Lecture 11

Now, let's go back to BPSK and ~~to~~ try to get more insight into the modulation process and its parameters

$$s(t) = b(t)\,p(t)\,A\cos(2\pi f_c t) \qquad kT_b \le t \le (k+1)T_b$$

Where $b(t) = b_k$ is the k-th bit. $b_k = \pm 1$.

Assuming that $p(t)$ is normalized so that its energy is equal to unity we can represent $s(t)$ as

$$s(t) = \pm A\cos(2\pi f_c t) \qquad kT_b \le t \le (k+1)T_b$$

to find energy of a bit (energy we spend to transmit a bit) is:

$$E_b = \int_{kT_b}^{(k+1)T_b} s^2(t)\,dt = \int_{kT_b}^{(k+1)T_b} A^2\cos^2(2\pi f_c t)\,dt = \frac{A^2}{2}T_b$$

So, $A = \sqrt{\dfrac{2E_b}{T_b}}$ So,

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t) = \sqrt{E_b}\,\phi(t)$$

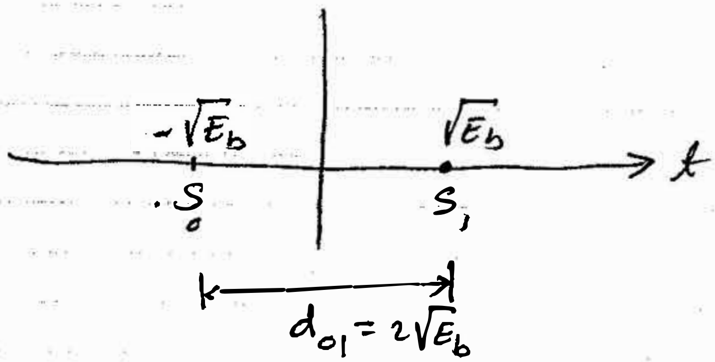
$$s_0(t) = -\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t) = -\sqrt{E_b}\,\phi(t)$$

where $\phi(t) = \sqrt{\dfrac{2}{T_b}}\cos(2\pi f_c t)$

Note that

$$\int_0^{T_b} \phi^2(t)\,dt = 1$$

We can show $s_1(t)$ ~~as~~ and $s_0(t)$ as



Note that:

$$d^2(s_0(t), s_1(t)) = \int_0^{T_b} (s_1(t) - s_0(t))^2\, dt$$

$$= \int_0^{T_b} \frac{8E_b}{T_b} \cos^2(2\pi f_c t)\, dt$$

$$= \frac{4E_b}{T_b} T_b = 4E_b$$

or $d(s_0(t), s_1(t)) = 2\text{sqrt}(E_b)$

Detection of BPSK in AWGN ~~noise~~

Assume that

$$s_i(t) = \pm\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$$

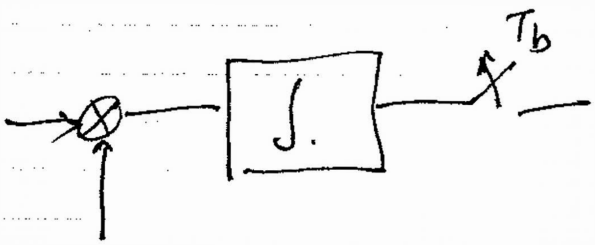
is transmitted and channel adds noise $n(t)$ to it. So, the received signal will be

$$r(t) = \pm\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) + n(t)$$

We say that $n(t)$ is AWGN if its samples are independent and distributed according to Gaussian law. In order for a Gaussian signal to have independent samples, its samples have to be uncorrelated, i.e.,

$$E[n(t)\,n(u)] = \frac{N_0}{2}\,\delta(t-u)$$

At the receiver $r(t)$ is multiplied by $\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$ and integrated over a T_b seconds interval



$$\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$$

The first term will be

$$\pm\int_0^{T_b}\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t)\,\mathrm{sqrt}(2/T_b)\cos(2\pi f_c t)\,dt = \pm\sqrt{E_b}$$

The second term will be

$$z = \int_0^{T_b} n(t)\sqrt{\frac{2}{T_b}}\cos 2\pi f_c t\,dt$$

z is a Gaussian random variable with zero-mean and variance $\frac{N_0}{2}$

$$E[z] = \int_0^{T_b} E[n(t)] \sqrt{\frac{2}{T_b}} \cos 2\pi f_c t = 0$$

$$E[z^2] = E\left[\int_0^{T_b} n(t) \sqrt{\frac{2}{T_b}} \cos 2\pi f_c t \, dt \int_0^{T_b} n(u) \sqrt{\frac{2}{T_b}} \cos 2\pi f_c u\right]$$

$$= \frac{2}{T_b} \int_0^{T_b} \int_0^{T_b} \cos 2\pi f_c t \cos 2\pi f_c u \, E[n(t) n(u)] \, d$$

$$= \frac{2}{T_b} \frac{N_0}{2} \int_0^{T_b} \int_0^{T_b} \cos 2\pi f_c t \cos 2\pi f_c u \, \delta(t-u) \, d$$

$$= \frac{2}{T_b} \cdot \frac{N_0}{2} \int_0^{T_b} \cos^2 2\pi f_c t \, dt$$

$$= \frac{2}{T_b} \cdot \frac{N_0}{2} \frac{T_b}{2} = \frac{N_0}{2}$$

So, when 1 is transmitted we get

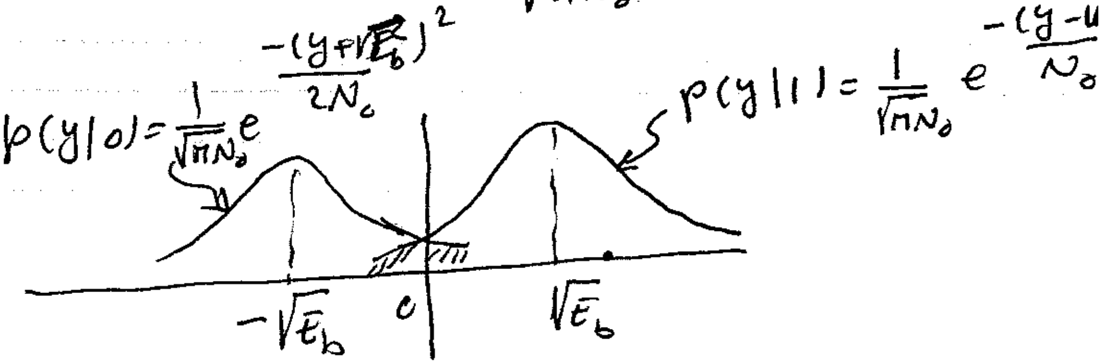
$$y = \sqrt{E_b} + z$$

and when zero is sent we get

$$y = -\sqrt{E_b} + z$$

where

$$p(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}} = \frac{1}{\sqrt{\pi N_o}} e^{-\frac{z^2}{N_o}}$$



$$P_e = \frac{1}{2} P[y > 0 \mid b = 0] + \frac{1}{2} P[y < 0 \mid b = 1]$$

$$= P[y > 0 \mid b = 0] = \int_0^\infty \frac{1}{\sqrt{\pi N_o}} e^{-\frac{(y+\sqrt{E_b})^2}{N_o}} dy$$

let $u = \frac{y + \sqrt{E_b}}{\sqrt{N_o/2}}$ then $du = \frac{1}{\sqrt{\frac{N_o}{2}}} dy$

$$P_e = \int_{\sqrt{\frac{2E_b}{N_o}}}^\infty \frac{1}{\sqrt{\pi N_o}} e^{-u^2/2} du \sqrt{\frac{N_o}{2}}$$

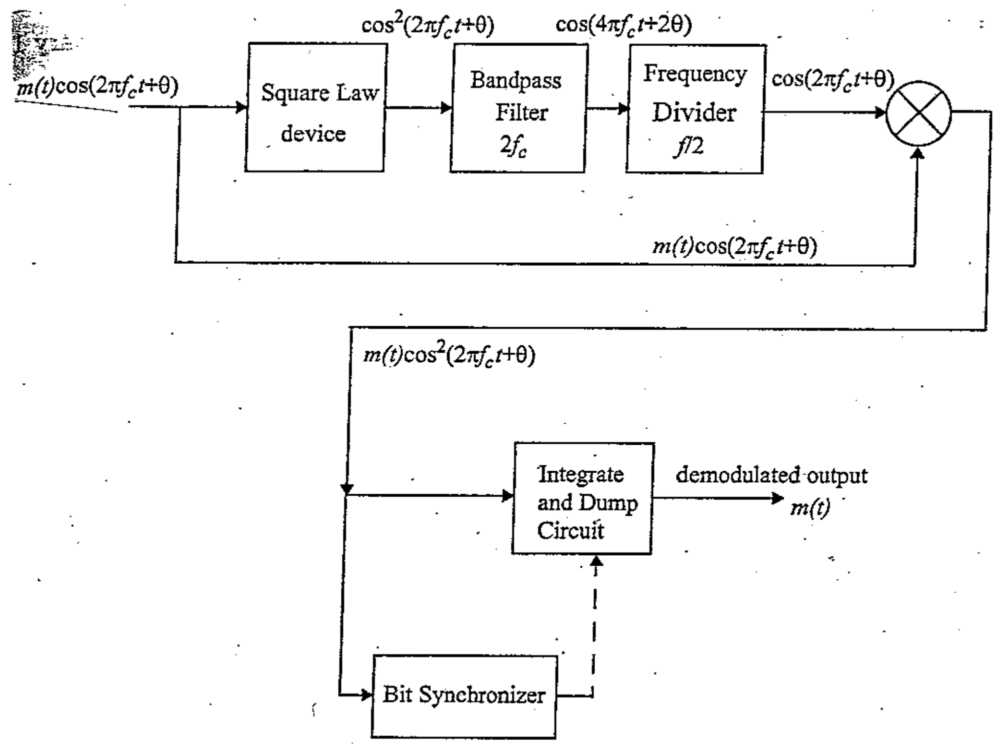
$$= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2E_b}{N_o}}}^\infty e^{-u^2/2} du = Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$$

$P_e = Q\left(\frac{d_{ij}}{\sqrt{2N_o}}\right) \longrightarrow Q\left(\frac{d_{ij}}{\sqrt{2N_o}}\right)$ in general where d_{ij} is the distance between the two points.

BPSK Receiver

If no multipath impairments are induced by the BPSK signal can be expressed as

$$S_{\text{BPSK}}(t) = m(t)\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t + \theta_c + \theta_{ch})$$
$$= m(t)\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t + \theta)$$



BPSK receiver with carrier recovery circuits.

$$P_{e,\text{BPSK}} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

QPSK (4-phase PSK)

Each of the four symbols, separated by $\frac{\pi}{2}$ in phase represent two bits. So, BW requirement is

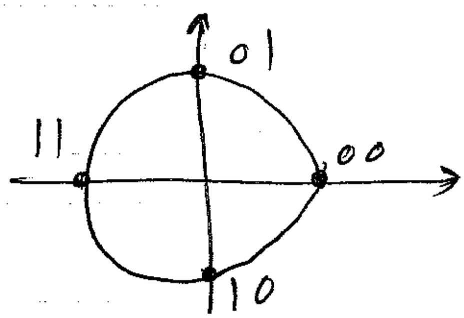
$$B = \frac{R_b}{2}(1+\alpha)$$

in general for M-ary PSK (or any M-ary two-dimensional constellation)

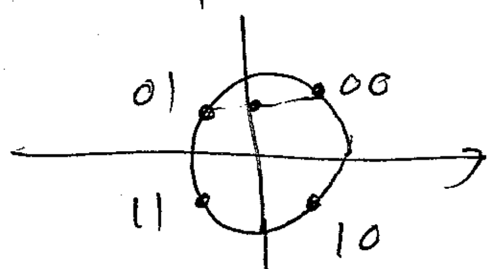
$$B = \frac{R_b}{\log_2 M}(1+\alpha)$$

$$S_i(t) = A\cos(2\pi f_c t + \frac{\pi}{2} i + \lambda) \qquad i = 0, 1, 2, 3$$

let $\lambda = 0$



and for $\lambda = \frac{\pi}{4}$



Energy per Symbol is

$$E_s = \int_0^{T_s} s_i^2(t)\,dt = \frac{A^2}{2} T_s$$

or (for $\lambda = \frac{\pi}{4}$)

$$s_i(t) = \sqrt{\frac{2E_s}{T_s}} \cos(2\pi f_c t + \frac{\pi}{2} i + \frac{\pi}{4})$$

$$= \sqrt{E_s} \cos(\frac{\pi}{2} i + \frac{\pi}{4}) \sqrt{\frac{2}{T_s}} \cos(2\pi f_c t)$$

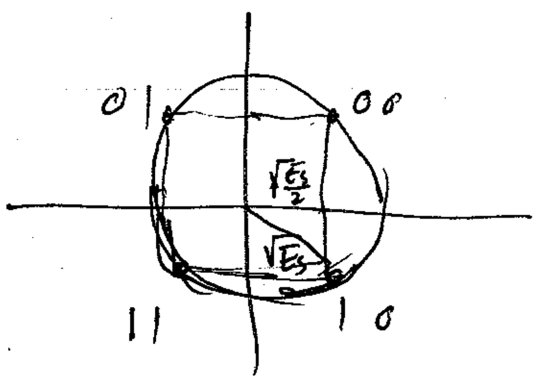
$$- \sqrt{E_s} \sin(\frac{\pi}{2} i + \frac{\pi}{4}) \sqrt{\frac{2}{T_s}} \sin(2\pi f_c t)$$

$$= \sqrt{E_s} \cos(\frac{\pi}{2} i + \frac{\pi}{4})\, \phi_1(t)$$

$$+ \sqrt{E_s} \sin(\frac{\pi}{2} i + \frac{\pi}{4})\, \phi_2(t)$$

So, the points will be:

$$\left(\pm\sqrt{\frac{E_s}{2}}, \pm\sqrt{\frac{E_s}{2}}\right)$$



$$P_{\text{QPSK}} = \frac{E_s}{2}\left[\left(\frac{\sin \pi (f-f_c)T_s}{\pi (f-f_c)T_s}\right)^2 + \left(\frac{\sin \pi (-f-f_c)T_s}{\pi (-f-f_c)T_s}\right)^2\right]$$
$$= E_b\left[\left(\frac{\sin 2\pi (f-f_c)T_b}{2\pi (f-f_c)T_b}\right)^2 + \left(\frac{\sin 2\pi (-f-f_c)T_b}{2\pi (-f-f_c)T_b}\right)^2\right]$$

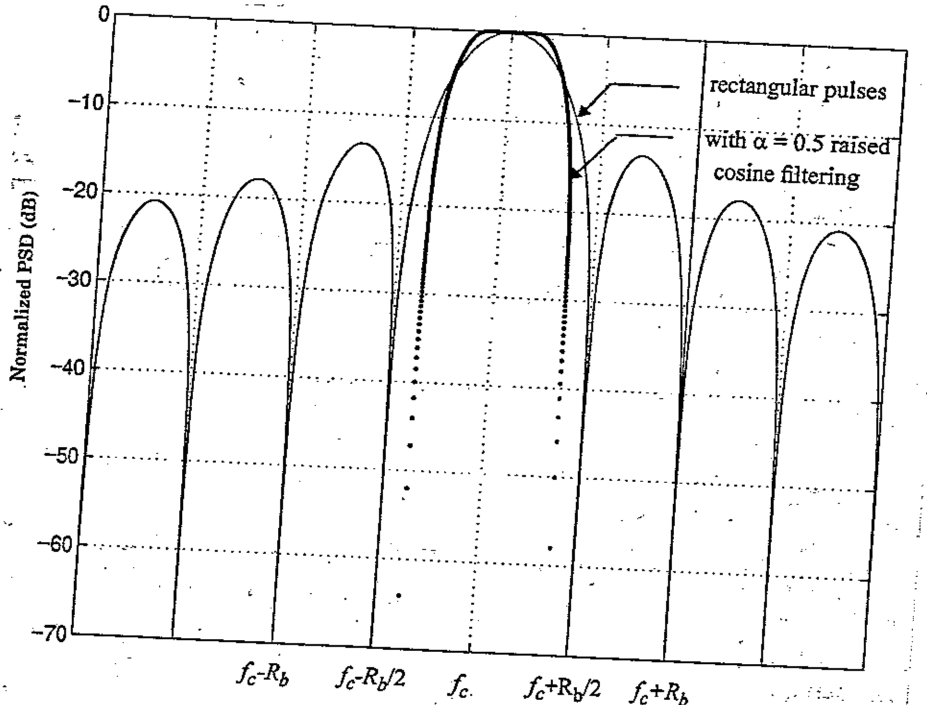


Figure 5.27
Power spectral density of a QPSK signal.

$$S_{\text{QPSK}}(t) = \sqrt{\frac{2E_s}{T_s}}\cos\left[(i-1)\frac{\pi}{2}\right]\cos(2\pi f_c t) - \sqrt{\frac{2E_s}{T_s}}\sin\left[(i-1)\frac{\pi}{2}\right]\sin(2\pi f_c t)$$

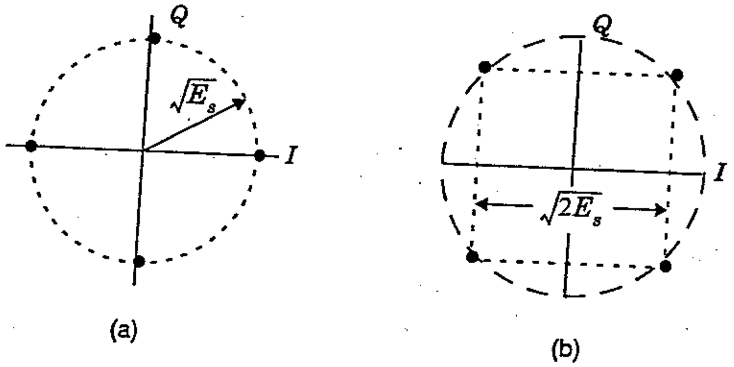


Figure 5.26
(a) QPSK constellation where the carrier phases are 0, π/2, π, 3π/2.
(b) QPSK constellation where the carrier phases are π/4, 3π/4, 5π/4, 7π/4.

$$P_{e,\text{QPSK}} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Bit Error Probability is (for the least significant bit)

$$d = 2\sqrt{\frac{E_s}{2}} = \sqrt{2E_s}$$

$$P\left[\text{noise} < \sqrt{\frac{E_s}{2}}\right] = Q\left(\frac{d}{\sqrt{2N_0}}\right) = Q\left(\frac{\sqrt{2E_s}}{\sqrt{2N_0}}\right)$$

$$= Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

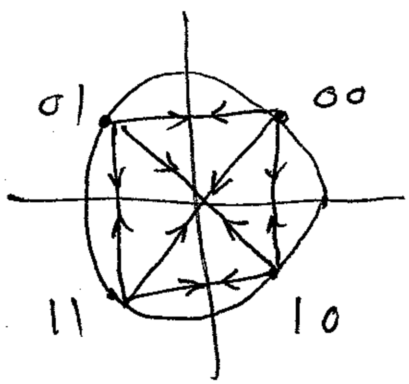
$$\boxed{= Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$

since $E_s = 2E_b$.

The same thing is true for the ~~Esb~~ most significant bit.

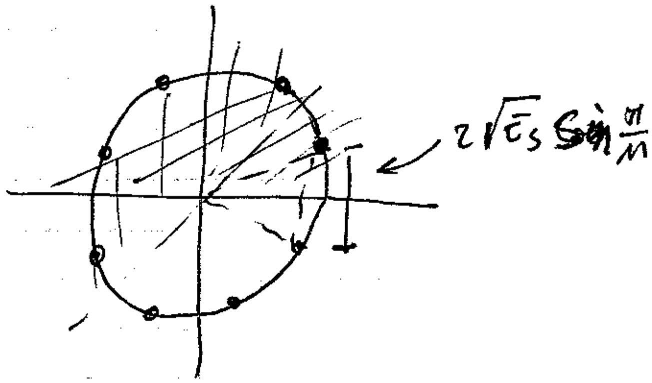
Offset QPSK (OQPSK)

The problem with QPSK is the abrupt phase changes of 90° and 180°



When the waveform passes through non-linear amplifier, the amplitude gets distorted resulting in increased error probability. To avoid this the I and Q components of the data stream are shifted (offset) by $T_s/2 = T_b$.

M-ary PSK



$$S_i(t) = \sqrt{E_s}\sqrt{\frac{2}{T_s}}\cos\left(2\pi f_c t + i\frac{2\pi}{M} + \lambda\right) \quad i = 0, \ldots, M-$$

$$\cancel{P(\varepsilon) \approx 2Q\left(\sqrt{\frac{2E_s}{N_0}}\right.}$$

$$P(\varepsilon) > Q\left(\frac{d_{ij}}{\sqrt{2N_0}}\right) = Q\left(\frac{2\sqrt{E_s}\sin\frac{\pi}{M}}{\sqrt{2N_0}}\right)$$

$$P(\varepsilon) > Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right)$$

also

$$P(\varepsilon) < 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right)$$

$$P(\varepsilon) \simeq 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right)$$

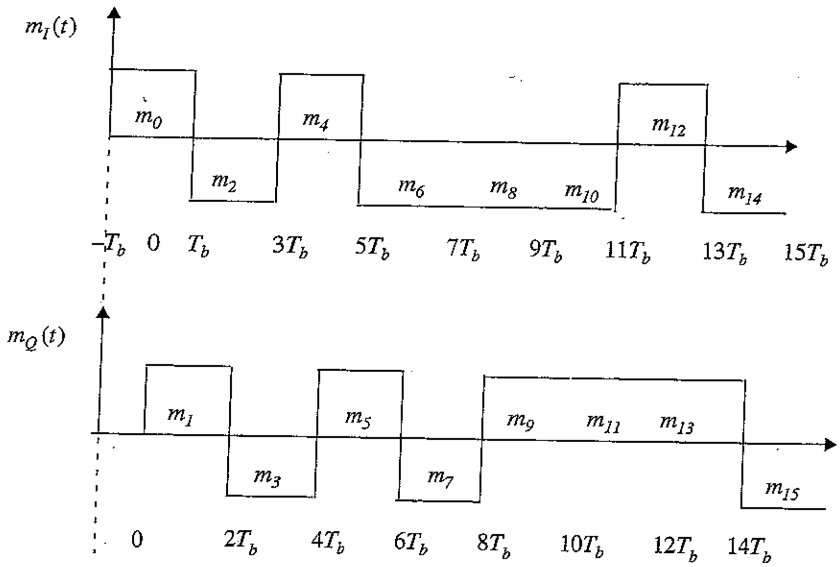


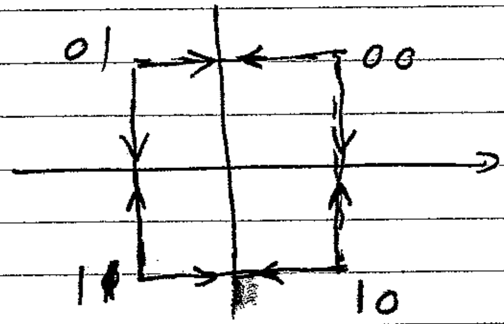
Figure 6.30 The time offset waveforms that are applied to the in-phase and quadrature arms of an OQPSK modulator. Notice that a half-symbol offset is used.

Due to the time alignment of $m_I(t)$ and $m_Q(t)$ in standard QPSK, phase transitions occur only once every $T_s = 2T_b$ s, and will be a maximum of 180° if there is a change in the value of both $m_I(t)$ and $m_Q(t)$. However, in OQPSK signaling, bit transitions (and, hence, phase transitions) occur every T_b s. Since the transition instants of $m_I(t)$ and $m_Q(t)$ are offset, at any given time only one of the two bit streams can change values. This implies that the maximum phase shift of the transmitted signal at any given time is limited to ±90°. Hence, by switching phases more frequently (i.e., every T_b s instead of $2T_b$ s) OQPSK signaling eliminates 180° phase transitions.

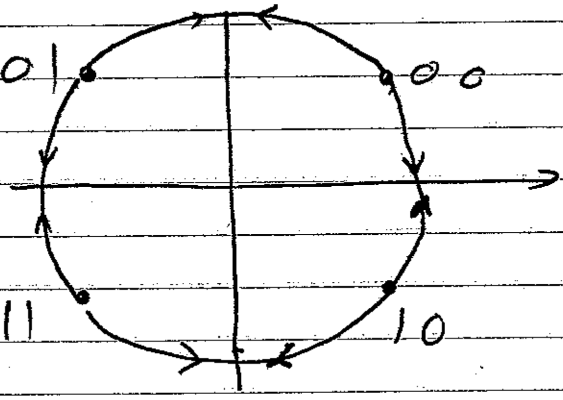
Since 180° phase transitions have been eliminated, bandlimiting of (i.e., pulse shaping) OQPSK signals does not cause the signal envelope to go to zero. Obviously, there will be some amount of ISI caused by the bandlimiting process, especially at the 90° phase transition points. But the envelope variations are considerably less, and hence hardlimiting or nonlinear amplification of OQPSK signals does not regenerate the high frequency sidelobes as much as in QPSK. Thus, spectral occupancy is significantly reduced, while permitting more efficient RF amplification.

The spectrum of an OQPSK signal is identical to that of a QPSK signal, hence both signals occupy the same bandwidth. The staggered alignment of the even and odd bit streams does not change the nature of the spectrum. OQPSK retains its bandlimited nature even after nonlinear amplification, and therefore is very attractive for mobile communication systems where bandwidth efficiency and efficient nonlinear amplifiers are critical for low power drain. Further, OQPSK signals also appear to perform better than QPSK in the presence of phase jitter due to noisy reference signals at the receiver [Chu87].

Now, we have made the allowed changes as shown below:



We can make the changes smoother by changing the phase gradually,



That is, instead of making the phase changes abruptly at kT_b time instants make it gradually between kT_b and $(k+1)T_b$

This will be done by multiplying m_I by $\cos\left(\frac{\pi t}{2T_b}\right)$ and m_Q by $\sin\left(\frac{\pi t}{2T_b}\right)$.

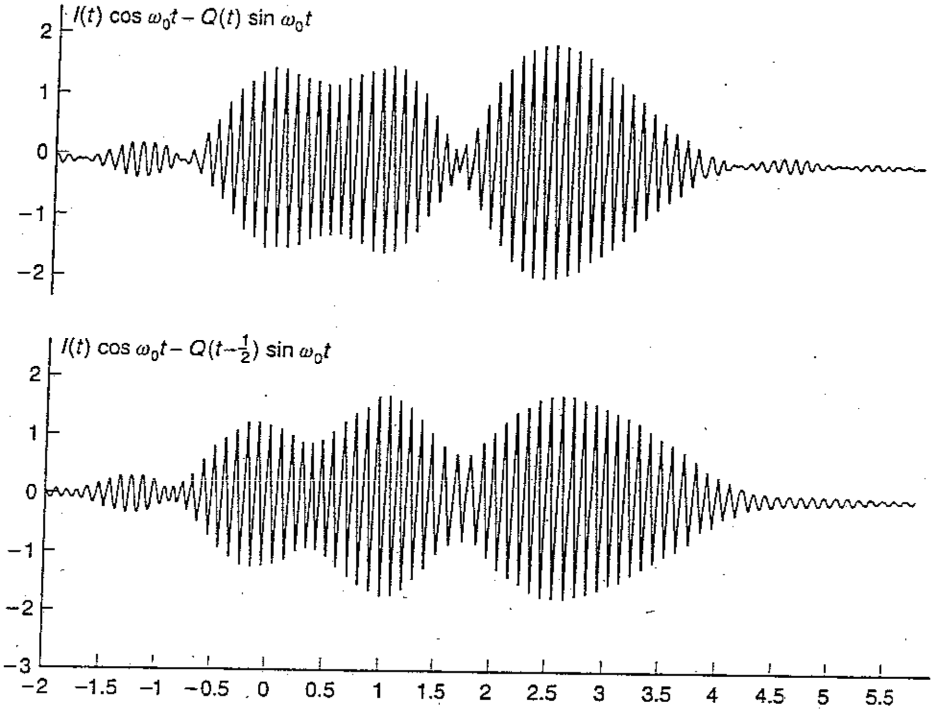


Figure 3.12 Comparison of QPSK (top) and offset QPSK (bottom) with same data and 30% root RC pulses. Data as in Figs. 3.1, 3.2, and 3.11.

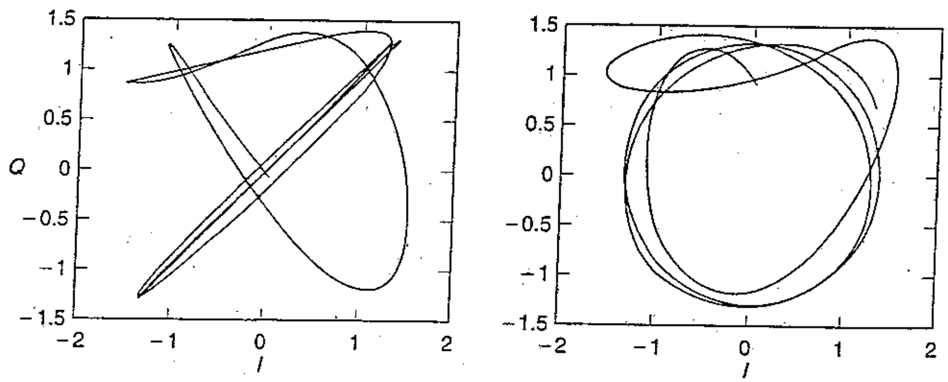


Figure 3.13 The I/Q plots for QPSK (left) and offset QPSK (right) for I data $(+-+-+--++-)$ and Q data $(+-+-++++--+)$. There are five 180 phase changes.

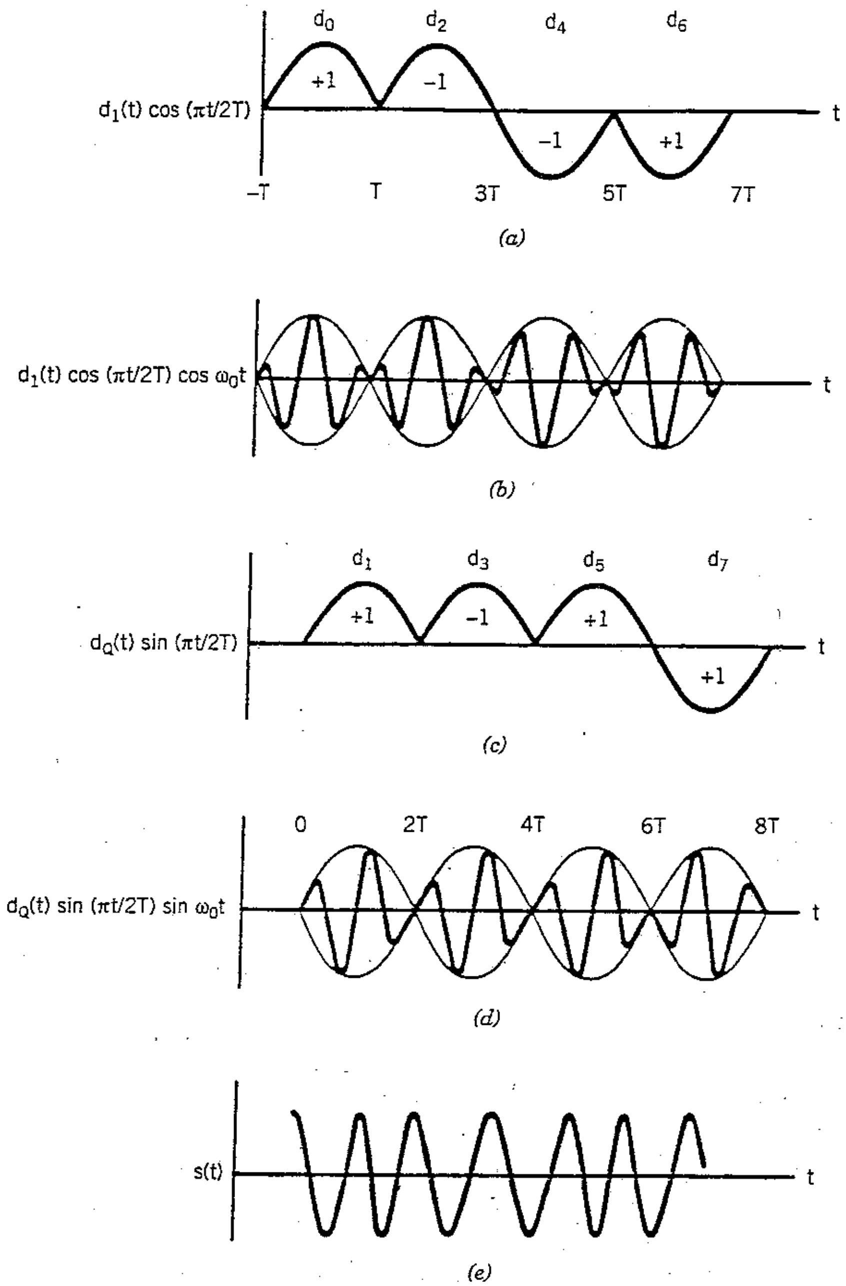


Figure 7.25 MSK waveform composition. (*a*) Modified *I* bit stream. (*b*) *I* bit stream times carrier. (*c*) Modified *Q* bit stream. (*d*) *Q* bit stream times carrier. (*e*) MSK waveforms. (From [Pas79] © IEEE.)

So, we will have

$$s_{MSK}(t) = m_I(t)\cos\left(\frac{\pi t}{2T}\right) \cancel{\cos_Q}(w_c t) + m_Q(t)\sin\left(\frac{\pi t}{T}\right)\sin w$$

if $m_I(t) = m_Q(t) = 1$

we have

$$s_{MSK}(t) = \cos\left(\frac{\pi t}{2T_b}\right)\cos(w_c t) + \sin\left(\frac{\pi t}{T}\right)\sin(w_c t)$$

$$= \cos\left(W_c - \frac{\pi}{2T_b}\right)t = \cos 2\pi\left(f_c - \frac{1}{4T_b}\right)t$$

if $m_I(t) = 1$ and $m_Q(t) = -1$

then

$$s_{MSK}(t) = \cos 2\pi\left(f_c + \frac{1}{4T_b}\right)t$$

if $m_I = m_Q = -1$

$$s_{MSK}(t) = -\cos 2\pi\left(f_c - \frac{1}{4T_b}\right)t = \cos\left[2\pi f_c t - \frac{\pi t}{2T_b} + \pi\right)$$

and for $m_I = -1$ and $m_Q = +1$

$$s_{MSK}(t) = -\cos 2\pi\left(f_c + \frac{1}{4T_b}\right)t$$

So

$$s_{MSK}(t) = \cos\left[2\pi f_c t - m_I(t)m_Q(t)\frac{\pi t}{2T_b} + \Phi_k\right]$$

where $\Phi_k = 0$ $\quad m_I(t) = 1$

$\Phi_k = \pi$ $\quad m_I(t) = -1$

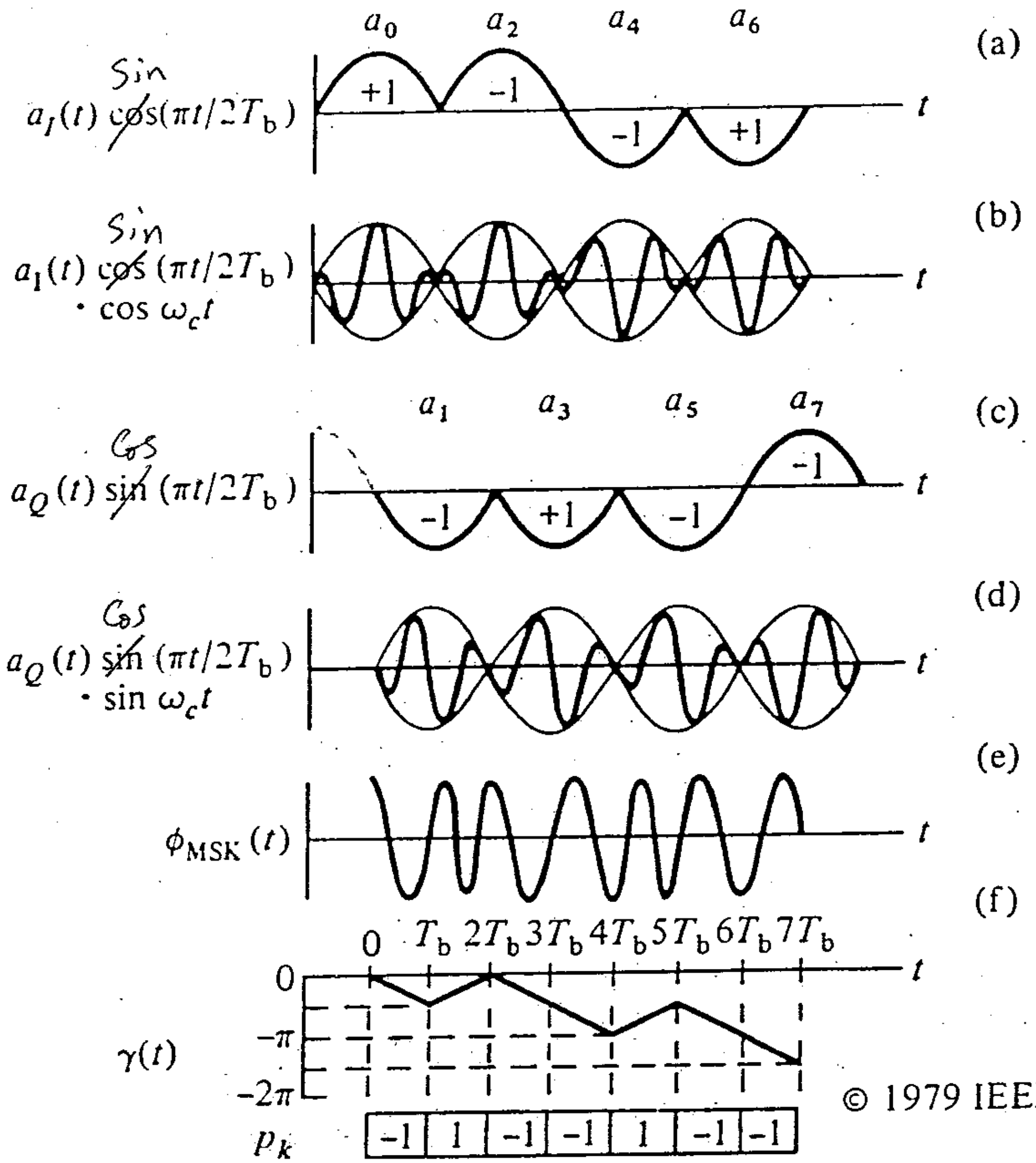


Figure 10.30 MSK waveforms.†

$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$

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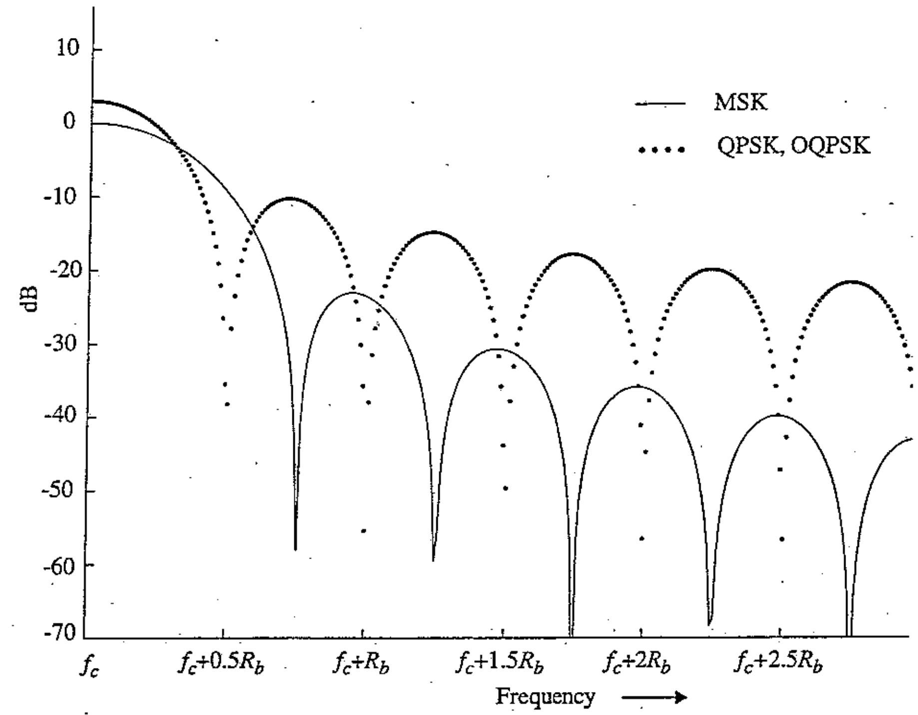
MSK Power Spectrum

$$P_{\text{MSK}} = \frac{16}{\pi^2}\left(\frac{\cos 2\pi(f+f_c)T}{1.16f^2T^2}\right)^2 + \frac{16}{\pi^2}\left(\frac{\cos 2\pi(f-f_c)T}{1.16f^2T^2}\right)^2$$



Power spectral density of MSK signals as compared to QPSK and OQPSK signals.

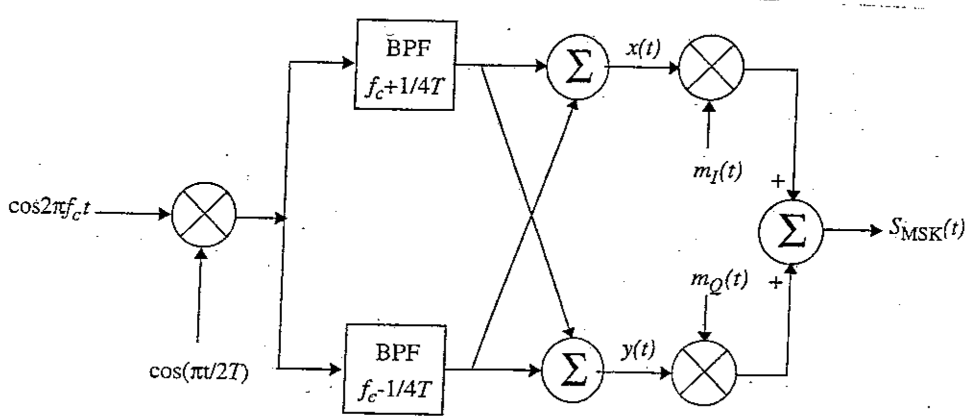


Figure 5.39
Block diagram of an MSK transmitter.

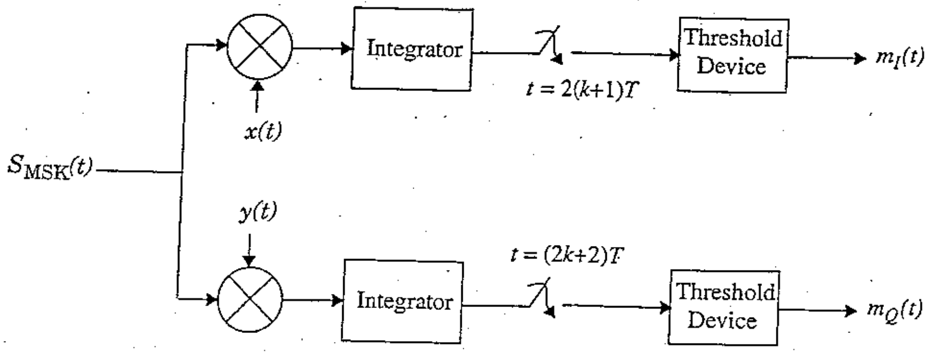


Figure 5.40
Block diagram of an MSK receiver.

Gaussian Pulse-shaping Filter

It is also possible to use non-Nyquist techniques for pulse shaping. Prominent among such techniques is the use of a Gaussian pulse-shaping filter which is particularly effective when used in conjunction with Minimum Shift Keying (MSK) modulation, or other modulations which are well suited for power efficient nonlinear amplifiers. Unlike Nyquist filters which have zero-crossings at adjacent symbol peaks and a truncated transfer function, the Gaussian filter has a smooth transfer function with no zero-crossings. The impulse response of the Gaussian filter gives rise to a transfer function that is highly dependent upon the 3-dB bandwidth. The Gaussian lowpass filter has a transfer function given by

$$H_G(f) = \exp(-\alpha^2 f^2)$$

The parameter α is related to B, the 3-dB bandwidth of the baseband gaussian shaping filter,

$$\alpha = \frac{\sqrt{ln 2}}{\sqrt{2}B} = \frac{0.5887}{B} \tag{5.53}$$

As α increases, the spectral occupancy of the Gaussian filter decreases and time dispersion of the applied signal increases. The impulse response of the Gaussian filter is given by

$$h_G(t) = \frac{\sqrt{\pi}}{\alpha}\exp\left(-\frac{\pi^2}{\alpha^2}t^2\right)$$

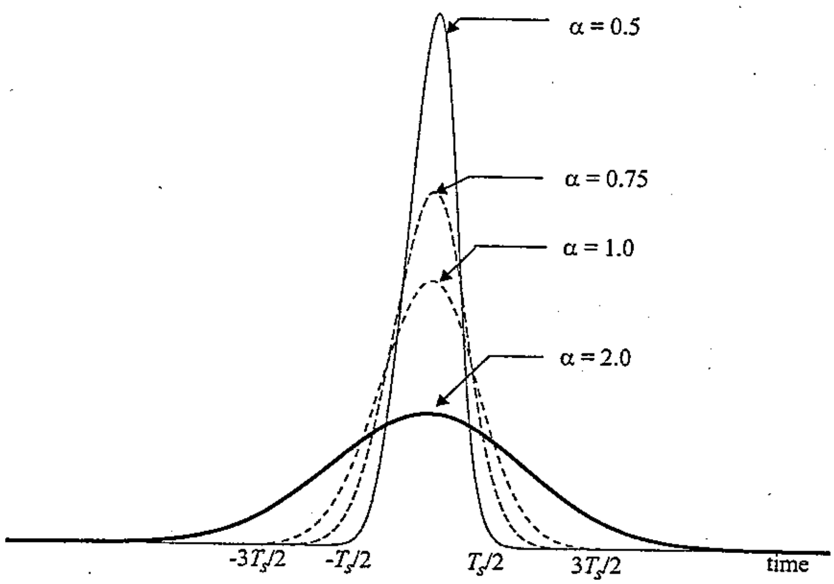


Figure 5.20
Impulse response of a Gaussian pulse- shaping filter.

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11-20

The GMSK premodulation filter has an impulse response given by

$$h_G(t) = \frac{\sqrt{\pi}}{\alpha}\exp\left(-\frac{\pi^2}{\alpha^2}t^2\right) \tag{5.108}$$

and the transfer function given by

$$H_G(f) = \exp(-\alpha^2 f^2) \tag{5.109}$$

The parameter α is related to B, the 3 dB baseband bandwidth of $H_G(f)$, by

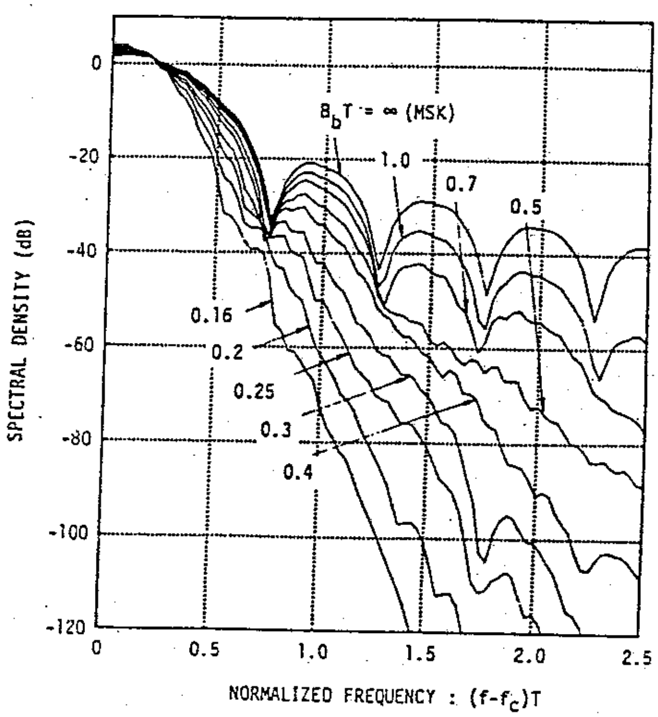
$$\alpha = \frac{\sqrt{ln2}}{\sqrt{2}B} = \frac{0.5887}{B} \tag{5.110}$$

and the GMSK filter may be completely defined from B and the baseband symbol duration T. It is therefore customary to define GMSK by its BT product.

Figure 5.41 shows the simulated RF power spectrum of the GMSK signal for various values of BT. The power spectrum of MSK, which is equivalent to GMSK with a BT product of infinity, is also shown for comparison purposes. It is clearly seen from the graph that as the BT product decreases, the sidelobe levels fall off very rapidly. For example, for a BT=0.5, the peak of the second lobe is more than 30dB below the main lobe, whereas for simple MSK, the second lobe is only 20 dB below main lobe. However, reducing BT increases the irreducible error rate produced by the low pass filter due to ISI. As shown in Section 5.11, mobile radio channels induce an irreducible error rate due to mobile velocity, so *as long as the GMSK irreducible error rate is less than that produced by the mobile channel, there is no penalty in using GMSK.* Table 5.3 shows occupied bandwidth containing a given percentage of power in a GMSK signal as a function of the BT product [Mur81].

Table 5.3 Occupied RF Bandwidth (for GMSK and MSK as a fraction of R_b) Containing a Given Percentage of Power [Mur81]. Notice that GMSK is spectrally tighter than MSK.

BT	90%	99%	99.9%	99.99%
0.2 GMSK	0.52	0.79	0.99	1.22
0.25 GMSK	0.57	0.86	1.09	1.37
0.5 GMSK	0.69	1.04	1.33	2.08
MSK	0.78	1.20	2.76	6.00



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GMSK Bit Error Rate

The bit error rate for GMSK was first found in [Mur81] for AWGN channels, and was shown to offer performance within 1 dB of optimum MSK when BT=0.25. The bit error probability is a function of BT, since the pulse shaping impacts ISI. The bit error probability for GMSK is given by

$$P_e = Q\left\{\sqrt{\frac{2\gamma E_b}{N_0}}\right\}$$

where γ is a constant related to BT by

$$\gamma \cong \begin{cases} 0.68 & \text{for GMSK with } BT = 0.25 \\ 0.85 & \text{for simple MSK } (BT = \infty) \end{cases}$$

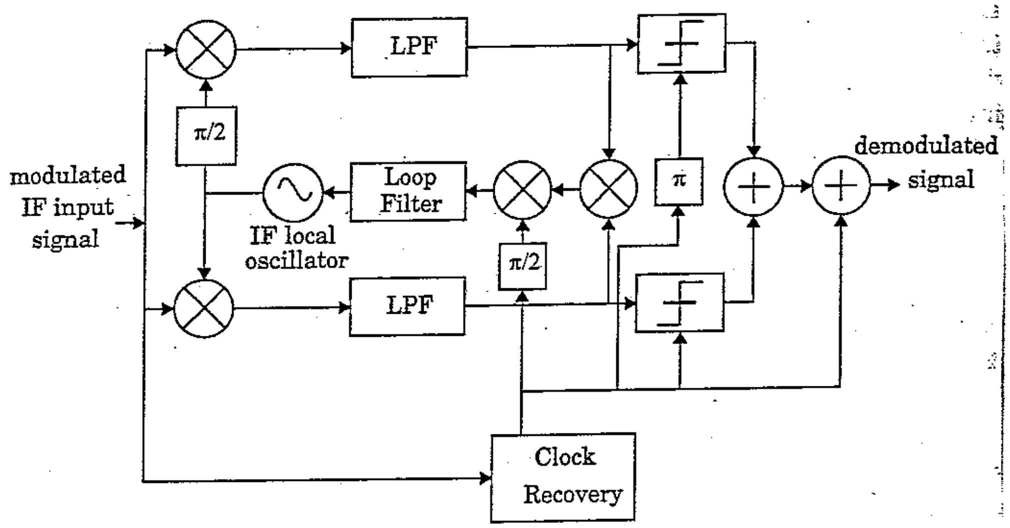

Figure 5.43
Block diagram of a GMSK receiver.

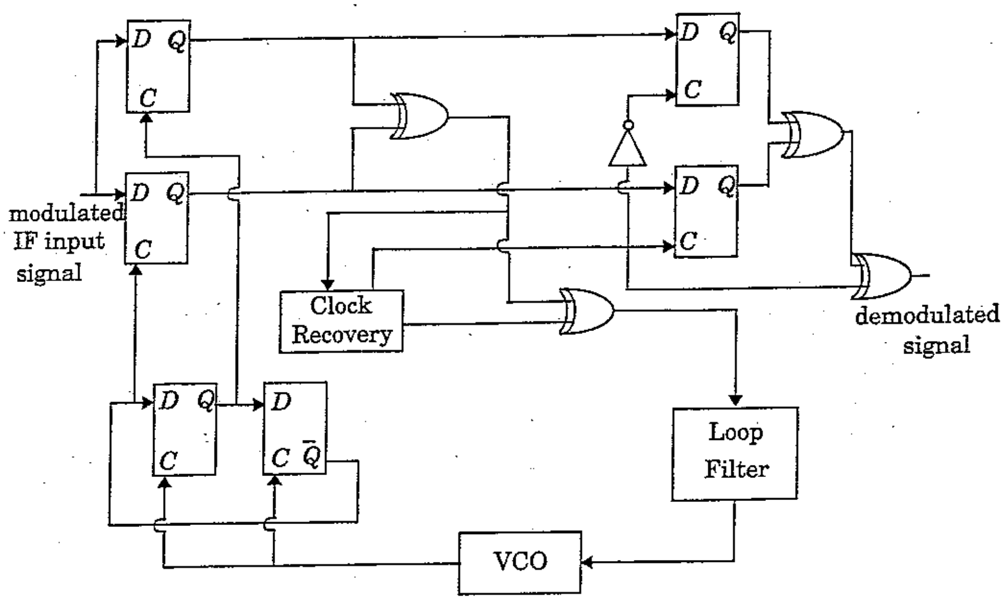

Figure 5.44
Digital logic circuit for GMSK demodulation [From [deB72] © IEEE].

5.7.2 Differential Phase Shift Keying (DPSK)

Differential PSK is a noncoherent form of phase shift keying which avoids the need for a coherent reference signal at the receiver. Noncoherent receivers are easy and cheap to build, and hence are widely used in wireless communications. In DPSK systems, the input binary sequence is first differentially encoded and then modulated using a BPSK modulator. The differentially encoded sequence $\{d_k\}$ is generated from the input binary sequence $\{m_k\}$ by complementing the modulo-2 sum of m_k and d_{k-1}. The effect is to leave the symbol d_k unchanged from the previous symbol if the incoming binary symbol m_k is 1, and to toggle d_k if m_k is 0. Table 5.1 illustrates the generation of a DPSK signal for a sample sequence m_k which follows the relationship $d_k = \overline{m_k \oplus d_{k-1}}$.

Table 5.1 Illustration of the Differential Encoding Process

$\{m_k\}$		1	0	0	1	0	1	1	0
$\{d_{k-1}\}$		1	1	0	1	1	0	0	0
$\{d_k\}$	1	1	0	1	1	0	0	0	1

A block diagram of a DPSK transmitter is shown in Figure 5.24. It consists of a one bit delay element and a logic circuit interconnected so as to generate the differentially encoded sequence from the input binary sequence. The output is passed through a product modulator to obtain the DPSK signal. At the receiver, the original sequence is recovered from the demodulated differentially encoded signal through a complementary process, as shown in Figure 5.25.

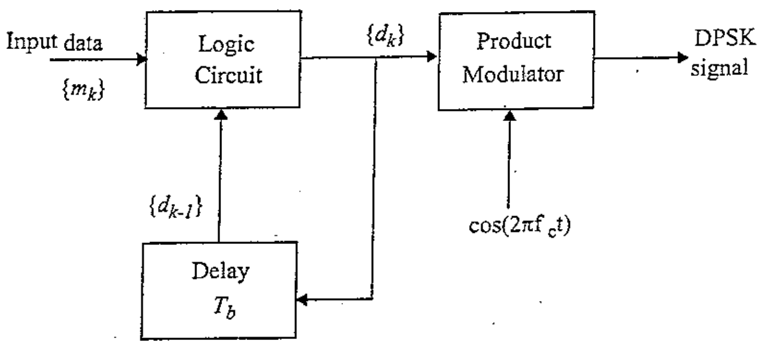

Figure 5.24
Block diagram of a DPSK transmitter.

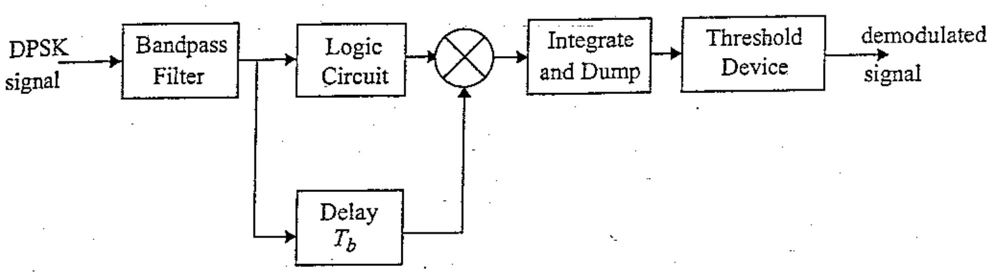

Figure 5.25
Block diagram of DPSK receiver.

While DPSK signaling has the advantage of reduced receiver complexity, its energy efficiency is inferior to that of coherent PSK by about 3 dB. The average probability of error for DPSK in additive white Gaussian noise is given by

$$P_{e,\,\mathrm{DPSK}} = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right) \qquad (5.75)$$

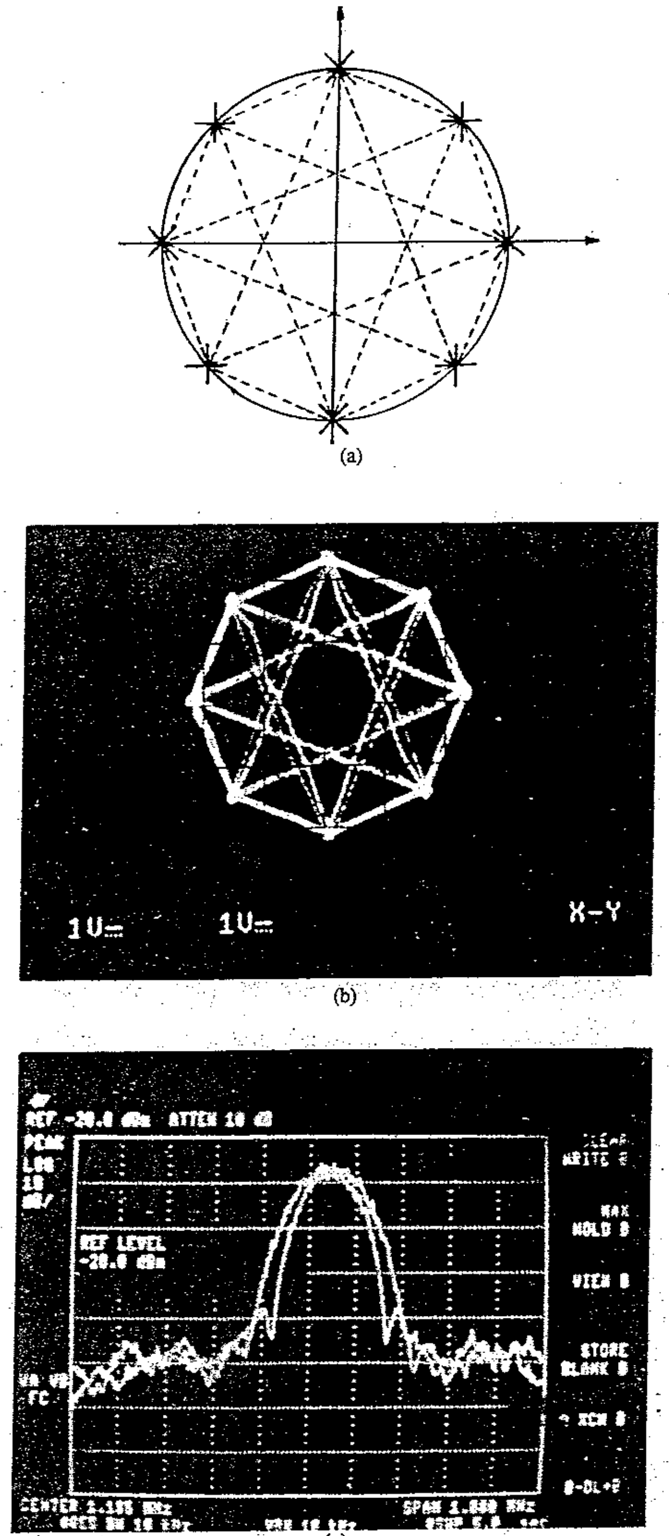

Figure 7.30 $\pi/4$-QPSK modulation. (*a*) Possible phase states of the $\pi/4$-QPSK modulated carrier at sampling instants. (*b*) The signal constellation with sinusoidal shaping. (*c*) Spectrum of the $\pi/4$-QPSK signal (*upper trace*) compared with that of an SQAM signal (*lower trace*). (From [Feh91] © IEEE.)