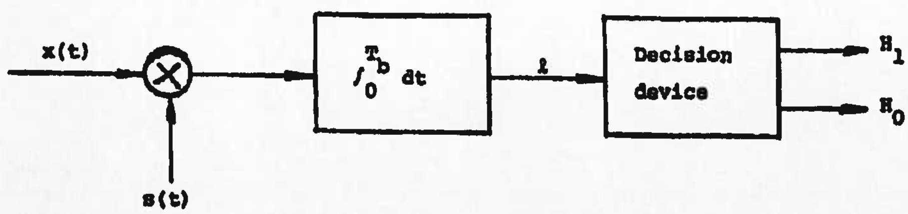
Solution to Assignment 4

Problem 1:

(a) <u>ASK with coherent reception</u>



Denoting the presence of symbol 1 or symbol 0 by hypothesis H_1 or H_0, respectively, we may write

H_1: $x(t) = s(t) + w(t)$

H_0: $x(t) = w(t)$

where $s(t) = A_c \cos(2\pi f_c t)$, with $A_c = \sqrt{2E_b/T_b}$. Therefore,

$$\ell = \int_0^{T_b} x(t)\, s(t)\, dt$$

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If $\ell > E_b/2$, the receiver decides in favor of symbol 1. If $\ell < E_b/2$, it decides in favor of symbol 0.

The conditional probability density functions of the random variable L, whose value is denoted by ℓ, are defined by

$$f_{L|0}(\ell|0) = \frac{1}{\sqrt{\pi N_0 E_b}} \exp\left(-\frac{\ell^2}{N_0 E_b}\right)$$

$$f_{L|1}(\ell|1) = \frac{1}{\sqrt{\pi N_0 E_b}} \exp\left[-\frac{(\ell - E_b)^2}{N_0 E_b}\right]$$

The average probability of error is therefore,

$$P_e = p_0 \int_{E_b/2}^{\infty} f_{L|0}(\ell|0)\, d\ell + p_1 \int_{-\infty}^{E_b/2} f_{L|1}(\ell|1)\, d\ell$$

$$= \frac{1}{2} \int_{E_b/2}^{\infty} \frac{1}{\sqrt{\pi N_0 E_b}} \exp\left(-\frac{\ell^2}{N_0 E_b}\right) d\ell + \frac{1}{2} \int_{-\infty}^{E_b/2} \frac{1}{\sqrt{\pi N_0 E_b}} \exp\left[-\frac{(\ell - E_b)^2}{N_0 E_b}\right] d\ell$$

$$= \frac{1}{\sqrt{\pi N_0 E_b}} \int_{E_b/2}^{\infty} \exp\left(-\frac{\ell^2}{N_0 E_b}\right) d\ell$$

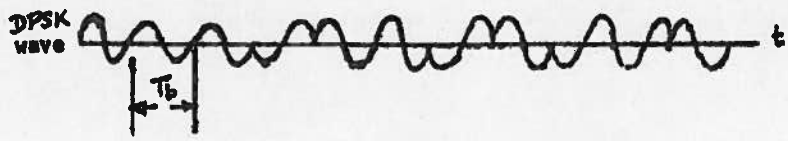
$$= \frac{1}{2} \operatorname{erfc}\left(\frac{1}{2}\sqrt{E_b/N_0}\right)$$

Problem 2

(a)

b_k		1	1	0	0	1	0	0	0	1	0
d_{k-1}		1	1	1	0	1	1	0	1	0	0
d_k	1	1	1	0	1	1	0	1	0	0	1
Transmitted phase	0	0	0	π	0	0	π	0	π	π	0

The waveform of the DPSK signal is thus as follows:



(b) Let

x_I = output of the integrator in the in-phase channel

x_Q = output of the integrator in the quadrature channel

x_I' = one-bit delayed version of x_I

x_Q' = one-bit delayed version of x_Q

l_I = in-phase channel output

$= x_I x_I'$

l_Q = quadrature channel output

$= x_Q x_Q'$

$y = l_I + l_Q$

Transmitted phase (radians)	0	0	0	π	0	0	π	0	π	π	0
Polarity of x_I	+	+	+	-	+	+	-	+	-	-	+
Polarity of x_I'		+	+	+	-	+	+	-	+	-	-
Polarity of l_I		+	+	-	-	+	-	-	-	+	-
Polarity of x_Q	-	-	-	+	-	-	+	-	+	+	-
Polarity of x_Q'		-	-	-	+	-	-	+	-	+	+
Polarity of l_Q		+	+	-	-	+	-	-	-	+	-
Polarity of y		+	+	-	-	+	-	-	-	+	-
Reconstructed data stream		1	1	0	0	1	0	0	0	1	0

Problem 3:

The transmitted binary PSK signal is defined by

$$s(t) = \begin{cases} \sqrt{E_b}\phi(t), & 0 \le t \le T_b, \quad \text{symbol } 1 \\ -\sqrt{E_b}\phi(t), & 0 \le t \le T_b, \quad \text{symbol } 0 \end{cases}$$

where the basis function $\phi(t)$ is defined by

$$\phi(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$$

The locally generated basis function in the receiver is

$$\phi_{\text{rec}}(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t + \varphi)$$

$$= \sqrt{\frac{2}{T_b}}[\cos(2\pi f_c t)\cos\varphi - \sin(2\pi f_c t)\sin\varphi]$$

where φ is the phase error. The correlator output is given by

$$y = \int_0^{T_b} x(t)\varphi_{\text{rec}}(t)dt$$

where

$$x(t) = s_k(t) + w(t), \qquad k = 1, 2$$

Assuming that f_c is an integer multiple of $1/T_b$, and recognizing that $\sin(2\pi f_c t)$ is orthogonal to $\cos(2\pi f_c t)$ over the interval $0 \le t \le T_b$, we get

$$y = \pm\sqrt{E_b}\cos\varphi + W$$

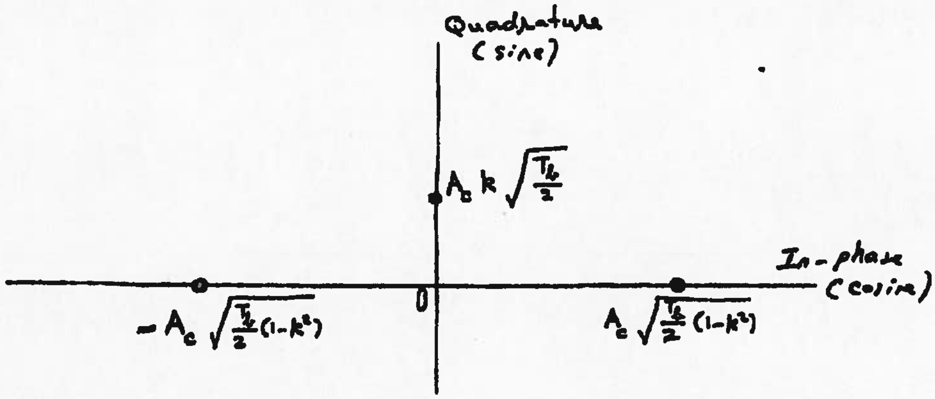
when the plus sign corresponds to symbol 1 and the minus sign corresponds to symbol 0, and W is a zero-mean Gaussian variable of variance $N_0/2$. Accordingly, the average probability of error of the binary PSK system with phase error φ is given by

$$P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b \cos\varphi}{N_0}}\right)$$

When $\varphi = 0$, this formula reduces to that for the standard PSK system equipped with perfect phase recovery. At the other extreme, when $\varphi = \pm 90°$, P_e attains its worst value of unity.

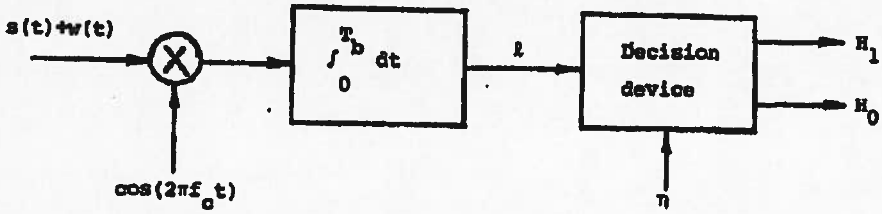
Problem 4:

(a) The signal-space diagram of the scheme described in this problem is two-dimensional, as shown by



This signal-space diagram differs from that of the conventional PSK signaling scheme in that it is two-dimensional, with a new signal point on the quadrature axis at $A_c k\sqrt{T_b/2}$. If k is reduced to zero, the above diagram reduces to the same form as that shown in Fig. 8.14.

(b)



The signal at the decision device input is

$$\ell = \pm\frac{A_c}{2}\sqrt{1-k^2}\, T_b + \int_0^{T_b} w(t)\cos(2\pi f_c t)dt \qquad (1)$$

Therefore, following a procedure similar to that used for evaluating the average probability of error for a conventional PSK system, we find that for the system defined by Eq. (1) the average probability of error is

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{E_b(1-k^2)/N_0}\right)$$

where $E_b = \frac{1}{2} A_c^2 T_b$.

(c) For the case when $P_e = 10^{-4}$ and $k^2 = 0.1$, we get

$$10^{-4} = \frac{1}{2}\,\mathrm{erfc}(u)$$

where $u^2 = \dfrac{0.9\,E_b}{N_0}$

Using the approximation

$$\mathrm{erfc}(u) = \frac{\exp(-u^2)}{\sqrt{\pi}\,u}$$

we obtain

$$\exp(-u^2) - 2\sqrt{\pi} \times 10^{-4}\,u = 0$$

The solution to this equation is $u = 2.64$. The corresponding value of E_b/N_0 is

$$\frac{E_b}{N_0} = \frac{(2.64)^2}{0.9} = 7.74$$

Expressed in decibels, this value corresponds to 8.9 dB.

(d) For a conventional PSK system, we have

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{E_b/N_0}\right)$$

In this case, we find that

$$\frac{E_b}{N_0} = (2.64)^2 = 6.92$$

Expressed in decibels, this value corresponds to 8.4 dB. Thus, the conventional PSK system requires 0.5 dB less in E_b/N_0 then the modified scheme described herein.

Problem 5:

The transmission bandwidth of 256-QAM signal is

$$B = \frac{2R_b}{\log_2 M}$$

where R_b is the bit rate given by $1/T_b$ and $M = 256$. Thus

$$B_{256} = \frac{2(1/T_b)}{\log_2 256} = \frac{2}{16T_b} = \frac{1}{8T_b}$$

The transmission bandwidth of 64-QAM is

$$B_{64} = \frac{2(1/T_b)}{\log_2 64} = \frac{2}{8T_b} = \frac{1}{4T_b}$$

Hence, the bandwidth advantage of 256-QAM over 64-QAM is

$$\frac{1}{4T_b} - \frac{1}{8T_b} = \frac{1}{8T_b}$$

The average energy of 256-QAM signal is

$$E_{256} = \frac{2(M-1)E_0}{3} = \frac{2(256-1)E_0}{3}$$

$$= 170E_0$$

where E_0 is the energy of the signal with the lowest amplitude. For the 64-QAM signal, we have

$$E_{64} = \frac{2(63)}{3}E_0 = 42E_0$$

Therefore, the increase in average signal energy resulting from the use of 256-QAM over 64-QAM, expressed in dBs, is

$$10\log_{10}\left(\frac{170E_0}{42E_0}\right) \approx 10\log_{10}(4)$$

$$= 6 \text{ dB}$$

Problem 5:

The probability of symbol error for 16-QAM is given by

$$P_e = 2\left(1-\frac{1}{\sqrt{M}}\right)\text{erfc}\left(\sqrt{\frac{3E_{av}}{2(M-1)N_0}}\right)$$

Setting $P_e = 10^{-3}$, we get

$$10^{-3} = 2\left(1-\frac{1}{4}\right)\text{erfc}\left(\sqrt{\frac{3E_{av}}{30N_0}}\right)$$

Solving this equation for E_{av}/N_0,

$$\frac{E_{av}}{N_0} = 58 = 17.6\text{dB}$$

The probability of symbol error for 16-PSK is given by

$$P_e = \text{erfc}\left(\sqrt{\frac{E}{N_0}}\sin(\pi/M)\right)$$

Setting $P_e = 10^{-3}$, we get

$$10^{-3} = \text{erfc}\left(\sqrt{\frac{E}{N_0}}\sin(\pi/16)\right)$$

Solving this equation for E/N_0, we get

$$\frac{E}{N_0} = 142 = 21.5\text{dB}$$

Hence, on the average, the 16-PSK demands 21.5 - 17.6 = 3.9 dB more symbol energy than the 16-QAM for $P_e = 10^{-3}$.

Thus the 16-QAM requires about 4 dB less in signal energy than the 16-PSK for a fixed N_0 and $P_e = 10^{-3}$, However, for this advantage of the 16-QAM over the 16-PSK to be realized, the channel must be linear.

Problem 7

The bit duration is

$$T_b = \frac{1}{2.5\times10^6 \text{ Hz}} = 0.4\ \mu s$$

The signal energy per bit is

$$E_b = \frac{1}{2} A_c^2\ T_b$$
$$= \frac{1}{2}(10^{-6})^2 \times 0.4 \times 10^{-6} = 2 \times 10^{-19} \text{ joules}$$

(a) <u>Coherent Binary FSK</u>

The average probability of error is

$$P_e = \frac{1}{2}\,\text{erfc}(\sqrt{E_b/2N_0})$$

$$= \frac{1}{2}\,\text{erfc}(\sqrt{2\times10^{-19}/4\times10^{-20}})$$

$$= \frac{1}{2}\,\text{erfc}(\sqrt{5})$$

Using the approximation

$$\text{erfc}(u) = \frac{\exp(-u^2)}{\sqrt{\pi}\,u}$$

we obtain the result

$$P_e = \frac{1}{2}\frac{\exp(-5)}{\sqrt{5\pi}} = 0.85 \times 10^{-3}$$

(b) <u>MSK</u>

$$P_e = \text{erfc}(\sqrt{E_b/N_0})$$

$$= \text{erfc}(\sqrt{10})$$

$$= \frac{\exp(-10)}{\sqrt{10\pi}}$$

$$= 0.81 \times 10^{-5}$$

(c) <u>Noncoherent Binary FSK</u> .

$$P_e = \frac{1}{2}\exp(-\frac{E_b}{2N_0})$$

$$= \frac{1}{2}\exp(-5)$$

$$= 3.37 \times 10^{-3}$$

Problem 8:

(a) The correlation coefficient of the signals $s_0(t)$ and $s_1(t)$ is

$$\rho = \frac{\int_0^{T_b} s_0(t)s_1(t)dt}{[\int_0^{T_b} s_0^2(t)dt]^{1/2}\ [\int_0^{T_b} s_1^2(t)dt]^{1/2}}$$

$$= \frac{A_c^2 \int_0^{T_b} \cos[2\pi(f_c+\frac{1}{2}\Delta f)t]\cos[2\pi(f_c-\frac{1}{2}\Delta f)t]\ dt}{[\frac{1}{2}A_c^2T_b]^{1/2}\ [\frac{1}{2}A_c^2T_b]^{1/2}}$$

$$= \frac{1}{T_b}\int_0^{T_b} [\cos(2\pi\Delta ft) + \cos(4\pi f_c t)]\ dt$$

$$= \frac{1}{2\pi T_b}[\frac{\sin(2\pi\Delta fT_b)}{\Delta f} + \frac{\sin(4\pi f_c T_b)}{2f_c}] \qquad (1)$$

Since $f_c >> \Delta f$, then we may ignore the second term in Eq. (1), obtaining

$$\rho = \frac{\sin(2\pi\Delta fT_b)}{2\pi T_b\ \Delta f} = \text{sinc}(2\Delta fT_b)$$

(b) The dependence of ρ on Δf is as shown in Fig. 1.

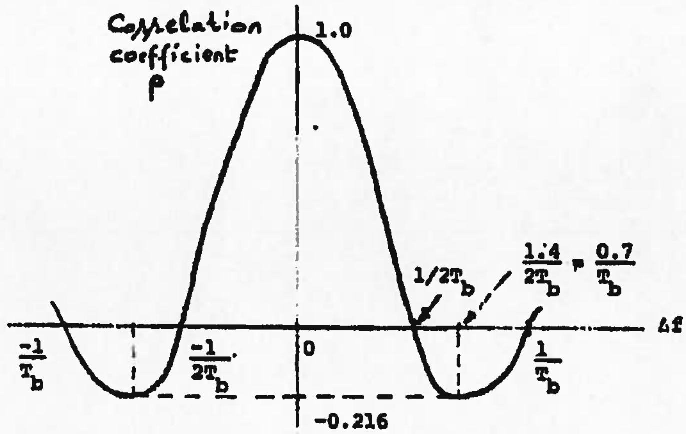

Correlation coefficient ρ
1.0
1/2Tb
1.4/2Tb = 0.7/Tb
Δf
-1/Tb
-1/2Tb
0
1/Tb
-0.216

Fig. 1

$s_0(t)$ and $s_1(t)$ are othogonal when $\rho = 0$. Therefore, the minimum value of Δf for which they are orthogonal, is $1/2T_b$.

(c) The average probability of error is given by

$$E_b = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{E_b(1-\rho)/2N_0}\right)$$

The most negative value of ρ is -0.216, occuring at $\Delta f = 0.7/T_b$. The minimum value of P_e is therefore

$$P_{e,min} = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{0.608E_b/N_0}\right)$$

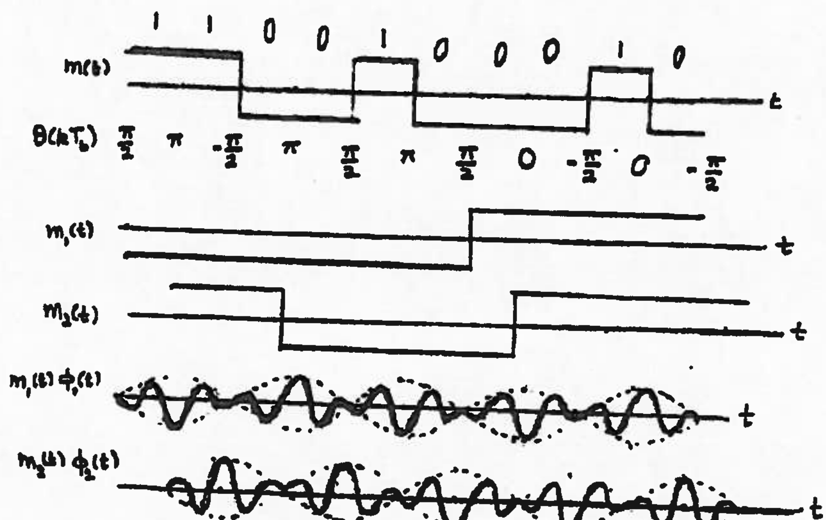
(d) For a coherent binary PSK system, the average probability of error is

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{E_b/N_0}\right)$$

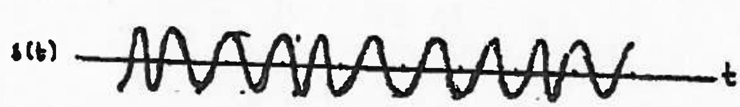
Therefore, the E_b/N_0 of this coherent binary FSK system must be increased by the factor $1/0.608 = 1.645$ (or 2.16 dB) so as to realize the same average probability of error as a coherent binary PSK system.

Problem 9:

(a)



(b)



Problem 10

(a) For a coherent PSK system, the average probability of error is

$$P_e = \frac{1}{2}\,\mathrm{erfc}[\sqrt{(E_b/N_0)_1}] = Q\left(\sqrt{2\left(\frac{E_b}{N_0}\right)_1}\right)$$

$$\simeq \frac{1}{2}\,\frac{\exp[-(E_b/N_0)_1]}{\sqrt{\pi}\,\sqrt{(E_b/N_0)_1}} \qquad (1)$$

For a DPSK system, we have

$$P_e = \frac{1}{2}\exp[-(E_b/N_0)_2] \qquad (2)$$

Let

$$\left(\frac{E_b}{N_0}\right)_2 = \left(\frac{E_b}{N_0}\right)_1 + \delta$$

Then, we may use Eqs. (1) and (2) to obtain

$$\sqrt{\pi}\,\sqrt{(E_b/N_0)_1} = \exp\delta$$

We are given that

$$\left(\frac{E_b}{N_0}\right)_1 = 7.2$$

Hence,

$$\delta = \ln[\sqrt{7.2\pi}]$$

$$= 1.56$$

Therefore,

$$10\log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10\log_{10}7.2 = 8.57 \text{ dB}$$

$$10\log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10\log_{10}(7.2 + 1.56)$$

$$= 9.42 \text{ dB}$$

The separation between the two (E_b/N_0) ratios is therefore $9.42 - 8.57 = 0.85$ dB.

(b) For a coherent PSK system, we have

$$P_e = \frac{1}{2}\,\mathrm{erfc}[\sqrt{(E_b/N_0)_1}]$$

$$\simeq \frac{1}{2}\,\frac{\exp[-(E_b/N_0)_1]}{\sqrt{\pi}\,\sqrt{(E_b/N_0)_1}} \quad (3)$$

For a QPSK system, we have

$$P_e = \mathrm{erfc}[\sqrt{(E_b/N_0)_2}]$$

$$\simeq \frac{\exp[-(E_b/N_0)_2]}{\sqrt{\pi}\,\sqrt{(E_b/N_0)_2}} \quad (4)$$

Here again, let

$$\left(\frac{E_b}{N_0}\right)_2 = \left(\frac{E_b}{N_0}\right)_1 + \delta$$

Then we may use Eqs. (3) and (4) to obtain

$$\frac{1}{2} = \frac{\exp(-\delta)}{\sqrt{1 + \delta/(E_b/N_0)_1}} \qquad (5)$$

Taking logarithms of both sides:

$$-\ln 2 = -\delta - 0.5 \ln[1 + \delta/(E_b/N_0)_1]$$

$$= -\delta - 0.5 \frac{\delta}{(E_b/N_0)_1}$$

Solving for δ:

$$\delta = \frac{\ln 2}{1 + 0.5/(E_b/N_0)_1}$$

$$= 0.65$$

Therefore,

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10 \log_{10}(7.2) = 8.57 \text{ dB}$$

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10 \log_{10}(7.2 + .65)$$

$$= 8.95 \text{ dB.}$$

The separation between the two (E_b/N_0) ratios is 8.95 - 8.57 = 0.38 dB.

(c) For a coherent binary FSK system, we have

$$P_e = \frac{1}{2} \text{erfc}[\sqrt{(E_b/2N_0)_1}]$$

$$= \frac{1}{2} \frac{\exp(-\frac{1}{2}(\frac{E_b}{N_0})_1)}{\sqrt{\pi}\sqrt{(E_b/2N_0)_1}} \qquad (6)$$

For a noncoherent binary FSK system, we have

$$P_e = \frac{1}{2} \exp(-\frac{1}{2}(\frac{E_b}{N_0})_2) \qquad (7)$$

Hence,

$$\sqrt{\frac{\pi}{2}\left(\frac{E_b}{N_0}\right)_1} = \exp(\frac{\delta}{2}) \qquad (8)$$

We are given that $(E_b/N_0)_1 = 13.5$. Therefore,

$$\delta = \ln\left(\frac{13.5\ \pi}{2}\right)$$

$$= 3.055$$

We thus find that

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10 \log_{10}(13.5)$$

$$= 11.3 \text{ dB}$$

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10 \log_{10}(13.5 + 3.055)$$

$$= 12.2 \text{ dB}$$

Hence, the separation between the two (E_b/N_0) ratios is 12.2 − 11.3 = 0.9 dB.

(d) For a coherent binary FSK system, we have

$$P_e = \frac{1}{2} \operatorname{erfc}\left[\sqrt{(E_b/2N_0)_1}\right]$$

$$= \frac{1}{2} \frac{\exp\left(-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_1\right)}{\sqrt{\pi}\ \sqrt{(E_b/2N_0)_1}} \qquad (9)$$

For a MSK system, we have

$$P_e = \frac{1}{2} \operatorname{erfc}\left[\sqrt{(E_b/2N_0)_2}\right] \qquad (10)$$

$$= \frac{\exp\left[-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_2\right]}{\sqrt{\pi}\ \sqrt{(E_b/2N_0)_2}} \qquad (10)$$

Hence, using Eqs. (9) and (10), we

$$\ln 2 - \frac{1}{2} \ln\left[1 + \frac{\delta}{(E_b/N_0)_1}\right] = \frac{1}{2}\delta \qquad (11)$$

Noting that

$$\frac{\delta}{(E_b/N_0)_1} \ll 1$$

we may approximate Eq. (11) to obtain

$$\ln 2 - \frac{1}{2}\left[\frac{\delta}{(E_b/N_0)_1}\right] = \frac{1}{2}\delta \qquad (11)$$

Solving for δ, we obtain

$$\delta = \frac{2 \ln 2}{1 + \frac{1}{(E_b/N_0)_1}}$$

$$= \frac{2 \times 0.693}{1 + \frac{1}{13.5}}$$

$$= 1.29$$

We thus find that

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10 \log_{10}(13.5) = 10 \times 1.13 = 11.3 \text{ dB}$$

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10 \log_{10}(13.5 + 1.29) = 11.7 \text{ dB}$$

Therefore, the separation between the two (E_b/N_0) ratios is 11.7 - 11.3 = 0.4 dB.