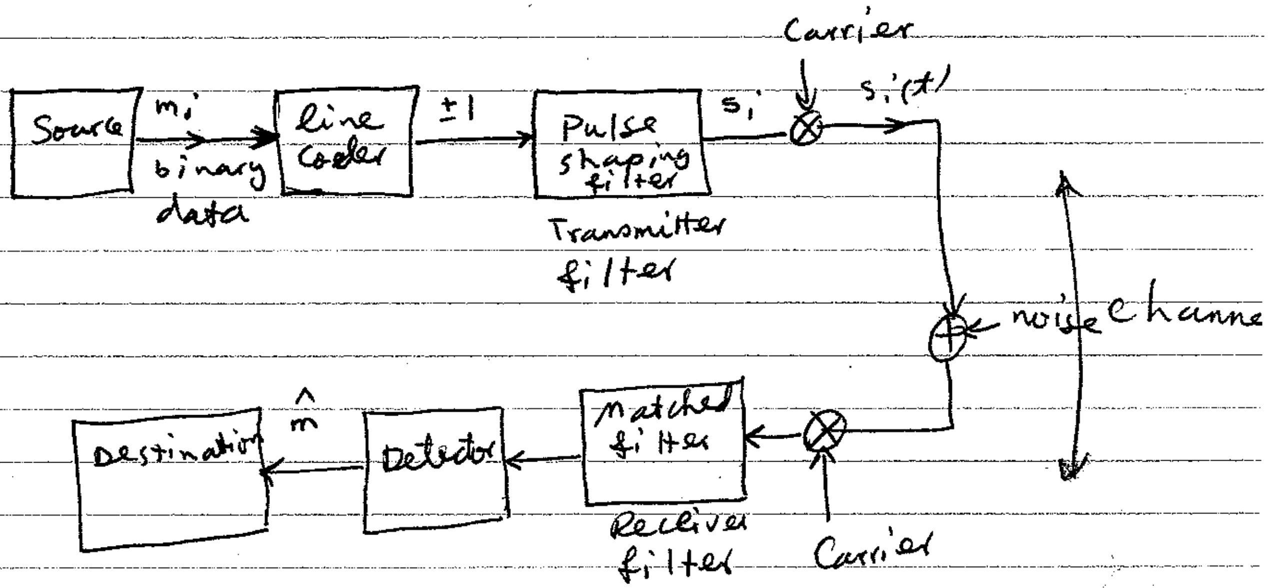
Lecture 6, May 19, 2011

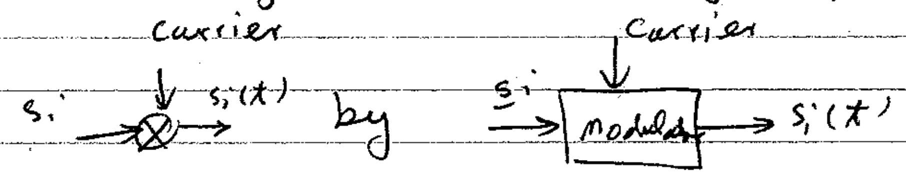
## Passband Data Transmission

(carrier modulation)

In Carrier (Passband) data transmission, the output of the pulse shaping filter is being frequency shifted by multiplying it by a sinusoidal carrier:




to be more general, we may represent




to allow for ~~modification of~~ varying any abitrary parameter of the carrier waveform.

Basic carrier Modulation shemes:

Phase Shift Keying (PSK)

Amplitude Shift Keying (ASK)

Frequency Shift Keying (FSK).

Phase Shift Keying (PSK)

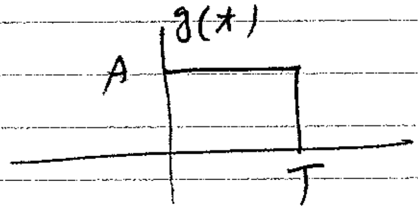
binary case:

$$s_1(t) = g(t)\cos(2\pi f_c t)$$

$$s_2(t) = g(t)\cos(2\pi f_c t + \pi) = -g(t)\cos(2\pi f_c t)$$

$g(t)$ is the pulse shaping filter, usually raised cosine filter.

For simplicity assume that $g(t)$ is an square pulse, i.e.,




Then

$$s_1(t) = A\cos(2\pi f_c t)$$

$$s_2(t) = -A\cos(2\pi f_c t)$$

$$E_b = \int_0^T s_1^2(t)\,dt = \frac{A^2}{2} T_b = \int_0^T s_2^2(t)\,dt$$

So: $A = \sqrt{\frac{2E_b}{T_b}}$

So:

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_c t$$

$$s_2(t) = -\sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_c t$$

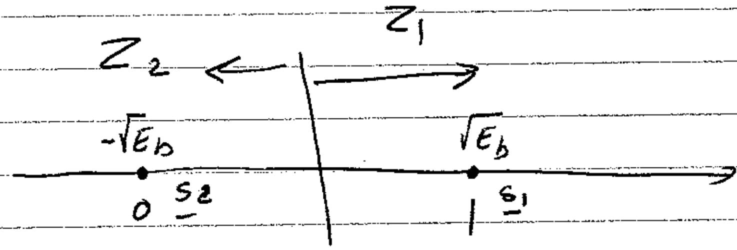
we only need one basis function for this modulation scheme, i.e.,

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t) \qquad 0 \leq t \leq T_b$$

and, then:

$$s_1(t) = \sqrt{E_b}\, \phi_1(t) \qquad 0 \leq t \leq T_b$$

$$s_2(t) = -\sqrt{E_b}\, \phi_1(t) \qquad 0 \leq t \leq T_b$$




$$P_e = p_0 P(1|0) + p_1 P(0|1)$$

for equally probable message bits:

$$P_e = \frac{1}{2} P(1|0) + \frac{1}{2} P(0|1)$$

$$P(1|0) = \int_0^{\infty} f_X(x|0)\,dx = \int_0^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0}(x+\sqrt{E_b})^2} dx$$

let $z = (x+\sqrt{E_b})/\sqrt{N_0/2}$

Then

$$P(1|0) = \int_{\sqrt{\frac{2E_b}{N_0}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Similarly

$$P(0|1) = \int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(x-\sqrt{E_b})^2}{N_0}} dx = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

So:

$$\boxed{P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$

The same as bipolar NRZ signalling.

Power Spectral Density of the PSK Signal

We can represent binary PSK as:

$$s_1(t) = g(t)\cos(2\pi f_c t)$$

$$s_2(t) = -g(t)\cos(2\pi f_c t)$$

where

$$g(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} = A & 0 \leq t < T_b \\ 0 & \text{otherwise} \end{cases}$$

The power spectral density of $s(t)$, $S_s(f)$ is given as

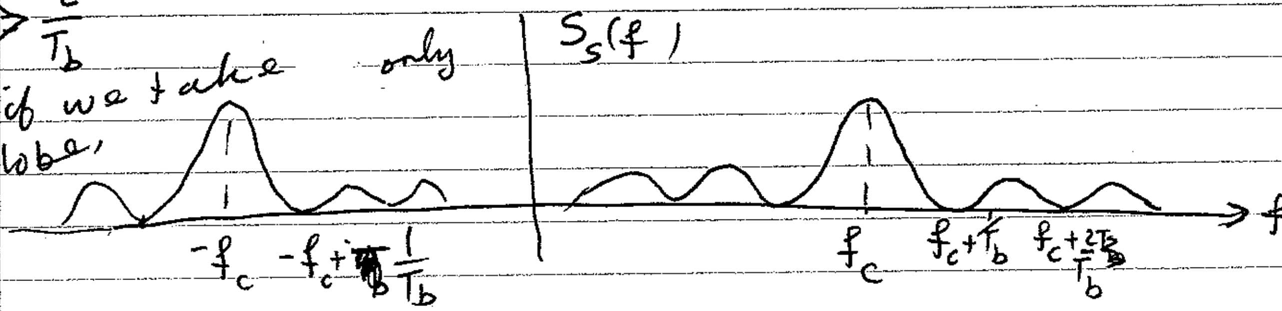
$$S_s(f) = \frac{1}{4}\left[S_B(f-f_c) + S_B(f+f_c)\right]$$

where $S_B(f)$ is the power spectral density of the $\pm g(t)$. (Remember: we know this from early lectures: random binary sequence).

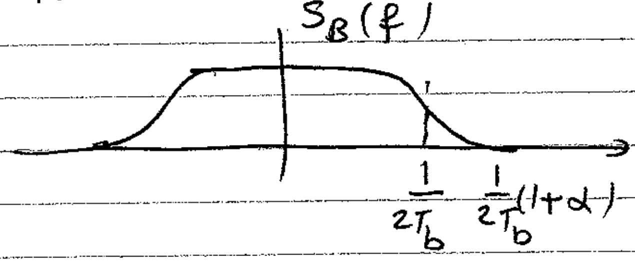
$$S_B(f) = \frac{2E_b \sin^2(\pi f T_b)}{(\pi f T_b)^2} = 2E_b \operatorname{sinc}^2(fT_b)$$

$BW > \frac{2}{T_b}$

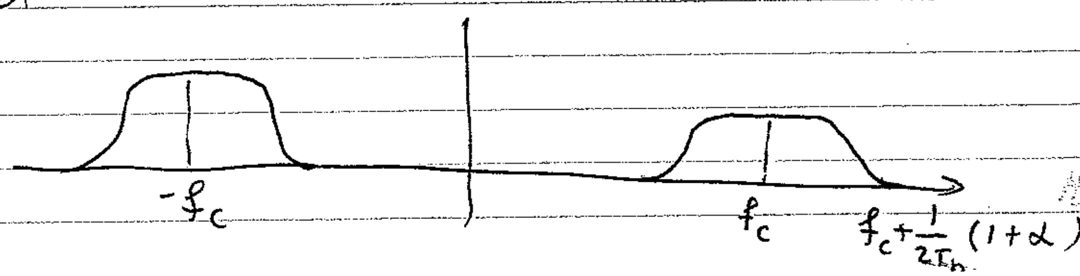
even if we take only the main lobe, we have $BW = \frac{2}{T_b}$




Howev, if we use for $g(t)$ the raised cosine filter, then




and



So, the require Bandwidth would ~~have been~~ be:

$$BW = \frac{1}{T_b}(1+\alpha) = R_b(1+\alpha) \quad Hz.$$

Bandwidth efficiency is

$$\eta = \frac{R_b}{BW} = \frac{1}{1+\alpha} \quad \text{bits/sec./Hz.}$$

Quaternary Phase-Shift Keying (QPSK).

$$s_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left[2\pi f_c t + (2i-1)\frac{\pi}{4}\right] & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases}$$

for $i = 1, 2, 3, 4$

Signal space representation

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t) \cos(2i-1)\frac{\pi}{4} - \sqrt{\frac{2E}{T}} \sin(2\pi f_c t) \times \sin(2i-1)\frac{\pi}{4}$$

Two orthonormal basis

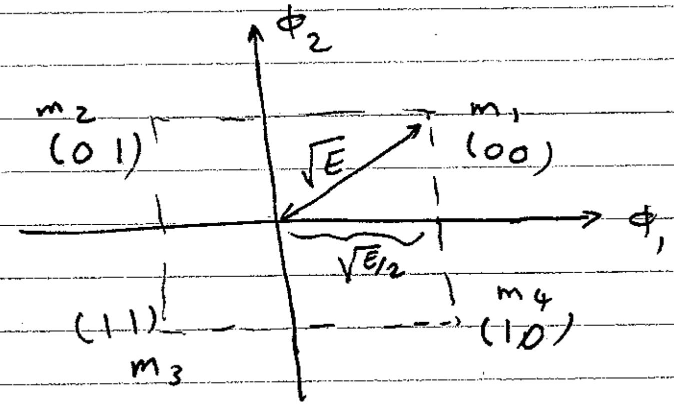
$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \qquad 0 \le t \le T$$

and

$$\phi_2(t) = -\sqrt{\frac{2}{T}} \sin(2\pi f_c t) \qquad 0 \le t \le T$$

are required to represent QPSK.

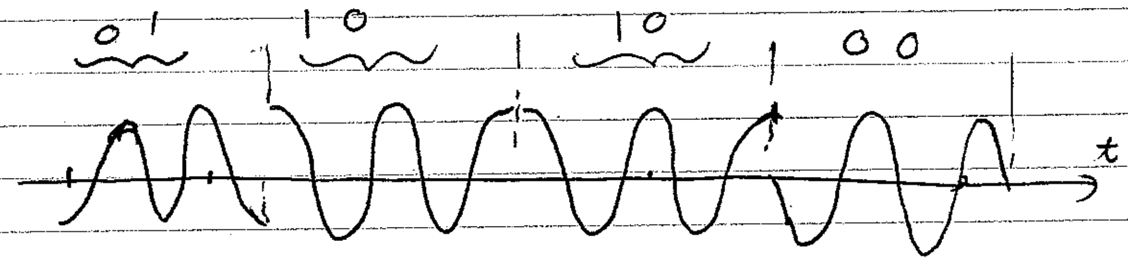
| | | $\phi_i$ | $s_{i1}$ | $s_{i2}$ |
|---|---|---|---|---|
| $i=1$ | 0 0 | $\pi/4$ | $+\sqrt{\frac{E}{2}}$ | $+\sqrt{E/2}$ |
| $i=2$ | 0 1 | $\frac{3\pi}{4}$ | $-\sqrt{E/2}$ | $+\sqrt{E/2}$ |
| $i=3$ | 1 1 | $\frac{5\pi}{4}$ | $-\sqrt{E/2}$ | $-\sqrt{E/2}$ |
| $i=4$ | 1 0 | $\frac{7\pi}{4}$ | $\sqrt{E/2}$ | $-\sqrt{E/2}$ |




Example:

Output corresponding to the sequence:

0 1 1 0 1 0 0 0



Probability of error for QPSK

With Gray coding, the probability of error for the two bits in a symbol are independent from each other.

Prob. of 2nd bit being in error is the probability that the noise in $\phi_1$ direction is greater than (less than) $\sqrt{E/2}$ $(-\sqrt{E/2})$ for 1 (0), respectively. Similarly, for the 1st. bit

$$P_B = \int_0^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(x+\sqrt{E/2})^2}{N_0}} dx$$

let $z = \dfrac{x+\sqrt{E/2}}{\sqrt{N_0/2}}$

to get

$$P_B = \int_{\sqrt{\frac{E}{N_0}}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2} dz = Q\left(\sqrt{\frac{E}{N_0}}\right)$$

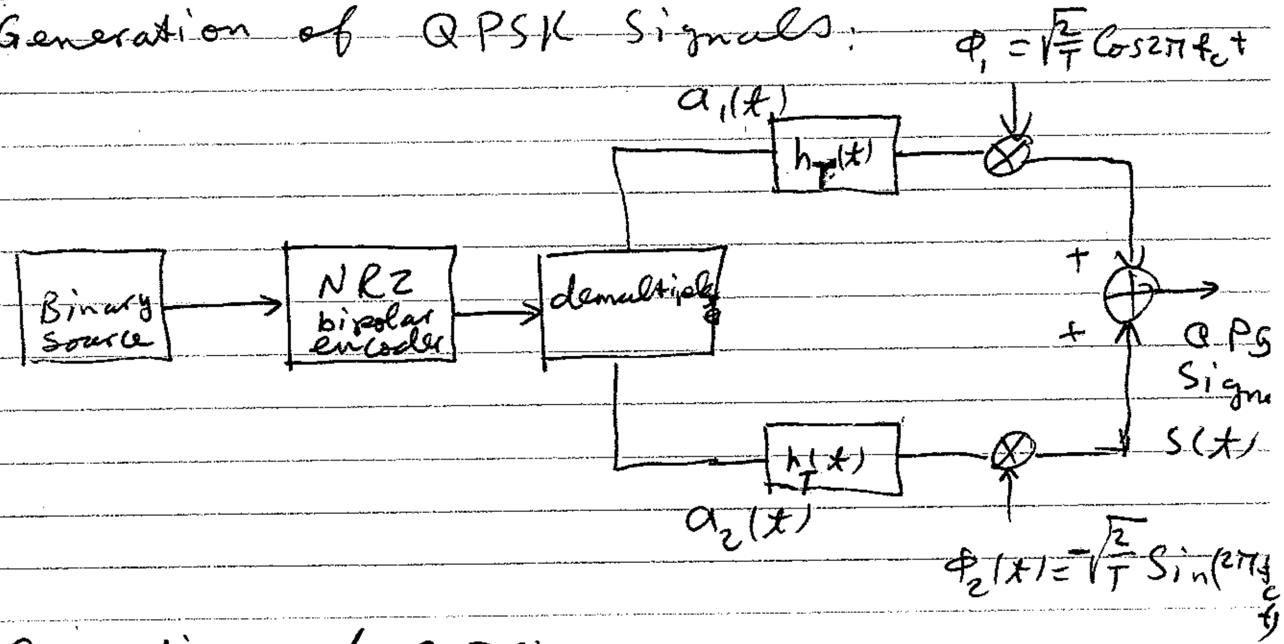
But, $E = 2E_b$ So:

$$P_B = Q\left(\sqrt{2E_b/N_0}\right)$$

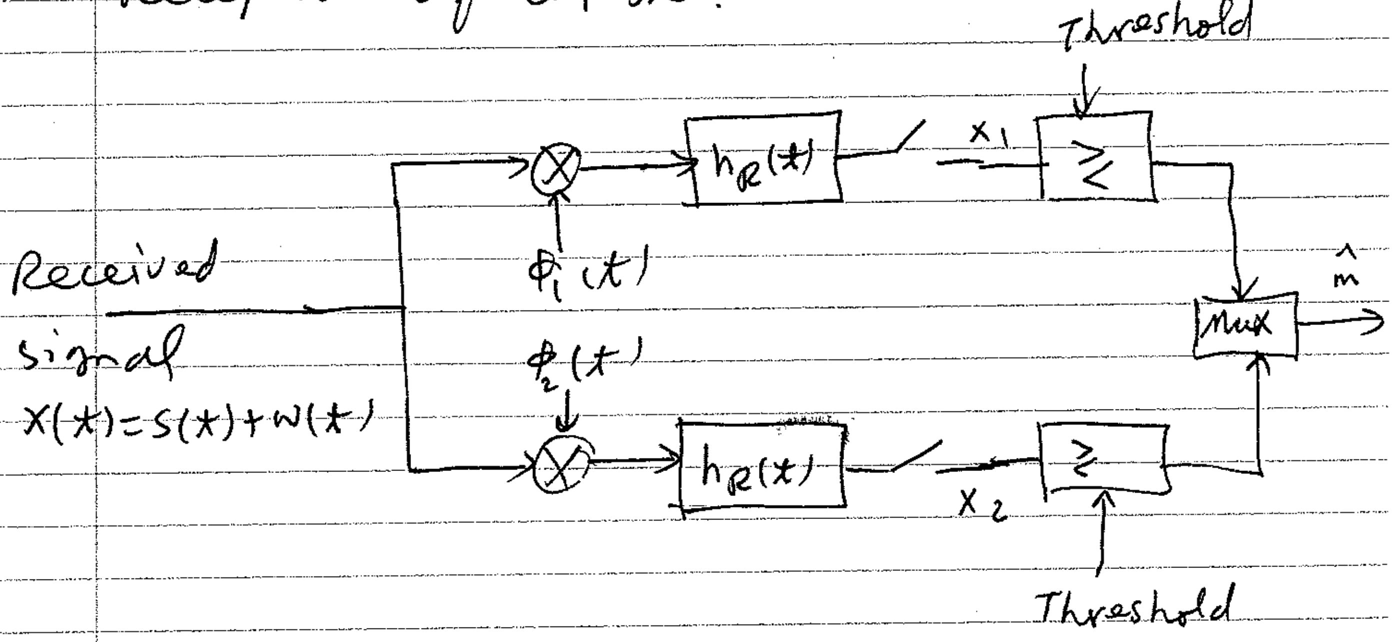
$P_B =$

So the, BER of BPSK and QPSK is the same.

Generation of QPSK Signals:




Reception of QPSK:

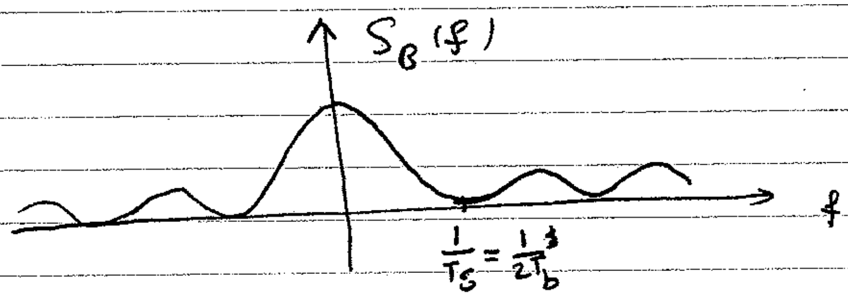


## Power Sepectra of QPSK Signal

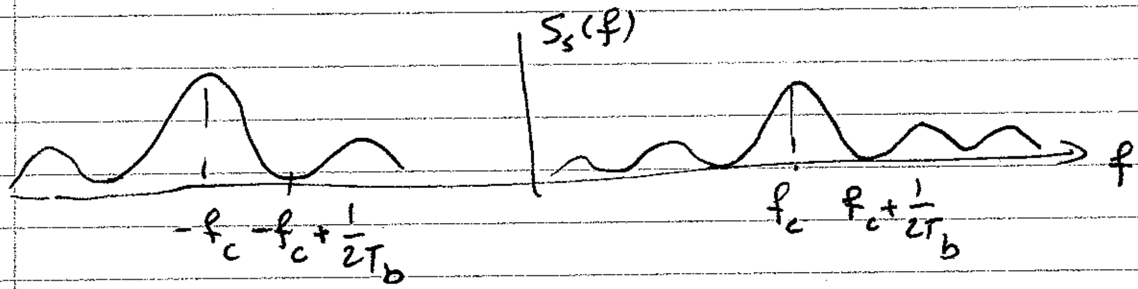
for unfiltered QPSK:

$$g(t) = \begin{cases} \sqrt{\frac{E_s}{T}} & 0 \le t \le T_s \\ 0 & \text{elsewhere} \end{cases}$$

$$S_B(f) = 2E_s \,\text{Sinc}^2(T_s f) = 4E_b \,\text{Sinc}^2(2T_b f)$$



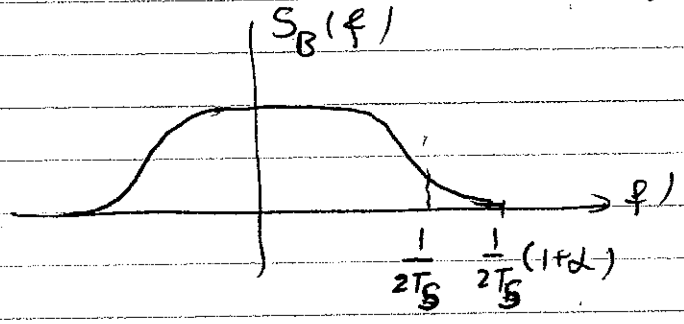

$$S_s(f) = \frac{1}{4}\left[ S_B(f - f_c) + S_B(f + f_c) \right]$$



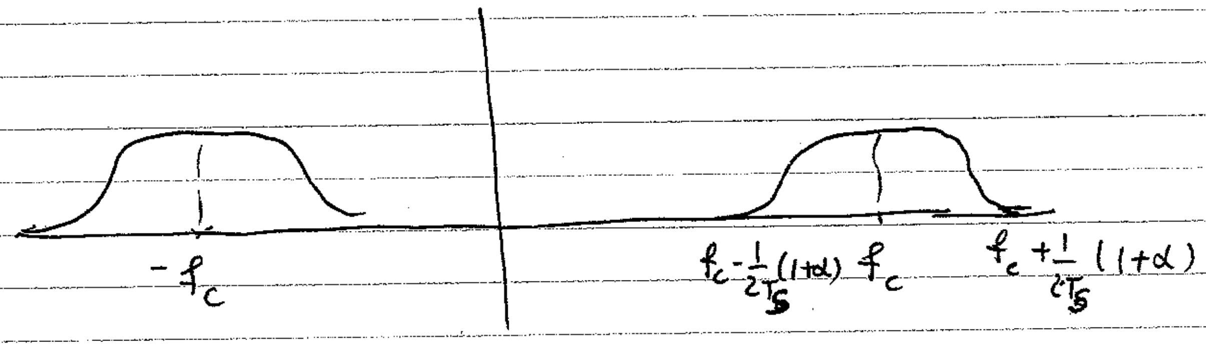

$BW = \frac{1}{T_b}$ if only ~~the~~ main lobe is considered if, instead, we use raised cosine filter

then

$$g(t) = \frac{\sin(2\pi W t)}{2\pi W t} \times \frac{\cos(2\pi \alpha W t)}{1 - 16\alpha^2 W^2 t^2}$$



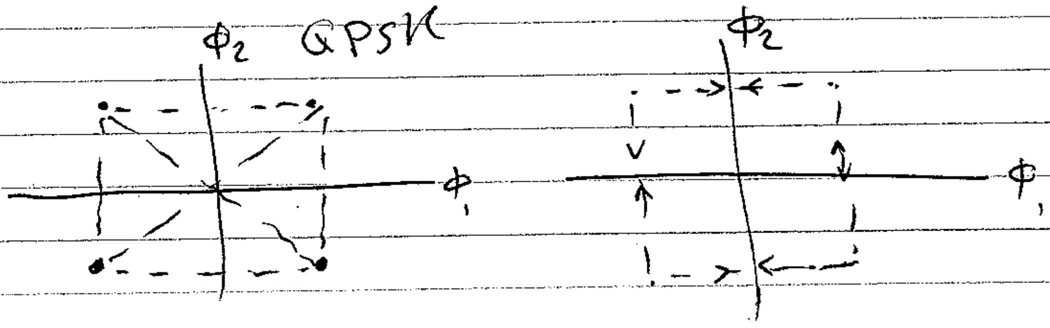

and



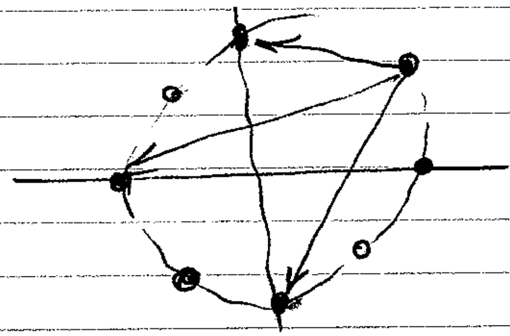

and $BW = \frac{1}{T_S}(1+\alpha) = \frac{1}{2T_b}(1+\alpha)$

## Offset QPSK (OQPSK)

in order to avoid $\pm 180°$ phase jumps in QPSK, one may delay I or Q data stream by half a symbol (1 bit).



$\frac{\pi}{4}$ - QPSK



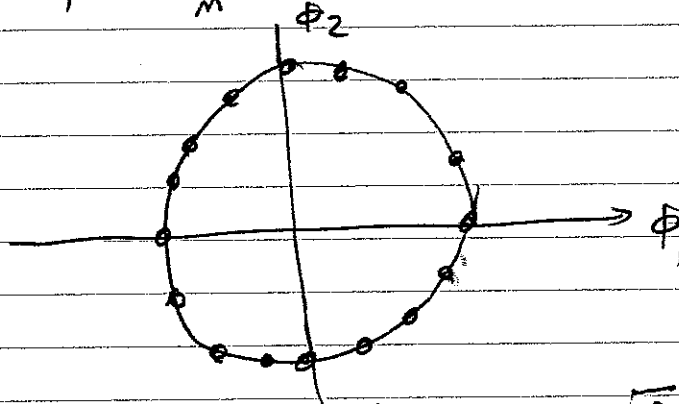
In this scheme, the signal points are alternatively taken from one of the two QPSK constellations.

So, the phase change is $\pm\frac{\pi}{4}$ and $\pm\frac{3\pi}{4}$ instead of $\pm\frac{\pi}{2}$ and $\pm\pi$.

M-ary PSK (MPSK)

$$s_i(t) = \sqrt{\frac{2E_s}{T}} \cos\left(2\pi f_c t + \frac{2\pi}{M}(i-1)\right) \qquad i = 1, 2, \ldots, M$$

$$= \sqrt{\frac{2E_s}{T}} \cos \frac{2\pi}{M}(i-1) \cos(2\pi f_c t) - \sqrt{\frac{2E_s}{T}} \sin\frac{2\pi}{M}(i-1) \sin(2\pi f_c t)$$




$\Phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)$ $\qquad$ $\Phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$

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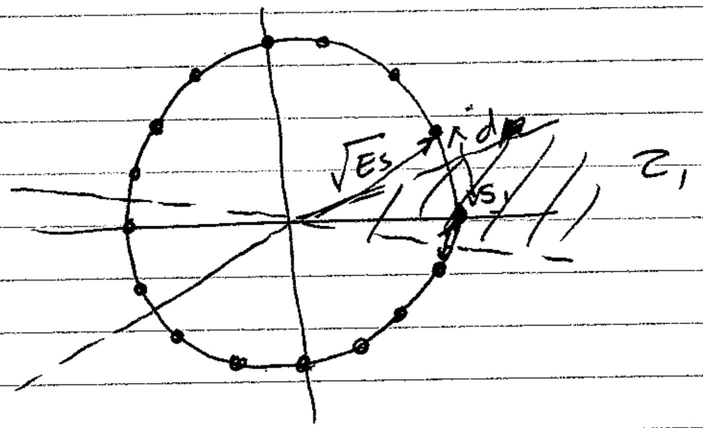
Probability of error for MPSK:




$$P_e = P_{e|s_1} = P(x \notin Z_1|s_1) > P(\cancel{\text{noise}} > \tfrac{d}{2})$$

$$= P(\text{noise} > \sqrt{E_s}\sin\tfrac{\pi}{M})$$

or $$P_e > P(\underline{s}_2|\underline{s}_1) = Q\left(\sqrt{\frac{2E_b}{N_0}}\sin\frac{\pi}{M}\right)$$

also

$$P_e < P(s_2|s_1) + P(s_3|s_1) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\sin\frac{\pi}{M}\right)$$

for large (also moderate) $M$, the upper and lower bounds are close and

$$P_e \approx 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right)$$

Note that this is the probability of symbol error.

The BER, assuming Gray coding is:

$$P_B = \frac{2}{\log_2 M}\left(\sqrt{\frac{2E_s}{N_o}}\ \sin\frac{\pi}{M}\right)$$

let $E_s = E_b \log_2 M$ to get:

$$P_B = \frac{2}{\log_2 M}\, Q\left(\sqrt{\frac{2E_b \log_2 M}{N_o}}\ \sin\frac{\pi}{M}\right)$$

Bandwidth requirement of MPSK:

unfiltered:

$$BW = \frac{2}{T_s} = \frac{2}{T_b \log_2 M} = \frac{2R_b}{\log_2 M}$$

with raised cousine filter

$$BW = \frac{1}{T_s}(1+\alpha) = \frac{1}{T_b \log_2 M}(1+\alpha) = \frac{R_b(1+\alpha)}{\log_2 M}$$

Bandwith efficiency.

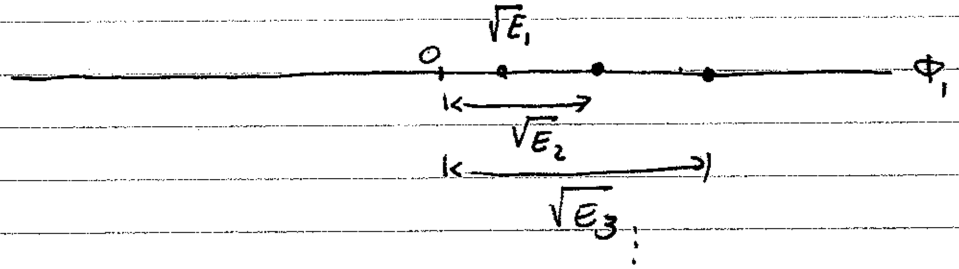
$$\boxed{\eta = \frac{R_b}{BW} = \frac{\log_2 M}{1+\alpha}}$$

## M-ary ASK ( M-ary PAM )

$$s(t) = A_i \cos(2\pi f_c t)$$

$$E_i = \frac{A_i^2 T_s}{2}$$

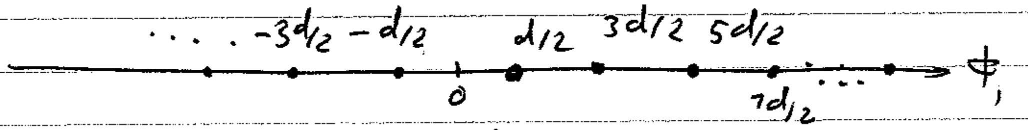
$$s(t) = \sqrt{\frac{2E_i}{T_s}} \cos(2\pi f_c t)$$




where

$$\phi_1(t) = \sqrt{\frac{2}{T_s}} \cos(2\pi f_c t)$$

let the distance between signals be the same and equal to $d$:




$$\pm\frac{d}{2}, \pm\frac{3d}{2}, \ldots\ldots, \pm\frac{(M-1)d}{2}$$

The average power is

$$E_{av} = \frac{(d/2)^2 \left[1 + 3^2 + 5^2 + \cdots + (M-1)^2\right] \times 2}{M}$$

$$E_{av} = \frac{d^2}{2} \times \frac{1}{\cancel{M}} \times \frac{\cancel{M}(M-1)(M+1)}{6} = \frac{d^2(M^2-1)}{12}$$

Probability of symbol error:

$$P_e = \left[(M-2)\,2Q\left(\frac{d}{2\sigma}\right) + 2 \times Q\left(\frac{d}{2\sigma}\right)\right] / M$$

$$P_e = \frac{2(M-1)}{M} Q\left(\frac{d}{2\sigma}\right)$$

$$d = \sqrt{\frac{12\,E_{av}}{(M^2-1)}} \qquad \sigma^2 = \frac{N_o}{2} \Rightarrow \sigma = \sqrt{\frac{N_o}{2}}$$

So,

$$P_e = \frac{2(M-1)}{M}\, Q\left(\sqrt{\frac{6\,E_{av}}{(M^2-1)N_o}}\right)$$

or

$$P_e = 2\left(1 - \frac{1}{M}\right) \times Q\left(\sqrt{\frac{6\,E_{av}}{(M^2-1)N_o}}\right)$$

M-ary Quadrature-Amplitude-Modulation (QAM)

This is a hybrid modulation scheme, where both phase and amplitude are varied.

like MPSK, we have

$$\phi_1(t) = \sqrt{\frac{2}{T_s}} \cos(2\pi f_c t)$$

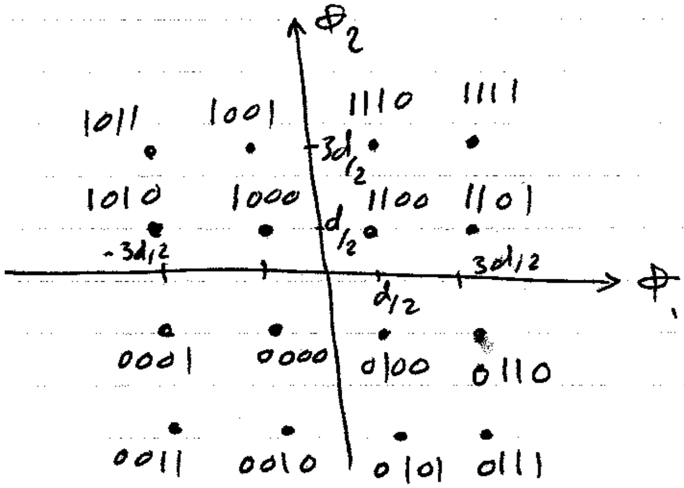
$$\phi_2(t) = \sqrt{\frac{2}{T_s}} \sin(2\pi f_c t)$$

but here

$$s_k(t) = \sqrt{\frac{2E_0}{T}}\, a_k \cos(2\pi f_c t) \cancel{+} \sqrt{\frac{2E_0}{2}}\, b_k \sin(2\pi f_c t)$$

$$k = 0, \pm 1, \pm 2,$$

i.e., the amplitude in I and Q dimension is also varied according to the symbol to be transmitted ($a_k$ and $b_k$). Here $\sqrt{E_0} = \frac{d}{2}$ as in the case of M-ASK.




## Probability of symbol error

$$P_e = 1 - P_c = 1 - (1 - P_{\sqrt{M}})^2 \approx 2 P_{\sqrt{M}}$$

where $P_{\sqrt{M}}$ is the probability ~~that~~ of error of a 1-dimensional scheme ($\sqrt{M}$-PAM or $\sqrt{M}$-ASK).

So:

$$P_e \cong 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_{av}}{2(M-1)N_0}}\right)$$

$$= 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_{av}}{(M-1)N_0}}\right)$$

Since

$$P_{\sqrt{M}} = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E'_{av}}{(M-1)N_0}}\right)$$

where $E'_{av} = \frac{1}{2} E_{av}$ is the average signal energy of

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the 1-dimensional scheme.

BER : Assuming Gray Coding

$$P_B = 4\left(1 - \frac{1}{\sqrt{M}}\right)\frac{1}{\log_2 M} \; Q\left(\sqrt{\frac{3\log_2 M}{(M-1)}\frac{E_b}{N_o}}\right)$$

Comparison with M-ary PSK

$$P_B \cong \frac{1}{\log_2 M} \quad Q\left(\sqrt{\frac{2E_b \log_2 M}{N_o}} \sin\frac{\pi}{M}\right)$$

for large M

$$P_B \cong \frac{1}{\log_2 M} \quad Q\left(\sqrt{\frac{2E_b \log_2 M}{N_o}} \frac{\pi}{M}\right)$$

while the exponent for M-QAM decreases as $\sqrt{\frac{\log_2 M}{M-1}}$ for M-PSK the decrease is as $\frac{\sqrt{\log_2 M}}{M}$ which is faster. So, the performance QAM is better.