d\tau_2 \right]$$

$$= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} h(\tau_2) E[X(t - \tau_1)X(u - \tau_2)] d\tau_2 \right] h(\tau_1) d\tau_1$$

for WSS Processes:

$$R_Y(t, u) = R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(\tau - \tau_1 - \tau_2) d\tau_1 d\tau_2$$

Power Spectral Density (PSD):

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} d\tau$$

and

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f \tau} df$$

Properties of PSD:

$$1) S_X(0) = \int_{-\infty}^{\infty} R_X(\tau) d\tau$$

$$2) E[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f) df$$

$$3) S_X(f) \geq 0 \quad \forall f$$

$$4) S_X(-f) = S_X(f)$$

Example:

$X(t) = A \cos(2\pi f_c t + \theta)$  where  $\theta$  is uniformly distributed over  $[-\pi, \pi]$

$$R_x(\tau) = \frac{A^2}{2} \cos(2\pi f_c \tau)$$

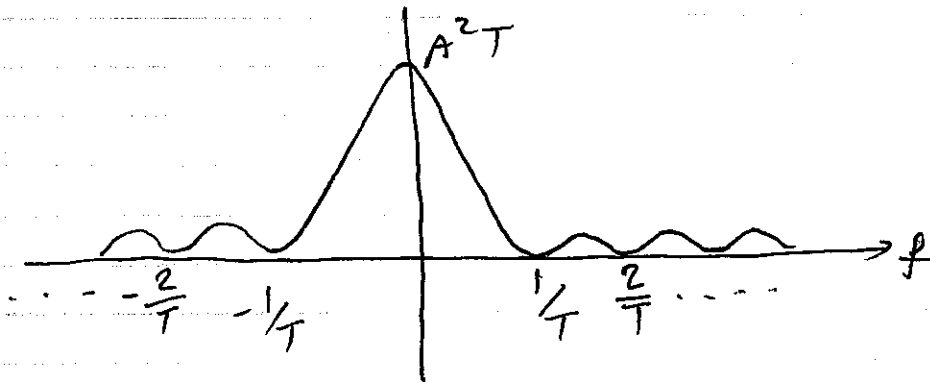
$$S_x(f) = \frac{A^2}{4} [\delta(f - f_c) + \delta(f + f_c)]$$

Example: Random binary pulse

$$R_x(\tau) = \begin{cases} A^2 \left(1 - \frac{|\tau|}{T}\right) & |\tau| < T \\ 0 & |\tau| \geq T \end{cases}$$

$$S_x(f) = \int_{-T}^T A^2 \left(1 - \frac{|\tau|}{T}\right) e^{-j2\pi f \tau} d\tau$$

$$S_x(f) = A^2 T \text{sinc}^2(fT)$$





Relating PSD of input and output of linear systems:

$$S_y(f) = \int_{-\infty}^{\infty} R_y(\tau) e^{-j2\pi f\tau} d\tau$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_x(\tau - \tau_1 - \tau_2) e^{-j2\pi f\tau} d\tau_1 d\tau_2$$

or

$$S_y(f) = H(f) H^*(f) S_x(f)$$

or

$$S_y(f) = |H(f)|^2 S_x(f)$$

Gaussian random Processes:

We say a random variable has Gaussian distribution if its density function is given by,

$$f_y(y) = \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}$$

where  $\mu_y$  is the mean and  $\sigma_y^2$  is the variance of the random variable.

When  $Y$  is normalized, i.e.,  $\mu_y = 0$  and  $\sigma_y^2 = 1$

we have

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

Central limit theorem:

Take random variables  $X_i, i=1, 2, \dots, N$   
such that

- 1)  $X_i$ 's are statistically independent.
- 2)  $X_i$ 's all have the same distribution

Then for the normalized version of  $\{X_i\}$  as

$$Y_i = \frac{1}{\sigma_X} (X_i - \mu_X) \quad i=1, 2, \dots, N$$

Now

$$E[Y_i] = 0$$

and

$$\text{Var}[Y_i] = 1$$

Define 
$$V_N = \frac{1}{\sqrt{N}} \sum_{i=1}^N Y_i$$

the Central limit theorem says that  $V_N$  approaches  
a random variable with distribution  $N(0, 1)$ , i.e.  
a zero-mean, unit-variance Gaussian variable.

## Properties of Gaussian Processes:

- 1) If a Gaussian random process is applied to a linear system, then the output is also Gaussian.
- 2) Given mean and variance all higher order statistics of samples of a Gaussian random process can be characterized.
- 3) If a Gaussian Process is Wide Sense Stationary then it is also Strict Sense Stationary.
- 4) The random variables  $X(t_1), X(t_2), \dots, X(t_n)$  obtained by sampling ~~that~~ a Gaussian Process  $X(t)$  are uncorrelated.

$$E[(X(t_k) - \mu_{X(t_k)})(X(t_i) - \mu_{X(t_i)})] = 0 \quad i \neq k$$

in case of Gaussian Process, these samples are also statistically independent.

## Thermal Noise

$$E[V_{TN}^2] = 4kTRB$$

where  $k = 1.38 \times 10^{-23}$  is the Boltzmann's Constant,  
 $B$  is the bandwidth and  $R$  is the resistance.

Maximum Power that can be delivered is to a match  
load of  $R \Omega$ , i.e.

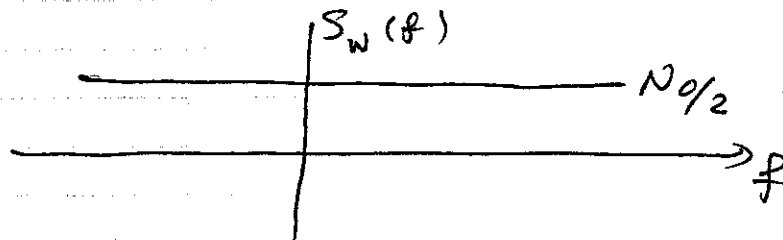
$$N = \left( \frac{\sqrt{4kTRB}}{R+R} \right)^2 R = kTB$$

if we assume that noise is uniformly distributed  
over all frequencies

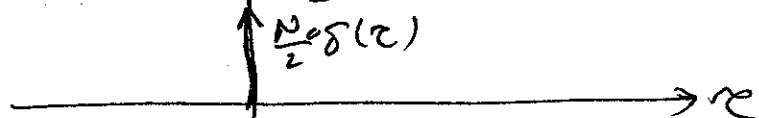
$$N_0 = \frac{N}{B} = kT$$

This is the starting point in defining the white  
noise, i.e., a process  $w$  with

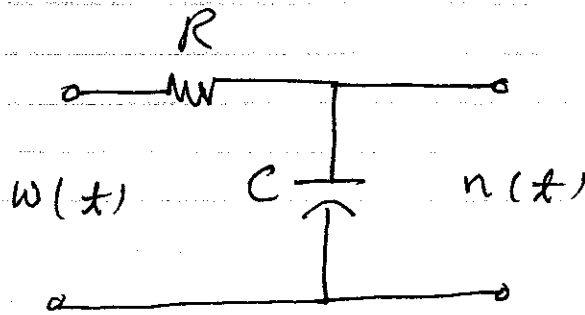
$$S_w(f) = \frac{N_0}{2} \quad \forall f$$



$$R_w(z) = \mathcal{F}^{-1} \left[ \frac{N_0}{2} \right] = \frac{N_0}{2} \delta(z)$$



Example: RC low pass filtered white noise



$$H(f) = \frac{1}{1 + j2\pi f RC}$$

$$|H(f)|^2 = H(f)H^*(f) = \frac{1}{1 + (2\pi f RC)^2}$$

$$S_N(f) = S_w(f) |H(f)|^2 = \frac{N_0/2}{1 + (2\pi f RC)^2}$$

$$R_N(\tau) = \frac{N_0}{4RC} e^{-\frac{|\tau|}{RC}}$$

where we have used

$$e^{-a|\tau|} \iff \frac{2a}{a^2 + (2\pi f)^2}$$

