Exercise 1:

Let \( \{h_{k,l}\} \) denote the impulse response of a general discrete-time linear filter. The output at time \( n \) due to the input signal is \( \sum_{l=1}^{n} h_{n,l}s_l \), and that due to noise is \( \sum_{l=1}^{n} h_{n,l}N_l \). Thus the output SNR at time \( n \) is

\[
SNR = \frac{|\sum_{l=1}^{n} h_{n,l}s_l|^2}{E\{(\sum_{l=1}^{n} h_{n,l}N_l)^2\}} = \frac{|h_n^T\bar{s}|^2}{h_n^T \Sigma_N h_n}
\]

where \( h_n = (h_{n,1}, h_{n,2}, \ldots, h_{n,n})^T \).

Since \( \Sigma_N > 0 \), we can write \( \Sigma_N = \Sigma_N^{1/2} \Sigma_N^{1/2} \) when \( \Sigma_N^{1/2} \) is invertible and symmetric. Thus,

\[
SNR = \frac{|(\Sigma_N^{1/2} h_n)^T \Sigma_N^{-1/2} \bar{s}|^2}{||\Sigma_N^{1/2} h_n||^2}
\]

By the Schwarz Inequality \( ||x^T y|| \leq ||x|| ||y|| \), we have

\[
SNR \leq ||\Sigma_N^{1/2} \bar{s}||^2
\]

with equality if and only if \( \Sigma_N^{1/2} h_n = \lambda \Sigma_N^{-1/2} \bar{s} \) for a constant \( \lambda \). Thus, max SNR occurs when \( h_n = \lambda \Sigma_N^{-1/2} \bar{s} \). The constant \( \lambda \) is arbitrary (it does not affect SNR), so we can take \( \lambda = 1 \), which gives the desired result.

Exercise 3:

a. From Exercise 15 of Chapter II, the optimum test here has critical regions:

\[
\Gamma_k = \{y \in \mathbb{R}^n \left| p_k(y) = \max_{0 \leq l \leq M-1} p_l(y) \right. \}.
\]
Since $p_l$ is the $N(\underline{s}_l, \sigma^2 \mathbf{I})$ density, this reduces to

$$\Gamma_k = \{ y \in \mathbb{R}^n \mid \| y - s_k \|^2 = \min_{0 \leq l \leq M-1} \| y - s_l \|^2 \}$$

$$= \{ y \in \mathbb{R}^n \mid s_k^T y = \max_{0 \leq l \leq M-1} s_l^T y \} .$$

b. We have

$$P_e = \frac{1}{M} \sum_{k=0}^{M} P_k(\Gamma_k^c),$$

and

$$P_k(\Gamma_k^c) = 1 - P_k(\Gamma_k) = 1 - P_k(\max_{0 \leq l \neq k \leq M-1} s_l^T Y < s_k^T Y) .$$

Due to the assumed orthogonality of $s_1, \ldots, s_n$, it is straightforward to show that, under $H_k$, $s_1^T Y, s_2^T Y, \ldots, s_n^T Y$, are independent Gaussian random variables with variances $\sigma^2 \| s_1 \|^2$, and with means zero for $l \neq k$ and mean $\| s_1 \|^2$ for $l = k$. Thus

$$P_k(\max_{0 \leq l \neq k \leq M-1} s_l^T Y < s_k^T Y)$$

$$= \frac{1}{\sqrt{2\pi \sigma \| s_1 \|}} \int_{-\infty}^{\infty} P_k(\max_{0 \leq l \neq k \leq M-1} s_l^T Y < z) \ e^{-(z-\| s_1 \|^2)/2\sigma^2 \| s_1 \|^2} \ dz .$$

Now

$$P_k(\max_{0 \leq l \neq k \leq M-1} s_l^T Y < z) = P_k(\bigcap_{0 \leq l \neq k \leq M-1} \{ s_l^T Y < z \})$$

$$= \prod_{0 \leq l \neq k \leq M-1} P_k(s_l^T Y < z)$$

$$= \left[ \Phi\left( \frac{z}{\sigma \| s_1 \|} \right) \right]^{M-1} .$$

Combining the above and setting $x = z/\sigma \| s_1 \|$ yields

$$1 - P_k(\Gamma_k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\Phi(x)]^{M-1} \ e^{-(x-\mu)^2/2\sigma^2} \ dx, \ k = 0, \ldots, M-1 ,$$

and the desired expression for $P_e$ follows.

**Exercise 6:**

Since $Y \sim N(\mu_n, \Sigma)$, it follows that $\hat{Y}_k$ is linear in $Y_1, \ldots, Y_{k-1}$, and that $\hat{\sigma}_{\hat{Y}_k}^2$ does not depend on $Y_1, \ldots, Y_{k-1}$. Thus, $\hat{I}$ is a linear transformation of $\hat{Y}$ and is Gaussian. We need only show that $E\{ I \} = 0$ and $cov(I) = \mathbf{I}$.

We have

$$E\{ I_k \} = \frac{E\{ Y_k \} - E\{ \hat{Y}_k \}}{\hat{\sigma}_k} .$$
Since $\hat{Y}_k = E\{Y_k|Y_1, \ldots, Y_{k-1}\}$, $E\{\hat{Y}_k\}$ is an iterated expectation of $Y_k$; hence $E\{Y_k\} = E\{\hat{Y}_k\}$ and $E\{I_k\} = 0, k = 1, \ldots, n$. To see that $cov(I) = I$, note first that

$$
\text{Var } (I_k) = E\{I_k^2\} = \frac{E\{(Y_k - \hat{Y}_k)^2\}}{\hat{\sigma}_Y^2} = \frac{\hat{\sigma}_Y^2}{\hat{\sigma}_Y^2} = 1.
$$

Now, for $l < k$, we have

$$
cov (I_k, I_l) = E\{I_k I_l\} = E\{(Y_k - \hat{Y}_k)(Y_l - \hat{Y}_l)\}.
$$

Noting that

$$
E\{(Y_k - \hat{Y}_k)(Y_l - \hat{Y}_l)\} = E\{E\{(Y_k - \hat{Y}_k)(Y_l - \hat{Y}_l)|Y_1, \ldots, Y_{k-1}\}\}
$$

$$
= E\{(E\{Y_k|Y_1, \ldots, Y_{k-1}\} - \hat{Y}_k)(Y_l - \hat{Y}_l)\} = E\{(\hat{Y}_k - \hat{Y}_k)(Y_l - \hat{Y}_l)\} = 0,
$$

we have $cov(I_k, I_l) = 0$ for $l < k$. By symmetry we also have $cov(I_k, I_l) = 0$ for $l > k$, and the desired result follows.

**Exercise 7:**

a. The likelihood ratio is

$$
L(y) = \frac{1}{2} e^{x^T \Sigma^{-1} y - d^2/2} + \frac{1}{2} e^{-x^T \Sigma^{-1} y - d^2/2}
$$

$$
= e^{-d^2/2} \cosh y^T \Sigma^{-1} y,
$$

which is monotone increasing in the statistic

$$
T(y) \equiv \left| y^T \Sigma^{-1} y \right|.
$$

(Here, as usual, $d^2 = \bar{s}^T \Sigma^{-1} \bar{s}$.) Thus, the Neyman-Pearson test is of the form

$$
\tilde{\delta}_{NP}(y) = \begin{cases} 
1 & \text{if } T(y) > \eta \\
\gamma, & \text{if } T(y) = \eta \\
0 & \text{if } T(y) < \eta.
\end{cases}
$$

To set the threshold $\eta$, we consider

$$
P_0(T(Y) > \eta) = 1 - P\left(-\eta \leq \bar{s}^T \Sigma^{-1} N \leq \eta\right) = 1 - \Phi(\eta/d) + \Phi(-\eta/d) = 2[1 - \Phi(\eta/d)],
$$

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where we have used the fact that $s^T\Sigma^{-1}N$ is Gaussian with zero mean and variance $d^2$. Thus, the threshold for size $\alpha$ is

$$\eta = d\Phi^{-1}(1 - \alpha/2).$$

The randomization is unnecessary.

The detection probability is

$$P_D(\delta_{NP}) = \frac{1}{2} P_1(T(\mathbf{Y}) > \eta|\Theta = +1) + \frac{1}{2} P_1(T(\mathbf{Y}) > \eta|\Theta = -1)$$

$$= \frac{1}{2} \left[ 1 - P\left( -\eta \leq -d^2 + s^T\Sigma^{-1}N \leq \eta \right) \right] + \frac{1}{2} \left[ 1 - P\left( -\eta \leq +d^2 + s^T\Sigma^{-1}N \leq \eta \right) \right]$$

$$= 2 - \Phi\left( \Phi^{-1}(1 - \alpha/2) + d \right) - \Phi\left( \Phi^{-1}(1 - \alpha/2) - d \right).$$

b. Since the likelihood ratio is the average over the distribution of $\Theta$ of the likelihood ratio conditioned on $\Theta$, we have

$$L(y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_\theta} e^{(\theta s^T y - n\theta^2 s^2/2)/\sigma^2} e^{-\theta^2/2\sigma^2} d\theta$$

$$= k_1 e^{k_2 |s^T y|} \frac{1}{\sqrt{2\pi} v} \int_{-\infty}^{\infty} e^{-(\theta - \mu)^2/2v^2} d\theta = k_1 e^{k_2 |s^T y|},$$

where

$$v^2 = \frac{\sigma_\theta^2 n s^2}{\sigma_\theta^2 \sigma^2 + ns^2},$$

$$\mu = \frac{v^2 s^T y}{2},$$

$$k_1 = \frac{v}{\sigma_\theta},$$

and

$$k_2 = \frac{v^2}{4}.$$

**Exercise 13:**

a. In this situation, the problem is that of detecting a Gaussian signal with zero mean and covariance matrix $\Sigma_s = \text{diag}\{A_{s_1}^2, A_{s_2}^2, \ldots, A_{s_n}^2\}$, in independent i.i.d. Gaussian noise with unit variance; and thus the Neyman-Pearson test is based on the quadratic statistic

$$T(y) = \sum_{k=1}^{n} \frac{A_{s_k}^2}{A_{s_k}^2 + 1} y_k^2.$$
b. Assuming $s_k \neq 0$, for all $k$, a sufficient condition for a UMP test is that $s_k^2$ is constant. In this case, an equivalent test statistic is the radiometer $\sum_{k=1}^{n} y_k^2$, which can be given size $\alpha$ without knowledge of $A$.

c. From Eq. (III.B.110), we see that an LMP test can be based on the statistic

$$T_{lo}(y) = \sum_{k=1}^{n} s_k^2 y_k^2.$$ 

**Exercise 15:**

Let $L_a$ denote the likelihood ratio conditioned on $A = a$. Then the undconditioned likelihood ratio is

$$L(y) = \int_0^\infty L_a(y)p_A(a)da = \int_0^\infty e^{-na^2/4\sigma^2} I_0(a^2 \hat{r}/\sigma^2) p_A(a) da,$$

with $\hat{r} \equiv r/A$, where $r = \sqrt{y_c^2 + y_s^2}$ as in Example III.B.5. Note that

$$\hat{r} = \sqrt{\left(\sum_{k=1}^{n} b_k \cos((k-1)\omega_c T_s) y_k\right)^2 + \left(\sum_{k=1}^{n} b_k \sin((k-1)\omega_c T_s) y_k\right)^2},$$

which can be computed without knowledge of $A$. Note further that

$$\frac{\partial L(y)}{\partial \hat{r}} = \frac{1}{\sigma^2} \int_0^\infty e^{-na^2/4\sigma^2} a^2 I_0'(a^2 \hat{r}/\sigma^2) p_A(a) da > 0,$$

where we have used the fact that $I_0$ is monotone increasing in its argument. Thus, $L(y)$ is monotone increasing in $\hat{r}$, and the Neyman-Pearson test is of the form

$$\tilde{\delta}_{NP}(y) = \begin{cases} 
1 & \text{if } \hat{r} > \tau' \\
\gamma, & \text{if } \hat{r} = \tau' \\
0 & \text{if } \hat{r} < \tau'
\end{cases}.$$

To get size $\alpha$ we choose $\tau'$ so that $P_0(\hat{R} > \tau') = \alpha$. From (III.B.72), we have that

$$P_0(\hat{R} > \tau') = e^{-(\tau')^2/na^2},$$

from which the size-$\alpha$ desired threshold is $\tau' = \sqrt{-na^2 \log \alpha}$.

The detection probability can be found by first conditioning on $A$ and then averaging the result over the distribution of $A$. (Note that we have not used the explicit form of the distribution of $A$ to derive any of the above results.) It follows from (III.B.74) that

$$P_D = \int_0^\infty Q\left(\frac{a}{\sigma} \sqrt{n/2}, \tau_0\right) p_A(a) da = \int_0^\infty \int_0^{\infty} xe^{-(x^2+na^2/2\sigma^2)/2} I_0(x) \frac{a}{A_0} e^{-a^2/2A_0^2} dx da$$

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\[
I_0(e^{-x^2/2}) = \int_0^\infty \int_0^\infty \frac{a}{A^2_0} e^{-a^2/2A^2_0} I_0(x) \frac{a}{\sigma} \sqrt{n/2} \, dx, \\
\text{where } a_0 = \sqrt{\frac{2A^2_0}{nA^2_0 + 2\sigma^2}}.
\]

On making the substitution \( y = a/a_0 \), this integral becomes

\[
P_D = \frac{a_0^2}{A^2_0} \int_0^\infty xe^{-x^2(1-b_0^2)/2} \int_0^\infty ye^{-(y^2+b_0^2x^2)/2} I_0(b_0xy) \, dy \, dx = \frac{a_0^2}{A^2_0} \int_0^\infty xe^{-x^2(1-b_0^2)/2} Q(b_0x, 0) \, dx,
\]

where \( b_0^2 = na_0^2/2\sigma^2 \). Since \( Q(b, 0) = 1 \) for any value of \( b \), and since \( 1-b_0^2 = a_0^2/A^2_0 \), the detection probability becomes

\[
P_D = \frac{a_0^2}{A^2_0} \int_0^\infty xe^{-x^2(1-b_0^2)/2} \, dx = e^{-\tau_0^2(1-b_0^2)/2} = \exp(-\frac{\tau_0^2}{2(1 + \frac{nA^2_0}{2\sigma^2})}) = \alpha x_0,
\]

where \( x_0 = \frac{1}{1 + \frac{nA^2_0}{2\sigma^2}} \).

**Exercise 16:**

The right-hand side of the given equation is simply the likelihood ratio for detecting a \( \mathcal{N}(\mu, \Sigma_S) \) signal in independent \( \mathcal{N}(0, \sigma^2 I) \) noise. From Eq. (III.B.84), this is given by

\[
\exp\left(\frac{1}{2\sigma^2} y^T \Sigma_S (\sigma^2 I + \Sigma_S)^{-1} y + \frac{1}{2} \log(|\sigma^2 I|/|\sigma^2 I + \Sigma_S|)\right).
\]

We thus are looking for a solution \( \hat{S} \) to the equation

\[
2\hat{S}^T y - \| \hat{S} \|^2 = y^T \Sigma_S (\sigma^2 I + \Sigma_S)^{-1} y + \sigma^2 \sum_{k=1}^n \log\left(\frac{\sigma^2}{\sigma^2 + \lambda_k}\right),
\]

where \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigenvalues of \( \Sigma_S \). On completing the square on the left-hand side of this equation, it can be rewritten as

\[
\| \hat{S} - y \|^2 = \| y \|^2 - y^T \Sigma_S (\sigma^2 I + \Sigma_S)^{-1} y - \sigma^2 \sum_{k=1}^n \log\left(\frac{\sigma^2}{\sigma^2 + \lambda_k}\right)
\]

\[
\equiv \sigma^2 \left[ y^T (\sigma^2 I + \Sigma_S)^{-1} y - \sum_{k=1}^n \log\left(\frac{\sigma^2}{\sigma^2 + \lambda_k}\right) \right],
\]

which is solved by

\[
\hat{S} = y \pm \frac{\sigma}{\| y \|} \left[ y^T (\sigma^2 I + \Sigma_S)^{-1} y - \sum_{k=1}^n \log\left(\frac{\sigma^2}{\sigma^2 + \lambda_k}\right) \right]^{1/2} \nu,
\]

for any nonzero vector \( \nu \).