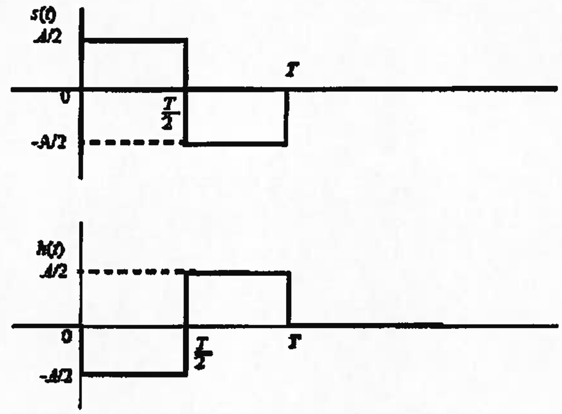
Solution to Assignment 3

Problem 1

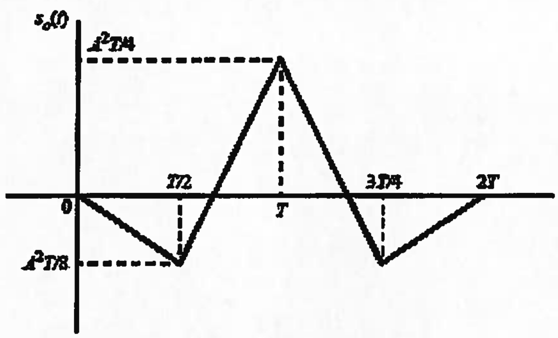
(a) The impulse response of the matched filter is

$$h(t) = s(T-t)$$

The $s(t)$ and $h(t)$ are shown below:

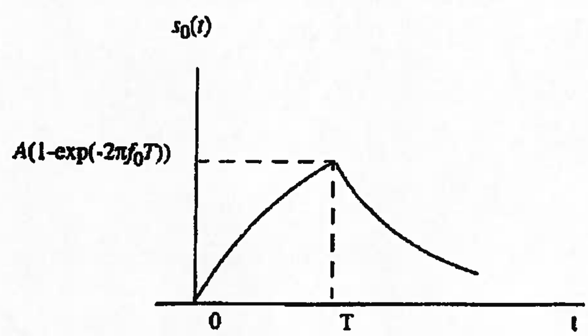


(b) The corresponding output of the matched filter is obtained by convolving $h(t)$ with $s(t)$. The result is shown below:



(c) The peak value of the filter output is equal to $A^2T/4$, occuring at $t = T$.

The output of the low-pass RC filter, produced by a rectangular pulse of amplitude A and duration T, is as shown below:



The peak value of the output pulse power is

$$P_{out} = A^2[1 - \exp(-2\pi f_0 T)]^2$$

where f_0 is the 3-dB cutoff frequency of the RC filter.

The average output noise power is

$$N_{\text{out}} = \frac{N_0}{2}\int_{-\infty}^{\infty} \frac{df}{1+(f/f_0)^2}$$

$$= \frac{N_0 \pi f_0}{2}$$

The corresponding value of the output signal-to-noise ratio is therefore

$$(\text{SNR})_{\text{out}} = \frac{2A^2}{N_0 \pi f_0}[1-\exp(-2\pi f_0 T)]$$

Differentiating $(\text{SNR})_0$ with respect to $f_0 T$ and setting the result equal to zero, we find that $(\text{SNR})_{\text{out}}$ attains its maximum value at

$$f_0 = \frac{0.2}{T}$$

The corresponding maximum value of $(\text{SNR})_{\text{out}}$ is

$$(\text{SNR})_{0,\max} = \frac{2A^2 T}{0.2\pi N_0}[1-\exp(-0.4\pi)]^2$$

$$= \frac{1.62 A^2 T}{N_0}$$

For a perfect matched filter, the output signal-to-noise ratio is

$$(\text{SNR})_{0,\text{matched}} = \frac{2E}{N_0}$$

$$= \frac{2A^2 T}{N_0}$$

Hence, we find that the transmitted energy must be increased by the ratio 2/1.62, that is, by 0.92 dB so that the low-pass RC filter with $f_0 = 0.2/T$ realizes the same performance as a perfectly matched filter.

Problem 3

The average probability of error is

$$P_e = p_1 \int_{-\infty}^{\lambda} f_Y(y\,|1)dx + p_0 \int_{\lambda}^{\infty} f_Y(y\,|0)dx \qquad (1)$$

An optimum choice of λ corresponds to minimum P_e. Differentiating Eq. (1) with respect to λ, we get:

$$\frac{\partial P_e}{\partial \lambda} = p_1 f_Y(\lambda\,|1) - p_0 f_Y(\lambda\,|0)$$

Setting $\frac{\partial P_e}{\partial \lambda} = 0$, we get the following condition for the optimum value of λ:

$$\frac{f_Y(\lambda_{opt}|1)}{f_Y(\lambda)_{opt}|0)} = \frac{p_0}{p_1}$$

which is the desired result.

Problem 4

(a) The average probability of error is

$$P_e = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

where $E_b = A^2T_b$. We may rewrite this formula as

$$P_e = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{A}{\sigma}}\right) \qquad (1)$$

where A is the pulse amplitude at $\sigma = \sqrt{N_0T_b}$. We may view σ^2 as playing the role of noise variance at the decision device input. Let

$$u = \sqrt{\frac{E_b}{N_0}} = \frac{A}{\sigma}$$

We are given that

$\sigma^2 = 10^{-2}$ volts^2, $\quad \sigma = 0.1$ volt

$P_e = 10^{-8}$

Since P_e is quite small, we may approximate it as follows:

$$\text{erfc}(u) \approx \frac{\exp(-u^2)}{\sqrt{\pi}u}$$

We may thus rewrite Eq. 1 as (with $P_e = 10^{-8}$)

$$\frac{\exp(-u^2)}{2}\sqrt{\pi}u = 10^{-8}$$

Solving this equation for u, we get

$u = 3.97$

The corresponding value of the pulse amplitude is

$A = \sigma u = 0.1 \times 3/97$

$= 0.397$ volts

(b) Let σ_i^2 denote the combined variance due to noise and interference; that is

$$\sigma_T^2 = \sigma^2 + \sigma_i^2$$

where σ^2 is due to noise and σ_i^2 is due to the interference. The new value of the average probability of error is 10^{-6}. Hence

$$10^{-6} = \frac{1}{2}\text{erfc}\left(\frac{A}{\sigma_T}\right)$$

$$= \frac{1}{2}\text{erfc}(u_T) \qquad (2)$$

where

$$u_T = \frac{A}{\sigma_T}$$

Equation (2) may be approximated as (with $P_e = 10^{-6}$)

$$\frac{\exp(-u_T^2)}{2\sqrt{\pi}u_T} \approx 10^{-6}$$

Solving for u_T, we get

$$u_T = 3.37$$

The corresponding value of σ_T^2 is

$$\sigma_T^2 = \left(\frac{A}{u_T}\right)^2 = \left(\frac{0.397}{3.37}\right)^2 = 0.0138 \text{ volts}^2$$

The variance of the interference is therefore

$$\begin{aligned}\sigma_i^2 &= \sigma_T^2 - \sigma^2 \\ &= 0.0138 - 0.01 \\ &= 0.0038 \text{ volts}^2\end{aligned}$$

Problem 5

The bandwidth B of a raised cosine pulse spectrum is $2W - f_1$, where $W = 1/2T_b$ and $f_1 = W(1 - \alpha)$. Thus $B = W(1 + \alpha)$. For a data rate of 56 kilobits per second, $W = 28$ kHz.

(a) For $\alpha = 0.25$,
$B = 28 \text{ kHz} \times 1.25$
$= 35 \text{ kHz}$

(b) $B = 28 \text{ kHz} \times 1.5$
$= 42 \text{ kHz}$

(c) $B = 49 \text{ kHz}$

(d) $B = 56 \text{ kHz}$

Problem 6

(a) For a unity rolloff, raised cosine pulse spectrum, the bandwidth B equals $1/T$, where T is the pulse length. Therefore, T in this case is 1/12 kHz. Quaternary PAM ensures 2 bits per pulse, so the rate of information is

$$\frac{2 \text{ bits}}{T} = 24 \text{ kilobits per second}$$

(b) For 128 quantizing levels, 7 bits are required to transmit an amplitude. the additional bit for synchronization makes each code word 8 bits. The signal is transmitted at 24 kilobits/s, so it must be sampled at

$$\frac{24 \text{ bits/s}}{8 \text{ bits/sample}} = 3 \text{ kHz}$$

The maximum possible value for the signal's highest frequency component is 1.5 kHz, in order to avoid aliasing.