CHAPTER 7

7.1 In this case $X_k = \sum_{n=0}^{N} w_{nk} \cdot y_{nk}$, let

$$\underline{Y}_k = [\, y_{0k} \;\; y_{1k} \;\; y_{2k} \;\cdots\; y_{Nk} \,]^T,$$

$$\underline{W}_k = [\, W_{0k} \;\; W_{1k} \;\; W_{2k} \;\cdots\; W_{Nk} \,]^T.$$

We have $X_k = \underline{Y}_k^T \cdot \underline{W}_k = \underline{W}_k^T \cdot \underline{Y}_k$, and

$$e_k = d_k - X_k = d_k - \underline{Y}_k^T \cdot \underline{W}_k = d_k - \underline{W}_k^T \cdot \underline{Y}_k.$$

We can see that the expression for X_k and e_k are the same as equation (7.11) and (7.12), thus the MSE are identical. Using the same method described in section 7.3, we have the optimum weight vector $\hat{w}$ for MMSE,

$$\underline{\hat{W}} = R^{-1} \cdot \underline{P}$$

where
$$R = E[\underline{Y}_k \cdot \underline{Y}_k^T] = E\begin{bmatrix} y_{0k}^2 & y_{0k} \cdot y_{1k} & \cdots & y_{0k} \cdot y_{Nk} \\ y_{1k} y_{0k} & y_{1k}^2 & \cdots & y_{1k} \cdot y_{Nk} \\ \cdots & \cdots & \cdots & \cdots \\ y_{Nk} \cdot y_{0k} & y_{Nk} \cdot y_{1k} & \cdots & y_{Nk}^2 \end{bmatrix}.$$

and $\underline{P} = E[d_k \cdot \underline{Y}_k] = E[\, d_k \, y_{0k} \;\; d_k \, y_{1k} \;\cdots\; d_k \, y_{Nk} \,]^T$

7.2 (a) Assume $N > 2$. We have

$$\underline{Y}_k = \left[\, \sin\frac{2\pi k}{N} \quad \sin\frac{2\pi(k-1)}{N} \,\right]^T. \qquad d_k = 2\cos\left(\frac{2\pi k}{N}\right).$$

$$\underline{W}_k = [\, W_0 \;\; W_1 \,]^T.$$

$$\Rightarrow R = E[\underline{Y}_k \cdot \underline{Y}_k^T] = E\begin{bmatrix} \sin^2\frac{2\pi k}{N} & \sin\frac{2\pi k}{N} \cdot \sin\frac{2\pi(k-1)}{N} \\ \sin\frac{2\pi(k-1)}{N} \cdot \sin\frac{2\pi k}{N} & \sin^2\frac{2\pi(k-1)}{N} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\cos\frac{2\pi}{N} \\ \frac{1}{2}\cos\frac{2\pi}{N} & \frac{1}{2} \end{bmatrix}$$

7.2 Cont'd

$$\underline{P} = E[d_k \cdot \underline{Y}_k] = E\left[2\cos\frac{2\pi k}{N}\cdot\sin\frac{2\pi k}{N} \quad 2\cos\left(\frac{2\pi k}{N}\right)\cdot\sin\frac{2\pi(k-1)}{N}\right]^T$$

$$= \left[0 \quad -\sin\frac{2\pi}{N}\right]^T$$

$$E[d_k^2] = E\left[4\cos^2\frac{2\pi k}{N}\right] = 2$$

$$\Rightarrow \quad MSE = E[|e_k|^2] = E[d_k^2] + \underline{W}_k^T \cdot \mathbb{R} \cdot \underline{W} - 2\underline{P}^T \cdot \underline{W}$$

$$= 2 + [W_0 \quad W_1] \cdot \frac{1}{2}\begin{bmatrix} 1 & \cos\frac{2\pi}{N} \\ \cos\frac{2\pi}{N} & 1 \end{bmatrix} \cdot \begin{bmatrix} W_0 \\ W_1 \end{bmatrix} - 2\cdot\left[0 \quad -\sin\frac{2\pi}{N}\right]\begin{bmatrix} W_0 \\ W_1 \end{bmatrix}$$

$$= \underline{\underline{2 + \frac{1}{2}\left(W_0^2 + W_1^2 + 2\cos\left(\frac{2\pi}{N}\right)\cdot W_0 W_1\right) + 2W_1\cdot\sin\frac{2\pi}{N}}}$$

(b) For $N>2$, we have

$$\underline{\hat{W}} = \mathbb{R}^{-1}\cdot\underline{P} = \frac{2\begin{bmatrix} 1 & -\cos\frac{2\pi}{N} \\ -\cos\frac{2\pi}{N} & 1 \end{bmatrix}}{\sin^2\frac{2\pi}{N}} \cdot \left[0 \quad -\sin\frac{2\pi}{N}\right]^T$$

$$= 2\begin{bmatrix} \cos\frac{2\pi}{N} / \sin\frac{2\pi}{N} \\ -1/\sin\frac{2\pi}{N} \end{bmatrix}$$

$$\Rightarrow W_0 = \frac{2\cos\frac{2\pi}{N}}{\sin\frac{2\pi}{N}}, \quad W_1 = \frac{-2}{\sin\frac{2\pi}{N}}$$

$$\Rightarrow MMSE = 2 + \frac{1}{2}\left(\frac{4\cos^2\frac{2\pi}{N}}{\sin^2\frac{2\pi}{N}} - \frac{8\cos^2\frac{2\pi}{N}}{\sin^2\frac{2\pi}{N}} + \frac{4}{\sin^2\frac{2\pi}{N}}\right) - 2\cdot\frac{2}{\sin\frac{2\pi}{N}}\cdot\sin\frac{2\pi}{N}$$

$$= \underline{\underline{0}}$$

(c) $$MSE = 2 + \frac{1}{2}\left(W_0^2 + 2\cos\left(\frac{2\pi}{N}\right)\cdot W_0\cdot W_1 + W_1^2\right) + 2W_1\cdot\sin\frac{2\pi}{N}$$

$$= 2 + \frac{1}{2}(0+0+4) + 2\times(-2)\cdot\sin\frac{2\pi}{4}$$

$$= \underline{\underline{0}}$$

7.2 Cont'd

(d)
$$e_k = d_k - (W_0 \cdot Y_k + W_1 \cdot Y_{k-1})$$
$$= 2\sin\left(\frac{2\pi k}{N}\right) - (-2)\cdot \sin\frac{2\pi(k-1)}{N}$$
$$= 2\cdot\left[\sin\left(\frac{\pi}{2}k\right) + \sin\left(\frac{\pi}{2}(k-1)\right)\right]$$
$$\Rightarrow E[|e_k|^2] = 2\cdot E\left[\sin^2\left(\frac{\pi}{2}k\right) + \sin^2\left(\frac{\pi}{2}(k-1)\right) + 2\cdot\sin\left(\frac{\pi}{2}k\right)\cdot\sin\left(\frac{\pi}{2}(k-1)\right)\right]$$
$$= 2\cdot\left(\frac{1}{2}+\frac{1}{2}+0\right) = \underline{\underline{2}}$$

$\boxed{7.3}$ $MSE = 2 + \frac{1}{2}\left[W_0^2 + 2\cos\left(\frac{2\pi}{N}\right)\cdot W_0 W_1 + W_1^2\right] + 2 W_1 \sin\frac{2\pi}{N} = 2^2 = 4$

$$\Rightarrow 2 + \frac{1}{2}\left[W_0^2 + 2\cos\left(\frac{2\pi}{5}\right)\cdot W_0 \cdot W_1 + W_1^2\right] + 2\cdot W_1 \cdot \sin\frac{2\pi}{5} = 4$$
$$\Rightarrow \underline{\underline{0.5 W_0^2 + 0.5 W_1^2 + 0.309 W_0 \cdot W_1 + 1.902 W_1 - 2 = 0}}$$

Any pair of W_0 and W_1 that can satisfy the above equation can have the rms value of $\varepsilon_k = 2$.

$\boxed{7.4}$ Let N denote the number of coefficients in the equalizer, and M the time required between each iteration.

(a) For LMS, $M = (2N+1)\cdot 10^{-6}\,(s) = \underline{\underline{2N+1\ \mu s}}$

(b) For Kalman RLS, $M = \underline{\underline{2.5N^2 + 4.5N\ \mu s}}$

(c) For square root RLS DFE, $M = \underline{\underline{1.5N^2 + 6.5N\ \mu s}}$

(d) For Gradient lattice DFE, $M = \underline{\underline{13N - 8\ \mu s}}$

7.5 For $f_d = 100\,Hz$, we have the coherence time

$$T_c = \sqrt{\frac{9}{16\pi f_d^2}} \doteq 4.23\ msec$$

Therefore the maximum time interval before retraining is 4.23 msec. Suppose $N=5$, for LMS algorithm, each updating of the equalizer needs time T_u, where

$$T_u = (2N+1) \times 10^{-6} \times \text{iteration numbers to converge}$$

$$= (2\times 5+1)\times 10^{-6}\times 10^3 = 11\times 10^{-3}\,S \doteq 11\ ms$$

Require $T_u < 10\%$ transmission overhead $= 10\% \times 4.23 = 0.423$ msec,

it is impossible to implement the LMS algorithm on such a low speed DSP chip. If the DSP chip can perform 27 Million multiplications per second. T_u becomes $0.41ms < 4.23\times 10\% = 0.423ms$. Therefore for $f_d = 100Hz$ and $N=5$, the LMS algorithm can be implemented on a 27 Million multiplications per second DSP chip

Suppose each time slot of the signal contains 162 symbol, in 0.41ms time duration, there should be $162\times 10\% \doteq 16$ symbols. Therefore the maximum symbol rate is

$$R_{Smax} = \frac{16}{0.41} \doteq 39.02\ KSps \quad \text{due to channel coherence.}$$

Similarly, for RLS algorithm using the low speed DSP chip.

$$T_u = (2.5N^2 + 4.5N)\times 10^{-6} \times 50 \text{ iterations}$$

$$= (2.5\times 5^2 + 4.5\times 5)\times 10^{-6}\times 50 = 4.25\times 10^{-3}\,S$$

$$= 4.25\ ms > 4.23\times 10\% = 0.423\ ms$$

7.5 Cont'd

Thus it is also impossible to implement the RLS algorithm on a DSP chip with 1 Million multiplications per second. The minimum speed of the DSP chip required is

$$\frac{4.25\ ms}{10\% \cdot T_c(ms)} = \frac{4.25}{0.1 \times 4.23} \doteq \underline{10.1 \text{ Million multiplications/sec.}}$$

$$\Rightarrow T_u = \frac{4.25ms}{10.1} \doteq 0.42ms, \quad R_{smax} = \frac{16}{0.42} \doteq \underline{\underline{38.02 \text{ ksps.}}}$$

If the DSP chip with <u>27 Million multiplications per second</u> is used, we have $T_u \doteq \frac{4.25\ ms}{27} \doteq 0.157ms < 0.423ms$. and $R_{smax} = \frac{16}{0.157} = \underline{\underline{101.65 \text{ KSPS}}}$. We can see that using the same speed DSP chip, RLS algorithm can handle higher data rate than LMS algorithm.

(b) For <u>$f_d = 1000Hz$</u>, $T_c = 0.423\ ms$, therefore the maximum time interval before retraining is <u><u>0.423 ms</u></u>. We can see that even using a DSP chip that can perform 100 Million multiplication per second, since $T_u = \frac{11ms}{100} = 0.11ms > 10\% \cdot T_c = 0.0423\ ms$, <u>it is impossible to implement the LMS algorithm</u>. <u>For the RLS algorithm</u>, the minimum speed of the DSP chip required is $\frac{4.25ms}{10\% \cdot T_c(ms)} = \frac{4.25}{0.0423} \doteq 101$ million multiplications/sec, $\Rightarrow T_u = \frac{4.25ms}{101} \doteq 0.042ms$. $R_{smax} = \frac{16}{0.042} \doteq \underline{\underline{380.26\text{ksps}}}$

(c) For <u>$f_d = 10000\ Hz$</u>, $T_c = 0.0423\ ms$. It's <u><u>impossible</u></u> for both the LMS and RLS algorithms to implement using the DSP chip with current technology.

$\boxed{7.6}$ (a) See the MATLAB program p7-06.m

7.6 Cont'd

(d) If the second ray is placed at $t = 25\ \mu s$, the maximum delay spread is greater than the delay that the equalizer can offer, therefore <u>the data cannot be recreated correctly</u>. See Fig. p7-06d (a), (b), (c) and (d).

(e) When the second ray is set equal to zero, although the error after convergence becomes very small, it still exists. That's the equalizer noise. This is shown in Fig. p7-06e.

$\boxed{7.7}$ (a) Since $-6dB = \frac{1}{4}$, $Pr[\gamma_i \le \frac{\gamma}{4}] = 1 - e^{-\frac{\gamma/4}{\Gamma}} = 0.2$, where γ is the SNR threshold, we have

$$\frac{\gamma}{4\Gamma} = -\ln 0.8 \Rightarrow \frac{\Gamma}{\gamma} = \frac{1}{-4\ln 0.8} \doteq 1.12 \doteq \underline{\underline{0.5\ dB}}$$

Therefore, the mean SNR of the Rayleigh fading signal is 0.5 dB above the SNR threshold. Using equation (7.59), we have

(b). P_2 (6dB below the mean SNR threshold) $= 0.2^2 = \underline{\underline{0.04}}$

(c) P_3(6dB below the mean SNR threshold) $= 0.2^3 = \underline{\underline{0.008}}$

(d) P_4 (6dB below the mean SNR threshold) $= 0.2^4 = \underline{\underline{0.0016}}$

(e) From the above we can see that for a M branch selection diversity receiver, the probability that the

7.7 Cont'd

SNR will be 6dB below the mean SNR threshold is $\underline{\underline{0.2^M}}$.

$\boxed{7.8}$ See the MATLAB program p7_08.m and Fig. p7_08

$\boxed{7.9}$ In the maximal ratio combiner, the signals in the M branches are cophased and added with appropriate branch weighting factors a_i. The resulting signal envelope is $r_m = \sum_{i=1}^{M} a_i \cdot r_i$, the total noise power is $N_T = N \cdot \sum_{i=1}^{M} a_i^2$, where N is the noise power per branch. Therefore, the resulting SNR is $\gamma_M = \frac{r_m^2}{2N_T} = \frac{\left(\sum_{i=1}^{M} a_i \cdot r_i\right)^2}{2 \cdot N \sum_{i=1}^{M} a_i^2}$. The weighting factors a_i, $i=1,2,\cdots M$, are the solutions for the set of equations $\frac{d\gamma_M}{da_i} = 0$, $i=1,2,\cdots M$.

It can be shown that if $a_i = r_i/N$, the γ_M will be maximized and $\gamma_M = \frac{1}{2} \frac{\sum_{i=1}^{M}(r_i^2/N)^2}{\sum_{i=1}^{M}(r_i^2/N^2)} = \frac{1}{2}\sum_{i=1}^{M}\frac{r_i^2}{N} = \sum_{i=1}^{M}\gamma_i$

where $\gamma_i = \frac{1}{2N} \cdot r_i^2$ is the branch SNR and can be represented as $\gamma_i = \frac{1}{2N}(x_i^2 + y_i^2)$, where x_i and y_i are independent Gaussian random variables of equal variance σ^2 and zero mean. Thus γ_M is a chi-square distribution of $2M$ Gaussian random variables with variance $\frac{\sigma^2}{2N} = \frac{1}{2}\Gamma$.

The density function is thus

$$p(\gamma_M) = \frac{\gamma_M^{M-1} e^{-\gamma_M/\Gamma}}{\Gamma^M (M-1)!}, \quad \gamma_M \geq 0$$

7.10 Cont'd

$$10\log_{10}(\bar{\gamma}_M/\Gamma) = 10\log_{10}(6) \doteq \underline{\underline{7.78\ dB}}$$

For 6-branch selection diversity, the average SNR improvement is

$$10\log_{10}(\bar{\gamma}_M/\Gamma) = 10\log_{10}\left(\sum_{k=1}^{6}\frac{1}{k}\right) = 10\log_{10}(2.45) \doteq \underline{\underline{3.9\,dB}}$$

If $\frac{\gamma}{\Gamma} = 0.01$, for 6-branch maximal ratio combining,

$$P(\gamma_M \le \gamma) = 1 - e^{-\gamma/\Gamma}\cdot\sum_{k=1}^{6}\frac{(\gamma/\Gamma)^{k-1}}{(k-1)!} \doteq 1 - e^{-0.01}\cdot\sum_{k=1}^{6}\frac{(0.01)^{k-1}}{(k-1)!}$$

$$\doteq 1 - e^{-0.01}\left[e^{0.01} - \frac{(0.01)^{7-1}}{(7-1)!}\right] \doteq \underline{\underline{1.37\times10^{-15}}}$$

For 6-branch selection diversity,

$$P(\gamma_M \le \gamma) = (1-e^{-\gamma/\Gamma})^M = (1-e^{-0.01})^6 \doteq \underline{\underline{9.7\times10^{-13}}}$$

For a single Rayleigh fading channel,

$$P(\gamma_1 \le \gamma) = 1 - e^{-\gamma/\Gamma} = 1 - e^{-0.01} \doteq \underline{\underline{9.95\times10^{-3}}}$$

$\boxed{7.11}$ (a) Based on the definition of y, (it should be more suitable to call y the compliment of the system reliability), we have

$$1 - y = \exp[-P^{-1}(x)/\gamma_0] \Longrightarrow \underline{\underline{\gamma_0 = \frac{-P^{-1}(x)}{\ln(1-y)}}}$$

(b) $y = \underline{\underline{\left[1 - e^{-\frac{P^{-1}(x)}{\gamma_0}}\right]^M}}$

7.11 Cont'd

(c) For BPSK, $P_e(\gamma) = Q(\sqrt{2\gamma})$. Given $X = 10^{-3}$, we have

$$\gamma_0 = \frac{-\frac{[Q^{-1}(x)]^2}{2}}{\ln(1-y)} \doteq \frac{-\frac{3.1^2}{2}}{\ln(1-10^{-3})} \doteq 4802.6 \doteq \underline{\underline{36.8\ dB}}$$

(d) In this case, $\gamma_0 = \frac{-P^{-1}(x)}{\ln(1-y^{\frac{1}{M}})}$, thus

$$\gamma_0 = \frac{-\frac{3.1^2}{2}}{\ln[1-(10^{-3})^{\frac{1}{4}}]} = 24.54 \doteq \underline{\underline{13.9\ dB}}$$

8.11 See the MATLAB program p8_11.m and Fig. p8_11.

The 64 quantization vectors are (each column contains one vector):

Columns 1 through 7

-0.5245	0.7293	0.5761	-1.0416	-0.5821	-1.4484	0.2083
0.1233	0.5841	-0.2705	-0.9283	-0.8850	1.9520	-0.7157

Columns 8 through 14

-0.9619	-2.4045	-2.6594	1.3634	-2.2150	-0.6623	0.8973
0.4884	-1.3398	1.3320	0.0562	-0.3274	0.8941	0.9270

Columns 15 through 21

0.6199	-0.1141	-0.4555	-1.4531	0.2642	0.9874	0.8000
1.0388	-2.4886	0.4939	-1.1477	-1.0350	-0.3348	1.6057

Columns 22 through 28

-0.0942	0.4185	0.8428	1.3229	0.1160	-0.1277	-0.3933
0.3561	0.1407	0.1858	1.1117	0.2152	0.7864	-0.5289

Columns 29 through 35

1.5744	0.0709	0.5237	0.3896	-0.0673	0.4072	-0.7746
0.5004	1.4680	-1.6375	0.6859	-1.4108	2.2418	-1.3458

Columns 36 through 42

-0.8210	0.6018	0.0829	-0.4342	0.3812	1.0026	1.0249
-0.3752	-0.9841	-0.0980	1.8908	0.4029	-1.0752	-2.0029

Columns 43 through 49

1.7521	-1.4234	-0.1661	0.3837	-1.0973	0.1494	3.1329
-1.3038	-0.2866	-0.9883	1.2358	1.1021	1.0070	0.0327

Columns 50 through 56

-0.2452	-1.6159	-0.1840	1.0793	0.6323	1.5192	-1.2874
-1.8059	0.4938	0.0660	0.6205	-0.6823	-0.6571	-1.8795

Columns 57 through 63

-0.2711	1.3955	-0.4755	-0.9577	0.2313	1.9427	2.0501
-0.2225	2.0920	1.2940	0.0914	-0.3363	1.1305	-0.1682

Column 64

0.0415
0.5773

The mean squared error distortion computed from the test sequence is 0.0275. We can see that the MSE distortion for the rate 3, 2-dimension vector quantizer is less than that for the rate 3 scalar quantizer.

Theoretical lower bound on MSE distortion for a rate 3 quantizer of large dimension $= 2^{-2R} \cdot \sigma^2 = 2^{-2\times 3} \times 1 \doteq \underline{0.0156}$. (See equation (64) in [Mak85]).

[Mak85] Makhoul, J., Roucos, S., and Gish., H., "Vector quantization in speech coding.", Proc. IEEE. pp. 1551-1582, Nov. 1985.

8.12 | See the MATLAB program p8.12.m and Fig. p8.12.

The 64 quantization vectors are (each column contains one vector)

Columns 1 through 7

0.0577	-1.8433	-0.6225	-1.1155	-2.0601	-0.1380	-0.0921
0.3076	-1.4062	-0.3863	-0.7736	-0.7074	-1.5222	-0.5689

Columns 8 through 14

0.2003	-1.5632	-0.2561	-1.9816	-0.9409	-0.8360	-0.9894
-0.2440	-1.9189	0.2848	-2.6935	-0.5288	-2.2484	-1.1809

Columns 15 through 21

1.2101	1.1565	2.0051	0.3935	0.4235	-0.0326	-0.6749
0.4419	1.8960	2.1411	-0.1250	1.7790	-0.1932	0.3061

Columns 22 through 28

-0.3221	0.1544	0.8663	-1.2890	0.4229	0.4424	2.4456
-0.3228	0.5541	-0.3847	-1.4615	1.3479	0.8415	1.3688

Columns 29 through 35

-0.2333	1.2679	-0.7651	0.1331	0.8701	0.3033	-0.7017
1.3738	-0.1312	0.8026	0.9053	0.1702	0.3488	-0.7243

Columns 36 through 42

0.4031	1.1372	0.0144	0.5464	-1.4392	-0.7975	-0.1514
-0.5522	0.8577	-0.9260	0.3412	-1.0051	-0.2046	0.0083

Columns 43 through 49

1.5928	1.4233	-0.4340	-0.1647	-2.5761	-0.7477	1.9103
1.4323	0.8812	-0.9211	0.7469	-1.8792	-1.0637	0.5321

Columns 50 through 56

0.5527	0.5871	-0.4250	-1.2357	0.7594	-0.4568	-0.4889
0.0519	-1.0419	-1.2822	-0.3979	2.4654	-0.5975	0.0831

Columns 57 through 63

-1.4924	-0.7798	0.7855	0.7272	-0.9640	-0.3679	0.1247
0.1370	-1.6209	0.6134	1.0414	0.1578	0.5105	0.0767

Column 64

1.0224
1.3258

The mean squared error distortion computed from the test sequence is 0.0249, which is less than that for the uncorrelation case in problem 7.11.

Comparing Fig. p8.12 and Fig. p8.11, we can see that the quantization vectors in this problem become more condense due to the correlation between the samples.

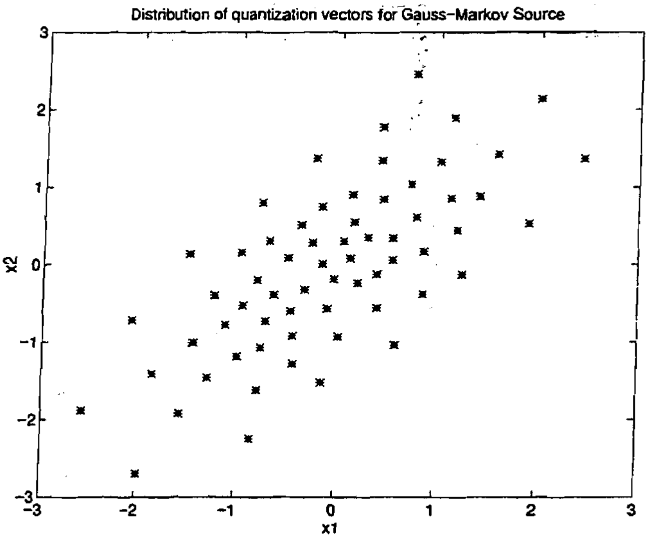
Distribution of quantization vectors for Gauss-Markov Source
3
2
1
0
-1
-2
-3
x2
-3
-2
-1
0
1
2
3
x1

Fig. p8_12

8.12 Cont'd

For the length of the training sequence, the relative correlation, and the dimensions of the vector quantizer, increase one of the three factors and fix the other two, the MSE distortion will decrease.