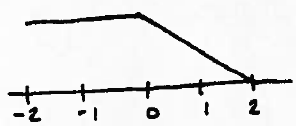
Solutions to Assignment 1

1)

$$P[|X|>x] = P[\{X>x\}\cup\{X<-x\}]$$
$$= P[X>x] + P[X<-x]$$
$$= 1 - F_X(x) + F_X(-x)$$

$$F_X(x) = \begin{cases} 0 & x\le -2 \\ \frac{x+2}{4} & -2\le x\le 2 \\ 1 & x\ge 2 \end{cases}$$

$$P[|X|>x] = \begin{cases} 1 & x\le 0 \\ 1-\left(\frac{x+2}{4}\right)+\left(\frac{-x+2}{4}\right) = 1-\frac{x}{2} & 0\le x\le 2 \\ 0 & x\ge 2 \end{cases}$$



2)

a) $P[X>4] = 1-F_X(4) = 1-\Phi\left(\frac{4-5}{4}\right) = 1-\Phi\left(-\frac{1}{4}\right) = \Phi\left(\frac{1}{4}\right) = 0.598$

$P[X\ge 7] = 1-F_X(7) = 1-\Phi\left(\frac{7-5}{4}\right) = 1-\Phi\left(\frac{1}{2}\right) = 0.308$

$P[6.72<X<10.16] = \Phi\left(\frac{10.16-5}{4}\right) - \Phi\left(\frac{6.72-5}{4}\right) = \Phi(1.29)-\Phi(0.43) = 0.235$

$P[2<X<7] = \Phi\left(\frac{7-5}{4}\right) - \Phi\left(\frac{2-5}{4}\right) = \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{3}{4}\right) = 0.465$

$P[6\le X\le 8] = \Phi\left(\frac{8-5}{4}\right) - \Phi\left(\frac{6-5}{4}\right) = \Phi\left(\frac{3}{4}\right) - \Phi\left(\frac{1}{4}\right) = 0.175$

b) $P[X<a] = 0.8869$

$\Phi\left(\frac{a-5}{4}\right) = 0.8869 = 1-Q(x)$

$Q(x) = 0.1131 \rightarrow x = 1.2 = \frac{a-5}{4} \rightarrow a = 9.8$

c) $P[X>b] = 1-\Phi\left(\frac{b-5}{4}\right) = 0.11131$

$Q(x) = 0.11131 \rightarrow x = 1.2 = \frac{b-5}{4} \rightarrow b = 9.8$

d) $P[13<X\le c] = 0.0123$

$\Phi\left(\frac{c-5}{4}\right) - \Phi\left(\frac{13-5}{4}\right) = \Phi\left(\frac{c-5}{4}\right) - \Phi(2) = 0.0123$

$\Phi\left(\frac{c-5}{4}\right) = 0.0123 + 0.9772 = 0.9895$

$Q\left(\frac{c-5}{4}\right) = 0.0105 \rightarrow x = 2.3 = \frac{c-5}{4} \rightarrow c = 14.2$

3)
$$\begin{aligned} Q(-x) &= \frac{1}{\sqrt{2\pi}}\int_{-x}^{\infty} e^{t^2/2}dt = 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-x} e^{-t^2/2}dt \\ &= 1 - \frac{1}{\sqrt{2\pi}}\int_{\infty}^{x} e^{-t'^2/2}(-dt') \text{ where } t' = -t \\ &= 1 - \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} e^{-t'^2/2}dt' = 1 - Q(x) \end{aligned}$$

4) a) $F_Y(X+N \le y | X=+1) = F_N(y-1)$

$F_Y(X+N \le y | X=-1) = F_N(y+1)$

$f_Y(y|X=+1) = f_N(y-1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-1)^2/2\sigma^2}$

$f_Y(y|X=-1) = f_N(y+1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y+1)^2/2\sigma^2}$

b) $f_Y(y|X=-1)P[X=-1] > f_Y(y|X=+1)P[X=+1]$ decide "0"

$\frac{e^{-(y+1)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}(3p_1) > \frac{e^{-(y-1)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} p_1$

$3e^{-(y+1)^2/2\sigma^2} e^{(y-1)^2/2\sigma^2} > 1$

$3e^{-y^2-2y-1+y^2-2y+1/2\sigma^2} > 1$

$3e^{-4y/2\sigma^2} > 1$

$\frac{-4y}{2\sigma^2} > \ln\left(\frac{1}{3}\right)$

$y < -\frac{\sigma^2}{2}\ln\left(\frac{1}{3}\right)$ decide "0" $T = -\frac{\sigma^2}{2}\ln\left(\frac{1}{3}\right)$

c) $P[X+N<T|X=+1] = P[N<T-1] = \Phi\left(\frac{T-1}{\sigma^2}\right)$

$P[X+N \ge T|X=-1] = P[N \ge T+1] = \left(1-\Phi\left(\frac{T+1}{\sigma^2}\right)\right)$

d) $P[X+N<T|X=+1]P[X=+1] + P[X+N \ge T|X=-1]P[X=-1] =$

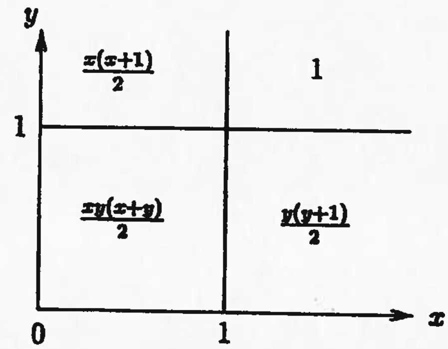
$= p_1\,\Phi\left(\frac{T-1}{\sigma^2}\right) + \left(1-\Phi\left(\frac{T+1}{\sigma^2}\right)\right)3p_1$

5)

$$1 = k\int_0^1\int_0^1 (x+y)\,dx\,dy = k\int_0^1\left(\frac{x^2}{2}+xy\right)_0^1 dy$$
$$= k\int_0^1\left(\frac{1}{2}+y\right)dy = k\left[\frac{1}{2}y+\frac{y^2}{2}\right]_0^1 = k$$

$\therefore k = 1$

b)



c) $$F_X(x) = \lim_{y\to\infty} F_{XY}(x,y) = F_{XY}(x,1) \quad 0<x<1$$
$$\Rightarrow f_X(x) = \frac{d}{dx}F_X(x) = x+\frac{1}{2}$$

Similarly

6)

$$f_Y(y) = y+\frac{1}{2}$$

$C_X(t_1,t_2) = 4e^{-2|t_1-t_2|}$

$X(t)$ & $X(t+s)$ are jointly Gaussian RVs with $COV(X(t),X(t+s)) = C_X(t,t+s) = 4e^{-2|s|}$

then $f_{X(t),X(t+s)}(x_1,x_2)$ can be achieved using Eq. 9.47 as follows:

$$f_{X(t),X(t+s)}(x_1,x_2) = \frac{e^{-\frac{1}{2}(x-m)^T K^{-1}(x-m)}}{2\pi\,|K|^{1/2}}$$

in which $m=0$ $x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $K=\begin{bmatrix} 4 & 4e^{-2|s|} \\ 4e^{-2|s|} & 4 \end{bmatrix}$

and $K^{-1} = \dfrac{1}{16(1-e^{-4|s|})}\begin{bmatrix} 4 & -4e^{-2|s|} \\ -4e^{-2|s|} & 4 \end{bmatrix}$ $\quad |K| = 16(1-e^{-4|s|})$

So

$$f_{X(t),X(t+s)}(x_1,x_2) = \frac{e^{\left\{-\frac{1}{32(1-e^{-4|s|})}\left(4x_1^2+4x_2^2-8e^{-4|s|}x_1x_2\right)\right\}}}{8\pi\sqrt{1-e^{-4|s|}}}$$

7) $Z(t) = X(t)\cos\omega t + Y(t)\sin\omega t$

a) $\mathcal{E}[Z(t)] = m_X(t)\cos\omega t + m_Y(t)\sin\omega t = 0$

$$
\begin{aligned}
C_Z(t_1,t_2) &= \mathcal{E}[(X(t_1)\cos\omega t_1 + Y(t_1)\sin\omega t_1) \\
&\quad (X(t_2)\cos\omega t_2 + Y(t_2)\sin\omega t_2)] \\
&= \mathcal{E}[X(t_1)X(t_2)]\cos\omega t_1\cos\omega t_2 \\
&\quad + \underbrace{\mathcal{E}[X(t_1)Y(t_2)]}_{0}\cos\omega t_1\sin\omega t_2 \\
&\quad + \underbrace{\mathcal{E}[Y(t_1)X(t_2)]}_{0}\sin\omega t_1\cos\omega t_2 + \mathcal{E}[Y(t_1)Y(t_2)]\sin\omega t_1\sin\omega t_2 \\
&= C(t_1,t_2)\cos\omega t_1\cos\omega t_2 + C(t_1,t_2)\sin\omega t_1\sin\omega t_2 \\
&= C(t_1,t_2)\cos\omega(t_1-t_2)
\end{aligned}
$$

b) $f_{Z(t)}(z) = \dfrac{1}{\sqrt{2\pi C(t,t)}}e^{-z^2/2C(t,t)}$

8)

a) $S_{YX}(f) = H(f)S_X(f) = \dfrac{N_0/2}{1+j2\pi f}$

$R_{YX} = \mathcal{F}^{-1}[S_{YX}(f)] = \dfrac{N_0}{2}e^{-\tau} \qquad \tau > 0$

b) $S_Y(f) = |H(f)|^2 S_X(f) = \dfrac{N_0/2}{1+4\pi^2 f^2}$

$R_Y(\tau) = \mathcal{F}^{-1}[S_Y(f)] = \dfrac{N_0}{4}e^{-|\tau|}$

c) $R_Y(0) = \dfrac{N_0}{4}$

9)

$Z(t) = X(t) - Y(t) \qquad Y(t) = h(t) \star X(t)$

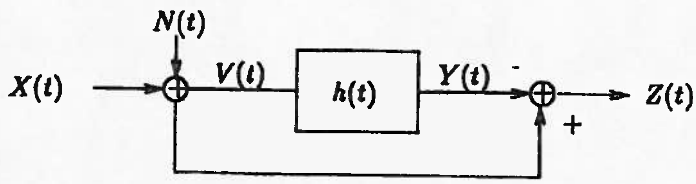
a)
$$
\begin{aligned}
E[Z(t)Z(t+\tau)] &= \mathcal{E}[(X(t)-Y(t))(X(t+\tau)-Y(t+\tau))] \\
&= R_X(\tau) + R_Y(\tau) - R_{YX}(-\tau) - R_{XY}(-\tau) \\
&= R_X(\tau) + R_Y(\tau) - R_{XY}(\tau) - R_{XY}(-\tau) \\
S_Z(f) &= S_X(f) + S_Y(f) - S_{XY}(f) - S^*_{XY}(f) \\
&= S_X(f) + |H(f)|^2 S_X(f) - H^*(f)S_X(f) - H(f)S_X(f) \\
&= \left\{1 + |H(f)|^2 - \underbrace{(H^*(f)+H(f))}_{2\mathrm{Re}[H(f)]} S_X(f)\right\} \qquad (f) \\
&= |\{1-H(f)\}|^2 S_X(f)
\end{aligned}
$$

b)
$$
\begin{aligned}
\mathcal{E}[Z(t)^2] &= R_X(0) + R_Y(0) - 2R_{XY}(0) \\
&= \mathcal{E}[X^2(t)] + \int\int_{-\infty}^{\infty} h(s)h(r)R_X(s-r)\,ds\,dr \\
&\quad -2\int_{-\infty}^{\infty} h(r)R_X(-r)\,dr
\end{aligned}
$$

Also

$$\mathcal{E}[Z^2(t)] = \int_{-\infty}^{\infty} |1-H(f)|^2 S_X(f)\,df$$

10)



$$
\begin{aligned}
R_Z(\tau) &= \mathcal{E}[Z(t+\tau)Z(t)] = \mathcal{E}[(X(t+\tau)-Y(t+\tau))(X(t)-Y(t))] \\
&= R_X(\tau)+R_Y(\tau)-R_{XY}(\tau)-R_{YX}(\tau) \\
R_{XY}(\tau) &= \mathcal{E}[X(t+\tau)Y(t)] = \mathcal{E}\left[X(t+\tau)\int_{-\infty}^{\infty} h(\lambda)V(t-\lambda)d\lambda\right] \\
&= \int_{-\infty}^{\infty} h(\lambda)R_{XV}(\tau+\lambda)d\lambda \\
&= \int_{-\infty}^{\infty} h(\lambda)R_X(\tau+\lambda)d\lambda \quad \text{since} \quad \begin{aligned} R_{XV}(\tau) &= \mathcal{E}[X(t+\tau)(X(t)+N(t))] \\ &= R_X(\tau) \end{aligned} \\
&= h(-\tau) \star R_X(\tau) \\
S_Z(f) &= S_X(f)+S_Y(f)-S_{XY}(f)-S_{YX}(f) \\
&= S_X(f)+|H(f)|^2(S_X(f)+S_N(f))-H(f)S_X(f)-H^*(f)S_X(f) \\
S_Z(f) &= |1-H(f)|^2 S_X(f)+|H(f)|^2 S_N(f) \qquad (*)
\end{aligned}
$$

Comments: If we view $Y(t)$ as our estimate for $X(t)$, then $S_Z(f)$ is the power spectral density of the error signal $Z(t) = Y(t) - X(t)$. Equation (*) suggests the following:

if	$S_X(f) >> S_N(f)$	let	$H(f) \approx 1$
if	$S_X(f) << S_N(f)$	let	$H(f) \approx 0$

i.e. select $H(f)$ to "pass" the signal and reject the noise.